Math 561 - Homework 6

SOLUTIONS

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1. Let $\mathbf{V}$ be a vector space and let $T\colon \mathbf{V}\to\mathbf{V}$ be a linear operator. Prove that T is one-to-one if and only if $\lambda = 0$ is *not* an eigenvalue of T.

   Proof. We have:

   $$\begin{aligned} T \text{ is one-to-one} &\iff \mathbf{N}(T) = \{\mathbf{0}\}\\ &\iff \mathbf{N}(T-0I) = \{\mathbf{0}\}\\ &\iff \lambda = 0 \text{ is not an eigenvalue of } T, \end{aligned}$$

   with the last equivalence because λ is an eigenvalue of T if and only if $\mathbf{N}(T-\lambda I)\neq\{\mathbf{0}\}$. □

2. Let $\mathbf{V} = \mathbb{R}^\infty$ be the vector space of all real sequences, and let ρ and λ be the right and left shift operators on $\mathbf{V}$,

   $$\begin{aligned} \rho(a_1,a_2,a_3,\ldots) &= (0,a_1,a_2,\ldots),\\ \lambda(a_1,a_2,a_3,\ldots) &= (a_2,a_3,a_4,\ldots). \end{aligned}$$

   Recall that because $\mathbf{V}$ is infinite dimensional, neither ρ nor λ have characteristic polynomials.

   (i) Determine all eigenvalues of ρ and their corresponding eigenvectors, or prove that none exist.

   Answer. Since ρ is one-to-one, we know that 0 is not an eigenvalue of ρ. So let $\alpha\neq 0$ be a scalar, and suppose that $\mathbf{x}$ is a vector such that $\rho(\mathbf{x}) = \alpha\mathbf{x}$. We claim that in this case, we must have $\mathbf{x} = (0,0,0,\ldots)$, which will prove that ρ does not have any eigenvectors corresponding to α. That is, this will show that ρ does not have any eigenvectors.

   Letting $\mathbf{x} = (a_1,a_2,a_3,\ldots)$, $\rho(\mathbf{x}) = \alpha\mathbf{x}$ gives

   $$(0,a_1,a_2,a_3,\ldots) = (\alpha a_1,\alpha a_2,\alpha a_3,\ldots);$$

   we prove by induction that $a_n = 0$.

   As $\alpha a_1 = 0$ and $\alpha\neq 0$, it follows that $a_1 = 0$. Assuming that $a_k = 0$, we have that $\alpha a_{k+1} = \alpha_k = 0$, so $\alpha_{k+1} = 0$. Thus, $\mathbf{x} = (0,0,0,\ldots)$ as claimed.

   It follows that ρ does not have any eigenvalues and does not have any eigenvectors. □

   (ii) Determine all eigenvalues of λ and their corresponding eigenvectors, or prove that none exist.

   Answer. We claim that every scalar is an eigenvalue of λ.

   Indeed, let α be a scalar. We need a nonzero vector $\mathbf{x}$ such that $\lambda(\mathbf{x}) = \alpha\mathbf{x}$. This would mean:

   $$(a_2,a_3,a_4,\ldots) = (\alpha a_1,\alpha a_2,\alpha a_3,\ldots).$$

   This yields the equations $a_{n+1} = \alpha a_n$ for $n = 1,2,\ldots$. Inductively, we have $a_2 = \alpha a_1$; and if $a_{k+1} = \alpha^k a_1$, then

   $$\alpha_{k+2} = \alpha a_{k+1} = \alpha(\alpha^k a_1) = \alpha^{k+1}a_1;$$

   so $a_n = \alpha^{n-1}a_1$ for all n. Thus,

   $$\mathbf{x} = a_1(1,\alpha,\alpha^2,\alpha^3,\ldots).$$

   If $a_1\neq 0$, then this is an eigenvectors.

   Thus, the eigenvalues of λ are all real numbers; and for the scalar α, the eigenvectors are the nonzero scalar multiples of $(1,\alpha,\alpha^2,\alpha^3,\ldots)$. □

3. Let A be the following 3×3 matrix:

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}.$$

(i) Find the characteristic polynomial of A, and use it to determine all eigenvalues of A.
Answer. Taking the determinant of $A - tI$, we have:

$$\det(A - tI) = \det \begin{pmatrix} 3-t & 1 & 1 \\ 2 & 4-t & 2 \\ -1 & -1 & 1-t \end{pmatrix}$$
$$= (3-t)(4-t)(1-t) + (2)(-1)(1) + (-1)(1)(2) - (-1)(4-t) - (2)(1-t) - (3-t)(-1)(2)$$
$$= (-t^3 + 8t^2 - 19t + 12) - 2 - 2 + (4-t) - (2-2t) + (6-2t)$$
$$= -t^3 + 8t^2 - 20t + 16 = -(t-4)(t^2 - 4t + 4) = -(t-4)(t-2)^2.$$

