

MATH 561 – MIDTERM

SOLUTIONS

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- Let $\mathbf{V}$ be a finite dimensional vector space, and let $T: \mathbf{V} \rightarrow \mathbf{V}$ be a linear transformation. Prove that the following are equivalent:

- (i) T is one-to-one.
- (ii) T is surjective.
- (iii) T is invertible.

(8 points)

Proof. We know that T is invertible if and only if it is bijective. This means that (iii) implies both (i) and (ii).

From the Dimension Theorem (aka the Rank-Nullity Theorem), we have

$$\dim(\mathbf{V}) = \text{nullity}(T) + \text{rank}(T).$$

Thus, we have:

$$\begin{aligned} T \text{ is one-to-one} &\iff \mathbf{N}(T) = \{\mathbf{0}\} \\ &\iff \text{nullity}(T) = 0 \\ &\stackrel{(1)}{\iff} \text{rank}(T) = \dim(\mathbf{V}) \\ &\stackrel{(2)}{\iff} \mathbf{R}(T) = \mathbf{V} \\ &\iff T \text{ is surjective.} \end{aligned}$$

(We use finite dimensionality for $\stackrel{(1)}{\iff}$, and for $\stackrel{(2)}{\iff}$, so neither of (i) and (ii) imply the other in the infinite dimensional setting; you did not need to identify this, but I am adding it for your information.)

Thus, (i) and (ii) are equivalent.

Finally, suppose (i) is true; then (ii) is also true by the above, so (i) and (ii) both hold, telling us that T is a bijection. And since it is a bijection, it follows that T is invertible, yielding (iii).

Thus, (iii) implies both (i) and (ii); (i) is equivalent to (ii); and (i) implies (iii), giving the equivalence. $\square$

- Give an example of a vector space $\mathbf{V}$, and linear transformations $T, U: \mathbf{V} \rightarrow \mathbf{V}$ such that $UT = \text{Id}_{\mathbf{V}}$, but neither T nor U are invertible. Justify the last three assertions by calculations or proofs. (8 points)

Answer. From problem 1 we know that this requires an infinite dimensional example. There are many, but the one we saw in class is the following:

Let $\mathbf{V} = \mathbb{R}^{\infty}$ (or $\mathbb{R}^{\mathbb{N}}$) be the vector space of real sequences with term-wise addition and scalar multiplication. Let T be the right-shift operator,

$$T(a_0, a_1, a_2, \dots, a_n, \dots) = (0, a_0, a_1, a_2, \dots, a_n, \dots),$$

and U be the left-shift operator,

$$U(a_0, a_1, a_2, \dots, a_n, \dots) = (a_1, a_2, \dots, a_n, \dots).$$

These are linear transformations, and

$$UT(a_0, a_1, a_2, \dots, a_n, \dots) = U(0, a_0, a_1, a_2, \dots, a_n, \dots) = (a_0, a_1, a_2, \dots, a_n, \dots),$$

so $UT = \text{Id}_{\mathbf{V}}$. But T is not invertible, because it is not surjective: since the 0th coordinate of $T(\mathbf{x})$ is 0 for all $\mathbf{x} \in \mathbf{V}$, we have $(1, 0, 0, 0, \dots) \notin \mathbf{R}(T)$. Dually, U is not invertible because it is not one-to-one: $(1, 0, 0, 0, \dots) \neq \mathbf{0}$, yet $U(1, 0, 0, 0, \dots) = \mathbf{0}$, so $\mathbf{N}(U) \neq \{\mathbf{0}\}$. $\square$

3. Let $\mathbf{P}_2(\mathbb{R})$ be the vector space of polynomials of degree at most 2 and coefficients in the real numbers.

Find the coordinate matrix of the linear operator

$$T(p(x)) = p(x+1) + xp'(x)$$

relative to the standard ordered basis of $\mathbf{P}_2(\mathbb{R})$. (8 points)

Answer. The standard ordered basis of $\mathbf{P}_2(\mathbb{R})$ is $\beta = [1, x, x^2]$. We have:

$$T(1) = 1 + x(0) = 1$$

$$T(x) = (x+1) + x(1) = 1 + 2x$$

$$T(x^2) = (x+1)^2 + x(2x) = 1 + 2x + 3x^2.$$

Therefore,

$$[T]_{\beta}^{\beta} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}.$$

4. Let $\mathbf{V} = \mathbb{R}^2$, and let f_1 and f_2 be elements of $\mathbf{V}^*$ given by

$$f_1(a, b) = 2a - b$$

$$f_2(a, b) = a + b.$$

Then $\gamma = [f_1, f_2]$ is a basis for $\mathbf{V}^*$. You may take for granted that f_1 and f_2 are linear functionals, and that they give a basis for $\mathbf{V}^*$. You do not need to prove either of those facts and may use them freely in your solution.

