Math 561 - Homework 7

SOLUTIONS

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1. Griffins and Dragons live in the Enchanted Forest. The number of Dragons D and of Griffins G in the forest each year is determined by their populations the previous year, according to the formulas

$$\begin{aligned} D(n+1) &= 1.5D(n) + G(n) \\ G(n+1) &= D(n). \end{aligned}$$

If in year 1 there are 25 Dragons and no Griffins, what will the population be in year k? (Don't worry about fractional answers: mythical creatures do not have to come in whole units). In the long run, will the total population grow, shrink, or approach a nonzero equilibrium point? After a very long time, approximately what will be the ratio of Dragons to Griffins?

Answer. If $\mathbf{p}(n) = \binom{D(n)}{G(n)}$ is the population vector in year n, then the problem is of the form

$$\mathbf{p}(n+1) = A\mathbf{p}(n)$$

where

$$A = \begin{pmatrix} 1.5 & 1 \\ 1 & 0 \end{pmatrix}.$$

The characteristic polynomial of A is $p(t)$, where

$$p(t) = \det(A - tI) = \begin{vmatrix} 1.5-t & 1 \\ 1 & -t \end{vmatrix} = (1.5-t)(-t) - 1 = t^2 - 1.5t - 1 = \Big(t-2\Big)\Big(t+\frac{1}{2}\Big).$$

So the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = -\frac{1}{2}$. The matrix is diagonalizable, and we can proceed along our usual lines.

First we find a basis of eigenvectors: for E_2, we compute

$$\mathbf{N}(A - 2I) = \mathbf{N}\begin{pmatrix} -\frac{1}{2} & 1 \\ 1 & -2 \end{pmatrix} = \mathbf{N}\begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}.$$

A basis for this nullspace is given by $\binom{2}{1}$. (Indeed, $A\binom{2}{1} = \binom{4}{2} = 2\binom{2}{1}$; just doing a gut-check).

For $E_{-1/2}$, we have:

$$\mathbf{N}\left(A + \tfrac{1}{2}I\right) = \mathbf{N}\begin{pmatrix} 2 & 1 \\ 1 & \frac{1}{2} \end{pmatrix} = \mathbf{N}\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}.$$

A basis for this nullspace is given by $\binom{1}{-2}$. Indeed, $A\binom{1}{-2} = \binom{-0.5}{1} = -\frac{1}{2}\binom{1}{-2}$.

Now, we are told that $\mathbf{p}(0) = \binom{25}{0}$; expressing this as a linear combination of $\mathbf{b}_1$ and $\mathbf{b}_2$, we have:

$$\begin{pmatrix} 25 \\ 0 \end{pmatrix} = \alpha\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \beta\begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

This amounts to solving the system of linear equations:

$$\begin{aligned} 2\alpha + \beta &= 25 \\ \alpha - 2\beta &= 0. \end{aligned}$$

Using Gaussian elimination, we have:

$$\begin{pmatrix} 2 & 1 & 25 \\ 1 & -2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 0 \\ 2 & 1 & 25 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 0 \\ 0 & 5 & 25 \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 10 \\ 0 & 1 & 5 \end{pmatrix}.$$

Thus, $\alpha = 10$ and $\beta = 5$. So

$$\mathbf{p}(0) = 10\mathbf{b}_1 + 5\mathbf{b}_2.$$

Now, since

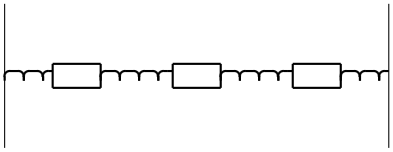
$$\mathbf{p}(n) = A^n\mathbf{p}(0) = A^n(10\mathbf{b}_1 + 5\mathbf{b}_2) = 2^n(10\mathbf{b}_1) + \left(-\tfrac{1}{2}\right)^n 5\mathbf{b}_2,$$

and $(-\frac{1}{2})^n \to 0$ as $n \to \infty$, in the long run only the contribution of $2^n(10\mathbf{b}_1)$ will matter; this will make the population grow as $n \to \infty$.