So the eigenvalues of A are $\lambda = 4$ and $\lambda = 2$ (the latter with algebraic multiplicity 2). □

(ii) Determine whether A is diagonalizable. If A is not diagonalizable, explain why. If A is diagonalizable, find an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.
Answer. The matrix A will be diagonalizable if and only if $\dim(E_2) = 2$. This is the nullity of $A - 2I$; we have:

$$\mathbf{N}(A - 2I) = \mathbf{N} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ -1 & -1 & -1 \end{pmatrix} = \mathbf{N} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This has dimension 2, spanned by $(-1, 1, 0)^T$ and $(-1, 0, 1)^T$.
For E_4, we have

$$\mathbf{N}(A - 4I) = \mathbf{N} \begin{pmatrix} -1 & 1 & 1 \\ 2 & 0 & 2 \\ -1 & -1 & -3 \end{pmatrix} = \mathbf{N} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ -1 & -1 & -3 \end{pmatrix} = \mathbf{N} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix},$$

which has dimension 1, spanned by $(-1, -2, 1)^T$.
Thus, an ordered basis for $\mathbb{F}^3$ consisting of eigenvectors is given by

$$\beta = [(-1, 1, 0)^T, (-1, 0, 1)^T, (-1, -2, 1)^T],$$

so one possibliity for Q, and the resulitng D, are:

$$Q = \begin{pmatrix} -1 & -1 & -1 \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}. \quad \square$$

4. Let $\mathbf{V} = \mathbb{F}^2$ and let $T(x, y) = (y, x)$.

(i) Find the characteristic polynomial of T.
Answer. Relative to the standard basis β, we have

$$[T]^\beta_\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

so the characteristic polynomial of T is

$$\begin{vmatrix} -t & 1 \\ 1 & -t \end{vmatrix} = t^2 - 1 = (t-1)(t+1).$$

(ii) Find all eigenvalues and corresponding eigenvectors of T.

Answer. The eigenvalues are $\lambda = 1$ and $\lambda = -1$.

If $2 \neq 0$ in $\mathbb{F}$, these are distinct eigenvalues; the eigenvectors corresponding to 1 are the nonzero vectors in $\mathbf{N}(T - I)$; using the matrix, we have:

$$\mathbf{N}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = \mathbf{N}\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix},$$

so $E_1 = \text{span}\{(1,1)\}$. The eigenvectors are the vectors of the form (a,a), $a \neq 0$.

For $\lambda = -1$, we have

$$\mathbf{N}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \mathbf{N}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

So $E_{-1} = \text{span}\{(1,-1)\}$. The eigenvectors are the vectors of the form $(a,-a)$, $a \neq 0$.

Note that if $2 = 0$, then $1 = -1$, so the two sets above are equal to each other. □

5. Let $\mathbf{V}$ be a finite dimensional vector space over $\mathbb{C}$. Prove that if $T \in \mathcal{L}(\mathbf{V})$, then T has eigenvectors.

Proof. Actually, this needs to assume that $\mathbf{V} \neq \{\mathbf{0}\}$. If $\mathbf{V} = \{\mathbf{0}\}$ then it clearly does not have eigenvectors.

So assume that $\dim(\mathbf{V}) > 0$. We know that the characteristic polynomial of T, $p(t)$, is a polynomial of degree $\dim(\mathbf{V})$, and that a scalar λ is an eigenvalue for T if and only if $p(\lambda) = 0$. Since $p(t)$ is a nonconstant polynomial with complex coefficients, by the Fundamental Theorem of Algebra it has at least one root λ, so T has at least one eigenvalue. But if λ is an eigenvalue of T, then there must exist an eigenvector corresponding to λ in $\mathbf{V}$, so T has eigenvectors, as claimed. □

6. Let $\mathbf{V} = \mathbb{F}^2$ and let $T(x,y) = (y,-x)$.

(i) Prove that T does not have eigenvalues if $\mathbb{F} = \mathbb{R}$.

Proof. Here, relative to the standard ordered basis β, we have

$$[T]_\beta^\beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The characteristic polynomial is

$$\det(T - tI) = \begin{vmatrix} -t & 1 \\ -1 & -t \end{vmatrix} = t^2 + 1.$$

If $\mathbb{F} = \mathbb{R}$, then $t^2 + 1$ has no roots, and therefore T has no eigenvalues. □

(ii) Determine all eigenvalues of T if $\mathbb{F} = \mathbb{C}$.

Answer. If $\mathbb{F} = \mathbb{C}$, then the characteristic polynomial factors as $t^2 + 1 = (t - i)(t + i)$. So the eigenvalues are $\lambda_1 = i$ and $\lambda_2 = -i$. □

(iii) Find a basis for $\mathbf{V}$ consisting of eigenvectors of T if $\mathbb{F} = \mathbb{C}$.

