# Zorn's Lemma

Math 561 – Fall 2026

*Prof. Arturo Magidin*

This handout provides the statement of Zorn's Lemma and the definitions needed to understand all the parts of the statement. I will not provide a proof of Zorn's Lemma here (I usually prove it in Math 565), because the proof is technical, though I will make some comments about the *idea* of one possible proof. Note that Zorn's Lemma is equivalent to the Axiom of Choice.

**Partial orders.** First, let us give the necessary definitions.

**Definition 1.** Let $P$ be a set. A partial order on $P$ is a binary relation $\leq$, which we will denote using infix notation, and that has the following three properties:

(i) Reflexivity: for all $x \in P$, $x \leq x$.
(ii) Antisymmetry: for all $x, y \in P$, if $x \leq y$ and $y \leq x$, then $x = y$.
(iii) Transitivity: for all $x, y, z \in P$, if $x \leq y$ and $y \leq z$, then $x \leq z$.

**Definition 2.** A partially order set or poset is an ordered pair, $(P, \leq)$, where $P$ is a set and $\leq$ is a partial order on $P$.

Often, if the order is understood from context, we will simply say *"$P$ is a partially ordered set"* instead of refering to $(P, \leq)$.

Note that there is no requirement that any two elements of $P$ be comparable for $P$ to be a partially ordered set (in fact, that's where the "partially" comes from).

If $(P, \leq)$ is a partially ordered set, and $x, y \in P$, we say that $x$ and $y$ are *comparable* if either $x \leq y$ or $y \leq x$ holds (or both, if and only if $x = y$). Otherwise, we say that $x$ and $y$ are *incomparable*.

The prototypical partially ordered set is $(\mathcal{P}(X), \subseteq)$, where $X$ is a set, $\mathcal{P}(X)$ is the power set of $X$ (the collection/set of all subsets of $X$), and $\subseteq$ is set containment.

**Definition 3.** We say that a partially ordered set $(P, \leq)$ is totally ordered if and only if for every $x, y \in P$, either $x \leq y$ or $y \leq x$; that is, any two elements of $P$ are comparable.

**Definition 4.** Let $(P, \leq)$ be a partially ordered set. An element $x \in P$ is said to be maximal (in $P$) if and only if for all $y \in P$, if $x \leq y$ then $x = y$. That is: there is no element of $P$ that is strictly larger than $x$.

Note that a maximal element need not be a *maximum*: a maximum is an element $m \in P$ such that $y \leq m$ for all $y \in P$: that is, it must be greater than or equal than all elements of $P$. A maximal element is only required to be greater than or equal to all elements that it can be compared to, but it could fail to be comparable to other elements of $P$. For example, consider the set

$$P = \Big\{\varnothing, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}\Big\},$$

partially ordered by inclusion (with $a, b, c$ pairwise distinct). Then the elements $\{a, b\}$, $\{a, c\}$, and $\{b, c\}$ are maximal in $P$, but none of them are a maximum element.

**Definition 5.** Let $(P, \leq)$ be a partially ordered set, and let $S \subseteq P$ be a subset of $P$. An element $x \in P$ is an UPPER BOUND for $S$ if and only if $s \leq x$ for all $s \in S$.

Note that if $S = \varnothing$, then any element of $P$ is an upper bound.

Finally, if $(P, \leq)$ is a partially ordered set, then any subset $C \subseteq P$ is also a partially ordered set, under the induced order $\leq\big|_{C\times C}$. Even if $P$ is not totally ordered, it is possible that $C$ itself is totally ordered. This situation has a special name:

**Definition 6.** Let $(P, \leq)$ be a partially ordered set. A subset $C \subseteq P$ is called a CHAIN (in $P$) if and only if $C$ is totally ordered under the induced order.

Note that the empty set is a chain by vacuity.

**Zorn's Lemma.** We can now state Zorn's Lemma:

**Theorem 7** (Zorn's Lemma)**.** *Let $(P, \leq)$ be a partially ordered set such that every chain $C \subseteq P$ has an upper bound in $P$. Then $P$ has maximal elements.*

One idea of why Zorn's Lemma is true goes as follows: start with $C = \varnothing$. This is a chain, so it must have an upper bound $p_0 \in P$. If $p_0$ is maximal, we are done. If not, then let $p_1 \in P$ be such that $p_0 < p_1$. If $p_1$ is maximal, we are done. If not, then let $p_2 \in P$ be such that $p_1 < p_2$. Continuing this inductively, we either find a maximal element, or we end up with a chain $p_0 < p_1 < \cdots < p_n < \cdots$. Letting $C = \{p_0, p_1, \ldots, p_n, \ldots\}$ we have a chain in $P$, which must have an upper bound. Let $q_0$ be that upper bound. If it is maximal in $P$, we are done, if not, then let $q_1 \in P$ be such that $q_0 < q_1$.

"Continuing this way" we must "eventually" get a maximal element, or else we would end up with a chain whose cardinality is strictly larger than the cardinality of $P$, which is impossible.

If you try to formalize this argument, you will likely run into several technical issues; it can be made to work, but it requires the use of transfinite induction and a few technical tweaks to ensure that everything we are doing is correct. It also requires the Axiom of Choice when we select either upper bounds, or witnesses to the non-maximality of an element.

In fact, Zorn's Lemma is *equivalent* to the Axiom of Choice in Zermelo-Fraenkel Set Theory. You can find a discussion and proof of the equivalence (as well as the equivalences with two other important equivalents of the Axiom of Choice, the Well-Ordering Principle and Bernstein's Theorem), in George Bergman's handout *The Axiom of Choice, Zorn's Lemma, and all that*, available at

`https://math.berkeley.edu/~gbergman/grad.hndts/AC+Zorn+.pdf`