Moreover, as $n \to \infty$, the value of $\mathbf{p}(n)$ will approach a multiple of $\mathbf{b}_1$, so the ratio of the populations of dragons to griffins will approach that given by $\mathbf{b}_1 = \binom{2}{1}$, which is twice as many dragons as there are griffins. So the ratio will approach 2.

The formula for year k can be read off the formula above for $\mathbf{p}(n)$: in year k, where will be $2^k(20) + (-\frac{1}{2})^k 5$ dragons, and $2^k(10) - 10(-\frac{1}{2})^k$ griffins. □

2. Consider three coupled blocks illustrated below; each block has mass $m = 1$ and each spring has spring constant $k = 1$. Write down the equations of motion for this coupled system, and describe the corresponding eigenvalues and eigenvectors.



Solution. Let x_1, x_2, x_3 be the horizontal displacement of the first, second, and third blocks, respectively. The first spring exerts a force of $-x_1$ on the first block; the second spring is stretched by $x_2 - x_1$, and exerts this force on the first block, and $x_1 - x_2$ on the second block. The third spring is stretched by $x_3 - x_2$, exerting that force on the second block, and $x_2 - x_3$ on the third block. The fourth spring is stretched by $-x_3$ and it exerts a force of $-x_3$ on the third block. Thus, the acceleration equations are:

$$\begin{array}{rcrcrcr} \frac{d^2x_1}{dt^2} & = & -2x_1 & + & x_2 & & \\ \frac{d^2x_2}{dt^2} & = & x_1 & - & 2x_2 & + & x_3 \\ \frac{d^2x_3}{dt^2} & = & & & x_2 & - & 2x_3 \end{array}$$

The corresponding matrix is therefore

$$A = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}.$$

The characteristic polynomial is:

$$\begin{aligned} \det(A - tI) &= \begin{vmatrix} -2-t & 1 & 0 \\ 1 & -2-t & 1 \\ 0 & 1 & -2-t \end{vmatrix} = (-2-t)\begin{vmatrix} -2-t & 1 \\ 1 & -2-t \end{vmatrix} - \begin{vmatrix} 1 & 0 \\ 1 & -2-t \end{vmatrix} \\ &= (-2-t)\Big((-2-t)^2 - 1\Big) - (-2-t) = (-t-2)\Big(t^2 + 4t + 3 - 1\Big) \\ &= -(t+2)(t^2+4t+2) = -(t+2)(t+2+\sqrt{2})(t+2-\sqrt{2}). \end{aligned}$$

Thus the eigenvalues are $\lambda_1 = -2$, $\lambda_2 = -2 + \sqrt{2}$, and $\lambda_3 = -2 - \sqrt{2}$.

Since the eigenvalues are all negative, the solutions to the differential equations will be sums of sines and cosines, so they describe oscillatory behavior (as might be expected).

For $\lambda_1 = -2$, an eigenvector is $(1, 0, -1)^t$. This corresponds to moving the first and third block towards the middle block by the same amount. The center block will remain stationary, and the other two will simply oscillate with the same frequency as mirror images.

For $\lambda_2 = -2 - \sqrt{2}$, we find an eigenvector $(1, -\sqrt{2}, 1)^t$. This amounts to moving the first and third block the same amount to the right, and the middle block $\sqrt{2}$ to the left. This makes the two outside blocks to move in unison, while the middle block moves in the opposite direction also in tandem.

For $\lambda_3 = -2 + \sqrt{2}$, an eigenvector is $(1, \sqrt{2}, 1)^t$. Here we move all three blocks to the right, the first and third the same amount and the middle $\sqrt{2}$. Now the two outside blocks move in tandem, and the middle block moves between them in the same direction, though slightly faster (to a greater oscilation) than the other two. This compensates for the delay in the force from the middle springs. □

3. The State University State football team gets confident after each win, and demoralized after each loss. After winning a game, it has a 90% chance of winning the next game, but after a loss it has only a 20% chance of winning the next game. In the long run, what fraction of the games can we expect the team to win?

