Math 566 - Homework 9
Due Wednesday April 17, 2024

1. Let K be an extension field of F.
   (i) Show that $[K : F] = 1$ if and only if $K = F$.
   (ii) Show that if $[K : F]$ is prime, and L is an intermediate field (that is, $F \subseteq L \subseteq K$), then either $F = L$ or $L = K$.
   (iii) Show that if $u \in K$ has degree n over F, and $[K : F]$ is finite, then n divides $[K : F]$.

2. Let $p(x) = x^3 - 6x^2 + 9x + 3 \in \mathbb{Q}[x]$.
   (i) Show that $p(x)$ is irreducible over $\mathbb{Q}$.
   (ii) Let u be a root of $p(x)$, and let $K = \mathbb{Q}(u)$. Express each of the following elements of K in terms of the basis $\{1, u, u^2\}$:
      (a) u^4.
      (b) u^5.
      (c) $3u^5 - u^4 + 2$.
      (d) $(u + 1)^{-1}$.

3. Let K be an extension of F, and let $u \in K$. Show that if $[F(u) : F]$ is finite and odd, then $F(u^2) = F(u)$.

4. Let E and F be field extensions of $\mathbb{Q}$. Prove that if $\sigma\colon E \to F$ is a nonzero field homomorphism, then $\sigma(q) = q$ for all $q \in \mathbb{Q}$.

5. Let $F = \mathbb{Q}(\sqrt{2})$.
   (i) Show that $x^2 - 3 \in F[x]$ is irreducible.
   (ii) Show that every element of $F(\sqrt{3})$ can be written uniquely in the form
   $$a_0 + a_2\sqrt{2} + a_3\sqrt{3} + a_6\sqrt{6}, \qquad a_i \in \mathbb{Q}.$$
   HINT: Note that $\{1, \sqrt{3}\}$ is a basis for $F(\sqrt{3})$ over F, and that $\{1, \sqrt{2}\}$ is a basis for F over $\mathbb{Q}$.
   (iii) Define $\sigma\colon F(\sqrt{3}) \to F(\sqrt{3})$ by
   $$\sigma(a_0 + a_2\sqrt{2} + a_3\sqrt{3} + a_6\sqrt{6}) = a_0 - a_2\sqrt{2} + a_3\sqrt{3} - a_6\sqrt{6}.$$
   Prove that σ is an isomorphism of $F(\sqrt{3})$ to itself which does not restrict to the identity on F.

6. Show that there is an isomorphism from $\mathbb{Q}(\sqrt{2})$ to $\mathbb{Q}(\sqrt{2}+1)$ that restricts to the identity on $\mathbb{Q}$, even though $\sqrt{2}$ and $\sqrt{2} + 1$ do not satisfy the same monic irreducible over $\mathbb{Q}$.

7. Let $\sigma\colon \mathbb{R} \to \mathbb{R}$ be a field automorphism.
   (i) Prove that σ must send positive reals to positive reals.
   (ii) Prove that if $a, b \in \mathbb{R}$ and $a < b$, then $\sigma(a) < \sigma(b)$.
   (iii) Show that if $q \in \mathbb{Q}$, then $\sigma(q) = q$.
   (iv) Show that $\sigma(r) = r$ for every $r \in \mathbb{R}$.