Find an ordered basis β for $\mathbf{V}$ such that γ is the dual basis of β . (8 points)

Answer. We want to find vectors (a, b) and (x, y) such that

$$f_1(a, b) = 1$$

$$f_2(a, b) = 0$$

$$f_1(x, y) = 0$$

$$f_2(x, y) = 1$$

That gives:

$$2a - b = 1$$

$$a + b = 0$$

$$2x - y = 0$$

$$x + y = 1$$

By adding the equations for (a, b) we get $3a = 1$, or $a = \frac{1}{3}$, and hence $b = -\frac{1}{3}$. Adding them for (x, y) gives $3x = 1$ and hence $x = \frac{1}{3}$, and thus $y = \frac{2}{3}$. So we have

$$\beta = \left[\left(\frac{1}{3}, -\frac{1}{3} \right), \left(\frac{1}{3}, \frac{2}{3} \right) \right].$$

5. Let $n \geq 1$, and let $A, B \in M_{n \times n}(\mathbb{R})$. Prove that if $AB = I_n$, then $B = A^{-1}$. (8 points)

Proof. Let $L_A, L_B: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the corresponding linear transformations obtained by left multiplying by the relevant matrix (viewing $\mathbb{R}^n$ as column vectors). Then we have that

$$L_A L_B = L_{AB} = L_{I_n} = \text{Id}_{\mathbb{R}^n}.$$

Since the composition of L_A and L_B is a bijection, it follows that L_B is one-to-one and L_A is surjective. By Problem 1, L_A and L_B are both bijections, and hence invertible. We know from class that L_A is invertible if and only if A is invertible, and in that case $(L_A)^{-1} = L_{A^{-1}}$. So A^{-1} exists. Then we have

$$B = I_n B = (A^{-1} A) B = A^{-1} (AB) = A^{-1} I_n = A^{-1},$$

which is what we wanted to prove. $\square$

6. Let $\mathbf{V} = \mathbf{P}_2(\mathbb{R})$, and let $\beta = [1, x, x^2]$ and $\gamma = [1 + x, x + x^2, 1 + x^2]$ be ordered bases for $\mathbf{V}$. You may take for granted that they are bases for $\mathbf{V}$, you do not need to prove it.

- (i) Find the change of basis matrix from γ to β . (2 points)

Answer. The change of basis matrix from γ to β is obtained by writing each vector in γ in terms of β ; in this case, this is straightforward since β is the standard basis, so we have

$$Q = [I_{\mathbf{V}}]_{\gamma}^{\beta} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

- (ii) Find the change of basis matrix from β to γ . (4 points)

Answer. This can be done either by finding an inverse to Q directly, or by writing each vector in β in terms of γ . For the latter, we have:

$$\begin{aligned} 1 &= \frac{1}{2}(1+x) - \frac{1}{2}(x+x^2) + \frac{1}{2}(1+x^2) \\ x &= \frac{1}{2}(1+x) + \frac{1}{2}(x+x^2) - \frac{1}{2}(1+x^2) \\ x^2 &= -\frac{1}{2}(1+x) + \frac{1}{2}(x+x^2) + \frac{1}{2}(1+x^2), \end{aligned}$$

which gives

$$Q^{-1} = [I_{\mathbf{V}}]_{\beta}^{\gamma} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

- (iii) Find the coordinate matrix of the linear transformation $T: \mathbf{V} \rightarrow \mathbf{V}$ given by $T(p(x)) = p'(x)$ relative to β . (2 points)

Answer. Since $T(1) = 0$, $T(x) = 1$, and $T(x^2) = 2x$, we have

$$[T]_{\beta}^{\beta} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

- (iv) Use the answers from (i), (ii), and (iii) to find $[T]_{\gamma}^{\gamma}$. (2 points)

Answer. We have

$$\begin{aligned}
[T]_{\gamma}^{\gamma} &= Q^{-1}[T]_{\beta}^{\beta}Q \\
&= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2} & \frac{3}{2} & 1 \\ -\frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & -\frac{1}{2} & -1 \end{pmatrix}
\end{aligned}$$