   Answer. If $x(n)$ is the probability of a win in game n, and $y(n)$ the probability of a loss, we have a Markov process:

$$\begin{aligned} x(n+1) &= 0.9x(n) + 0.2y(n) \\ y(n+1) &= 0.1x(n) + 0.8y(n). \end{aligned}$$

   The characteristic polynomial of the system is

$$(0.9 - t)(0.8 - t) - (0.2)(0.1) = t^2 - 1.7t + .7 = (t - 1)(t - 0.7),$$

   so the system is diagonalizable. Alternatively: the eigenvalues include $\lambda_1 = 1$ (because this is a probability matrix), and they add up to the trace, which is 1.7. So $\lambda_2 = 0.7$. An eigenvector with eigenvalue $\lambda = 1$ is given by

$$\mathbf{N}\begin{pmatrix} -0.1 & 0.2 \\ 0.1 & -0.2 \end{pmatrix} = \mathbf{N}\begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix},$$

   which has an eigenvector $\binom{2}{1}$. If we normalize this eigenvector so its entries add up to 1, we get $\binom{2/3}{1/3}$, and so we have that in the long run, any initial $\mathbf{x}(0)$ (the odds of winning and losing their first game) will approach $\binom{2/3}{1/3}$ (the entries of the initial vector add up to 1). So in the long run, we expect this team to win about two thirds of its games, and lose the remaining third. Note that the second eigenvalue is 0.7, which is fairly large comparatively speaking; it takes 15 games before the contribution of this eigenvalue is less than one half of one percent, so it may take a while for the team to settle into this pattern. □

4. The City State of Ankh-Morpork has a population of 9 million, of whom 5 million live in the Old City of Ankh, and 4 million live in the Morpork neighborhoods. Each year, only 20% of the Morporkians move to Ankh (and 80% remain in Morpork), while 40% of Ankhites move to Morpork (and 60% stay in Ankh). Citizens of Ankh-Morpork, however, must remain in Ankh-Morpork (they are not allowed to migrate outside the Twin Cities), and nobody *wants* to move *into* Ankh-Morpork. If we ignore the effects of births and deaths, what percentage of the total population will the Old City of Ankh have in the long run?

Answer. Let $a(n)$ be the population in Ankh on year n (in millions of people), and $m(n)$ the population of Morpork in year n (also in millions of people). Then we have

$$\begin{aligned} a(n+1) &= 0.6a(n) + 0.2m(n) \\ m(n+1) &= 0.4a(n) + 0.8m(n). \end{aligned}$$

If we let $\mathbf{x}(n+1) = (a(n), m(n))^t$, then $\mathbf{x}(n+1) = A\mathbf{x}(n)$, where

$$A = \begin{pmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{pmatrix}.$$

This is a probability matrix, so $\lambda_1 = 1$ is an eigenvalue; the eigenvalues add up to $tr(A) = 1.4$, so the second eigenvalue is $\lambda_2 = 0.4$.

For E_{λ_2}, we have:

$$\mathbf{N}\begin{pmatrix} 0.2 & 0.2 \\ 0.4 & 0.4 \end{pmatrix} = \mathbf{N}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

So $\mathbf{b}_2 = (1, -1)^t$ is an eigenvector of $\lambda_2 = 0.4$. To get the eigenvector corresponding to $\lambda_1 = 1$, we have:

$$\mathbf{N}\begin{pmatrix} -0.4 & 0.2 \\ 0.4 & -0.2 \end{pmatrix} = \mathbf{N}\begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix}.$$

So one eigenvector is $(1, 2)^t$; normalizing so that the entries add up to 1, we have $\mathbf{b}_1 = (\frac{1}{3}, \frac{2}{3})^t$.