Answer. For $\lambda_1 = i$, we have

$$\mathbf{N}(T - iI) = \mathbf{N}\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} = \mathbf{N}\begin{pmatrix} 1 & i \\ -1 & -i \end{pmatrix} = \mathbf{N}\begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix}.$$

So an eigenvector is $(-i,1)^T$.

For $\lambda_2 = -i$, we have

$$\mathbf{N}(T + iI) = \mathbf{N}\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix} = \mathbf{N}\begin{pmatrix} 1 & -i \\ -1 & i \end{pmatrix} = \mathbf{N}\begin{pmatrix} 1 & -i \\ 0 & 0 \end{pmatrix},$$

so an eigenvector is given by $(i,1)^T$.

Thus, one possible basis of $\mathbf{V}$ consisting of eigenvectors of T is $\gamma = [(-i,1)^T, (i,1)^T]$. □

7. Let $T\colon \mathbb{C}^2 \to \mathbb{C}^2$ be given by $T(x,y) = (x+y, y)$.

(i) Find all eigenvalues and eigenvectors of T.

Answer. Using the standard ordered basis β, we have:

$$[T]_\beta^\beta = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

So the characteristic polynomial is

$$\det([T - tI]_\beta^\beta) = \begin{vmatrix} 1-t & 1 \\ 0 & 1-t \end{vmatrix} = (1-t)^2.$$

So the only eigenvalue of T is $\lambda = 1$.

The eigenspace is then $\mathbf{N}(T - I)$:

$$\mathbf{N}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

So $E_1 = \operatorname{span}\{(1,0)\}$. The eigenvectors are the vectors of the form $(a,0)$ with $a \neq 0$. □

(ii) Determine whether T is diagonalizable. If T is not diagonalizable, explain why. If T is diagonalizable, then find a basis for $\mathbb{C}^2$ consisting of eigenvectors of T.

Answer. T is not diagonalizable, because the only eigenvalue is $\lambda = 1$, but $\dim(E_1) = 1$; that means that any two eigenvectors of T will both be in E_1, and two vectors in E_1 are always linearly dependent. So there can be no basis for $\mathbb{C}^2$ consisting of eigenvectors of T.

Alternatively, the geometric multiplicity of $\lambda = 1$ is 1, but the algebraic multiplicity is 2. Since they are not equal to each other, T cannot be diagonalizable. □

8. Let $\mathbf{V} = \mathsf{M}_{n\times n}(\mathbb{R})$ be given by $T(A) = A^t$, the transpose of A.

(i) Show that the only eigenvalues of T are $\lambda = -1$ and $\lambda = 1$, and describe the corresponding eigenvectors.

Answer. Consider a $n \times n$ matrix, and assume that $A^t = \lambda A$ for some nonzero matrix A.

Every diagonal entry of A must then satisfy $a_{ii} = \lambda a_{ii}$; thus, if $a_{ii} \neq 0$ for any i, it follows that $\lambda = 1$.

If all diagonal entries of A are zero, let $a_{ij} \neq 0$ with $i \neq j$. Then $a_{ji} = \lambda a_{ij}$ and $a_{ij} = \lambda a_{ji}$, so we get $a_{ij} = \lambda a_{ji} = \lambda(\lambda a_{ij}) = \lambda^2 a_{ij}$. Since $a_{ij} \neq 0$, it follows that $\lambda^2 = 1$, so either $\lambda = 1$ or $\lambda = -1$.

Thus, the only possible eigenvalues are $\lambda = 1$ and $\lambda = -1$. If $n = 1$, then $\lambda = 1$ is the only possible eigenvalue, as no nonzero 1×1 matrix satisfies $(a) = -(a)$; and (1) is an eigenvector corresponding to 1.

For $n > 1$, both are in fact eigenvalues: for $\lambda = 1$, the identity matrix is an eigenvector. For $\lambda = -1$, the matrix $A = (a_{ij})$ with

$$a_{ij} = \begin{cases} 0 & \text{if } i = j, \\ 1 & \text{if } i < j, \\ -1 & \text{if } i > j; \end{cases}$$

satisfies $A^t = -A$. So both of them are eigenvalues.

(ii) Find an ordered basis β for $\mathsf{M}_{2\times 2}(\mathbb{R})$ such that $[T]_\beta^\beta$ is a diagonal matrix.

