Math 566 - Homework 4

Due Wednesday February 21, 2024

1. Let R be a ring, and I an ideal of R. Show that if R is a principal ideal ring (a ring in which every ideal is principal), then R/I is a principal ideal ring. Do not assume R is commutative or has a unity.

2. Let $R = \mathbb{Z}[\sqrt{-5}] = \left\{a + b\sqrt{-5} \;\middle|\; a, b \in \mathbb{Z}\right\}$. This is a unital subring of $\mathbb{C}$ (you may take this for granted). Define $N\colon R \to \mathbb{Z}$ by

$$N\left(a + b\sqrt{-5}\right) = \left(a + b\sqrt{-5}\right)\left(a - b\sqrt{-5}\right) = a^2 + 5b^2.$$

   (i) Show that N is multiplicative: if $x, y \in R$, then $N(xy) = N(x)N(y)$.
   (ii) Show that $N(x) \geq 0$ for all $x \in R$, with equality if and only if $x = 0$.
   (iii) Show that $N(x) = 1$ if and only if x is a unit in R. Determine all units of R.
   (iv) Show that if $a, b \in R$ and $a \mid b$ in R, then $N(a) \mid N(b)$ in $\mathbb{Z}$.
   (v) Show that 2, 3, $1 + \sqrt{-5}$, and $1 - \sqrt{-5}$ are irreducible in R.
   (vi) Show that none of 2, 3, $1 + \sqrt{-5}$, and $1 - \sqrt{-5}$ are prime.

3. A complex number z is an *algebraic integer* if and only if there is a monic polynomial $p(x)$ with integer coefficients,

$$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \qquad a_i \in \mathbb{Z}$$

   such that $p(z) = 0$. The set $\mathbb{A}$ of all algebraic integers forms a subring of $\mathbb{C}$ (you may take this for granted).

   (i) Prove that the only rational numbers that are algebraic integers are the integers.
   (ii) Prove that $\mathbb{A}$ is not a field, but has no irreducible elements and no primes.

4. A proper ideal I of a commutative ring with unity R is said to be a *primary ideal* if and only if for all $a, b \in R$, if $ab \in I$, then either $a \in I$ or $b^n \in I$ for some $n \geq 1$. Determine the primary ideals of $\mathbb{Z}$.

5. Let R be a commutative ring with unity, and let X be a nonempty subset of R. We say that d is a greatest common divisor of X if and only if

   (i) For every $x \in X$, $d \mid x$; and
   (ii) If $c \in R$ is such that $c \mid x$ for all $x \in X$, then $c \mid d$.

   Prove that if R is a commutative principal ideal ring with unity, then every nonempty (possibly infinite) set of elements of R has a greatest common divisor.

6. Let R be a commutative ring with unity. Show that if $x \in R$ is nilpotent, then $1_R - x$ and $1_R + x$ are both units.

7. Let R be a commutative ring, and let $A \subseteq R$. Let

$$\sqrt{A} = \{r \in R \mid \text{there exists } n > 0 \text{ such that } r^n \in A\}.$$

   Prove that if I is an ideal of R, then $\sqrt{I}$ is an ideal of R that contains I. The ideal $\sqrt{I}$ is called the *radical of I*.

8. Let R be a commutative ring with unity. Show that $\sqrt{(0)}$ is the ideal of all nilpotent elements of R (we proved the set of all nilpotent elements is an ideal in Homework 3) and that it is contained in every prime ideal of R.