Now, we have $\mathbf{x}(0) = (5, 4)^t$. We want to express $\mathbf{x}(0)$ as a linear combination of $\mathbf{b}_1$ and $\mathbf{b}_2$. If $\mathbf{x}(0) = \alpha\mathbf{b}_1 + \beta\mathbf{b}_2$, then looking at the sum of the entries we must have $\alpha = 9$ (because the entries of $\mathbf{b}_2$ add up to 0, and the entries of $\mathbf{b}_1$ add up to 1). So we have:

$$\begin{pmatrix} 5 \\ 4 \end{pmatrix} = 9\begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix} + \beta\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Then $\beta = 2$.

Since

$$\mathbf{x}(n) = A^n\mathbf{x}(0) = A^n(9\mathbf{b}_1 + 2\mathbf{b}_2) = 9\mathbf{b}_1 + (0.4)^n 2\mathbf{b}_2,$$

we have that

$$\lim_{n\to\infty} \mathbf{x}(n) = 9\mathbf{b}_1 = \begin{pmatrix} 3 \\ 6 \end{pmatrix};$$

in the end the population will be distributed as in $\mathbf{b}_1 = (\frac{1}{3}, \frac{2}{3})$. That is, one third of the population will be living in the Old City of Ankh, and two thirds will be living in Morpork. In this case, we will end up with 3 million people in Ankh, and 6 million in Morpork. □

5. Three Knights of the Round Table sit down for dinner (at the Round Table, of course). They are each brought a bowl of porridge. Unhappy with their portions, each Knight takes a spoon in each hand, and simultaneously takes half the porridge from the Knight to their left and half the porridge from the Knight on their right, and places it on his own bowl (which has now been emptied by his two adjoining Knights). Describe what happens to the distribution of porridge among the bowls if this process is repeated indefinitely.

Answer. Let $x_i(n)$ be the amount of porridge in the bowl of the ith knight after n steps, and let $\mathbf{x}(n) = (x_1(n), x_2(n), x_3(n))^t$. Then we have $\mathbf{x}(n+1) = A\mathbf{x}(n)$, where

$$A = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

This is a probability matrix, but not a regular probability matrix. We know that the eigenvalues include $\lambda_1 = 1$, add up to 0 (the trace of A), and multiply to:

$$\det(A) = \frac{1}{8} + \frac{1}{8} - 0 = \frac{1}{4}.$$

So the two other eigenvalues λ_2 and λ_3 satisfy $\lambda_2 + \lambda_3 = -1$ and $\lambda_2\lambda_3 = \frac{1}{4}$. That is, the characteristic polynomial of A is

$$-(t-1)\left(t^2 + t + \frac{1}{4}\right) = -(t-1)\left(t + \frac{1}{2}\right)^2.$$

So the only eigenvalue of absolute value 1 is $\lambda_1 = 1$. Because there is an eigenvalue of algebraic multiplicity 2, we need to worry about diagonalizability of the matrix. But for $\lambda = -\frac{1}{2}$, the eigenspace is given as the nullspace of $A + \frac{1}{2}I$:

$$A + \frac{1}{2}I = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

This matrix has rank 1, so it has nullity 2; that means that $\dim(E_{-\frac{1}{2}}) = 2$, so the geometric multiplicity is the same as the algebraic multiplicity. That means A is diagonalizable, and we can proceed as usual.

An eigenvector corresponding to 1 is $(1,1,1)^t$, which normalized is equal to $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})^t$, so in the long run, the porridge will be evenly distributed among the three knights, with each of them having one third of the total porridge they group started with.

6. Repeat the previous problem with 4 Knights.

Answer. Proceeding as above, we have $\mathbf{x}(n+1) = A\mathbf{x}(n)$, with $\mathbf{x}(n) = (x_1(n), x_2(n), x_3(n), x_4(n))^t$ and

$$A = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix}.$$

This is a probability matrix, but is not a regular probability matrix.