Answer. A 2×2 matrix that is an eigenvector for $\lambda = -1$ is of the form

$$\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}, \qquad a \neq 0.$$

For $\lambda = 1$, we have that the matrix must be of the form

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}, \qquad a, b, c \in \mathbb{F}, \text{ not all zero.}$$

Thus, one possibility for β is:

$$\beta = \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right].$$

Then we will have

$$[T]_\beta^\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad \square$$

9. Let $\mathbf{V}$ be a finite dimensional vector space, and let $T \in \mathcal{L}(\mathbf{V})$. Prove that if $\text{rank}(T) = k$, then T has at most $k+1$ distinct eigenvalues. Give an example where T has strictly fewer than $k+1$ distinct eigenvalues, and one where T has exactly $k+1$ distinct eigenvalues.

Proof. By the Rank-Nullity Theorem, if $\dim(\mathbf{V}) = n$, then $\text{nullity}(T) = n - k$. Take an ordered basis $\gamma = [\mathbf{v}_1, \ldots, \mathbf{v}_{n-k}]$ for the nullspace of T, and extend to an ordered basis of $\mathbf{V}$, $\beta = [\mathbf{v}_1, \ldots, \mathbf{v}_{n-k}, \mathbf{v}_{n-k+1}, \ldots, \mathbf{v}_n]$. Since $\mathbf{v}_1, \ldots, \mathbf{v}_{n-k}$ are in the nullspace of T, then $T(\mathbf{v}_i) = \mathbf{0}$ for $i = 1, \ldots, n-k$. Therefore, the matrix of T with respect to β has the form

$$[T]_\beta^\beta = \begin{pmatrix} 0 & B \\ 0 & C \end{pmatrix},$$

where the top left corner is an $(n-k) \times (n-k)$ block of all zeros, B is an $(n-k) \times k$ block, C is a $k \times k$ block, and the bottom left is a block of $k \times (n-k)$ zeros. Taking the determinant of $[T]_\beta^\beta - tI$, we will have:

$$f(t) = \begin{vmatrix} -tI_{n-k} & B \\ 0 & C - tI_k \end{vmatrix} = (-t)^{n-k} g(t),$$

where I_{n-k} is the $(n-k) \times (n-k)$ identity matrix, I_k is the $k \times k$ identity matrix, and $g(t)$ is the characteristic polynomial of C. Since C is $k \times k$, we know that $g(t)$ is a polynomial of degree k, and hence has at most k distinct roots. Thus, $f(t)$, the characteristic polynomial of T, has at most $k+1$ roots: the up to k roots of $g(t)$, plus the root $t = 0$.

To figure out the examples, think about the argument above: T will have strictly fewer than $k+1$ distinct eigenvalues if $g(t)$ has fewer than k distinct roots, or one of its roots is 0. And T will have exactly $k+1$ distinct eigenvalues if $g(t)$ has exactly k distinct roots, all nonzero. So an example where T has strictly fewer than $k+1$ distinct eigenvalues is, for example, $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T(x, y, z) = (0, y, z)$. The standard matrix for T is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The rank of T is 2, but T has only the eigenvalues $\lambda = 0$ and $\lambda = 1$, fewer than $2 + 1 = 3$.

For an example where T has exactly $k+1$ distinct eigenvalues, let us take, in the same dimension, $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T(x, y, z) = (0, y, 2z)$. Here the standard matrix is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

so T has rank 2, and 3 distinct eigenvalues (namely $\lambda = 0$, $\lambda = 1$, and $\lambda = 2$). $\square$

10. Let $\mathbf{V}$ be a finite dimensional vector space over $\mathbb{R}$, let $T \in \mathcal{L}(\mathbf{V})$, and let λ be an eigenvalue of T. Prove that there exists $\delta > 0$ such that for all scalars α such that $0 < |\alpha - \lambda| < \delta$, the linear transformation $T - \alpha I_{\mathbf{V}}$ is invertible.

Proof. Let $p(t)$ be the characteristic polynomial of t. Then we know that for any real number α,

$$\begin{aligned} p(\alpha) = 0 &\iff \alpha \text{ is an eigenvalue of } T \\ &\iff \mathbf{N}(T - \alpha I) \neq \{\mathbf{0}\} \\ &\iff T - \alpha I \text{ is not invertible.} \end{aligned}$$

So $T - \alpha I$ is invertible if and only if α is not an eigenvalue of T, and equivalently, if and only if α is not a root of $p(t)$.

Let $\lambda = \lambda_1, \lambda_2, \ldots, \lambda_k$ be the distinct roots of $p(t)$ (that is, the distinct eigenvalues of T). Then $T - \alpha I_{\mathbf{V}}$ is invertible if and only if $\alpha \neq \lambda_i$ for $i = 1, \ldots k$. Let

$$\delta = \min\Big(\big\{|\lambda_1 - \lambda_i| \;\big|\; i = 2, \ldots, k\big\}\Big).$$

If $0 < |\alpha - \lambda| < \delta$, then α is not equal to any of the λ_i, so it is not an eigenvalue of T, and therefore $T - \alpha I_{\mathbf{V}}$ is invertible, as claimed. $\square$