The characteristic polynomial of this matrix is $t^4 - t^2 = t^2(t-1)(t+1)$; so there are two eigenvalues with absolute value equal to 1. The matrix is diagonalizable, since the nullspace is 2-dimensional with basis $[(0,1,0,-1)^t, (1,0,-1,0)^t]$. (The only possible obstacle to diagonalizability would be if the algebraic multiplicity of the eigenvalue λ_0 were strictly larger than its geometric multiplicity, which is not the case).

The eigenvectors that will matter in the long run correspond to eigenvalues with absolute value equal to 1; these are the ones in E_1 and E_{-1}. Since $(1,1,1,1)$ maps to itself under A, E_1 must be spanned by this vector; normalizing, we get the vector $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. So one stable situation is the one where all Knights have with the same amount of porridge.

The one corresponding to E_{-1} is a bit trickier. Here we have:

$$\mathbf{N}\begin{pmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 1 \end{pmatrix} = \mathbf{N}\begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \end{pmatrix} = \mathbf{N}\begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \end{pmatrix}$$

$$= \mathbf{N}\begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & -3 & -2 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & -2 & 0 & 2 \end{pmatrix} = \mathbf{N}\begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & -3 & -2 & 1 \\ 0 & -2 & 0 & 2 \end{pmatrix} = \mathbf{N}\begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 4 & 4 \end{pmatrix}$$

$$= \mathbf{N}\begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \mathbf{N}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

So a basis for the eigenspace E_{-1} is given by a vector $(x, y, z, w)^t$ with $x = z = -w$ and $y = w$. One such vector is $(1, -1, 1, -1)^t$. This vector maps to $(-1, 1, -1, 1)^t$ under A.

Instead of considering $(1, 1, 1, 1)^t$ and $(1, -1, 1, -1)^t$ separately, it is a little easier to consider $(1, 1, 1, 1)^t$ and the sum of the two, $(2, 0, 2, 0)^t$. The first vector maps to itself, the second to $(0, 2, 0, 2)^t$. Normalizing, we have the mode represented by $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})^t$, which is the mode where all the Knights have exactly the same amount of porridge; and the mode $(\frac{1}{2}, 0, \frac{1}{2}, 0)^t$ represents the case where the first and third Knights have all the porridge divided equally between the two of them; in the next iteration of the process, we obtain $(0, \frac{1}{2}, 0, \frac{1}{2})^t$, that is, now the second and fourth Knights have the entire porridge divided equally between them. This alternates indefinitely between them. The time average converges, as expected, to $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})^t$.

Thus, what happens in the long run? Suppose that $\mathbf{x}(0)$ is the initial porridge distribution, and we normalize it so that the entries add up to 1. If we write it as a linear combination of the neutrally stable modes $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})^t$ and $(1, -1, 1, -1)^t$, we have $\mathbf{x}(0)$ is given by $\mathbf{x}(0) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})^t + c_2(1, -1, 1, -1)^t$, with $0 \leq c_2 \leq \frac{1}{4}$ (since no Knight gets negative porridge). Part of the porridge will eventually be distributed evenly among all Knights, while the rest of the porridge will alternate between pairs of non-neighboring Knights indefinitely.

Alternatively, we can think of the normalized vector $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})^t$ and the vector $(1, -1, 1, -1)^t$. The latter represents a "swap" between the knights seated in seats 1 and 3, and the knights sitting in seats 2 and 4. If the initial portions are α_1, α_2, α_3, and α_4, then in the long run we will have two knighs with $\frac{\alpha_1+\alpha_3}{2}$ each, sitting opposite each other; the other two knights having $\frac{\alpha_2+\alpha_4}{2}$ each, sitting opposite each other, and the two averages swapping position each turn, without stabilizing (unless the two averages are equal). $\square$