MATH 566 – Spring 2024

MIDTERM

SOLUTIONS

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1. Give an example of each of the following; you do not need to prove that the example has the given properties (though I certainly hope you could if you needed to). *(2 points each, 10 points total)*

   (i) A ring R that is not commutative.

   Example. One example is any $n \times n$ matrix ring, with $n \geq 2$, over a nontrivial ring. For instance, $M_{2\times 2}(\mathbb{R})$. Of course there are many others.

   (ii) A ring R that does not have a unity.

   Example. The even integers $2\mathbb{Z}$ with their usual addition and multiplication. Alternatively, any nontrivial abelian group A with multiplication defined as $xy = 0$ for all $x, y \in \mathbb{A}$.

   (iii) A ring R and a left ideal I of R that is not a two-sided ideal of R.

   Example. The ideal of 2×2 matrices with first column 0 in $M_{2\times 2}(\mathbb{R})$. That is,

$$\left\{ \left(\begin{array}{cc} 0 & a \\ 0 & b \end{array} \right) \in M_{2\times 2}(\mathbb{R}) \,\middle|\, a, b \in \mathbb{R} \right\}.$$

   (iv) A ring R and a two-sided ideal I of R that is a prime ideal but not maximal.

   Example. The ideal (0) in $\mathbb{Z}$ is prime but not maximal. Also, the ideal (x) in $\mathbb{R}[x, y]$ is prime but not maximal.

   (v) A commutative ring R and an ideal I that is not principal.

   Example. The ideal $(2, x)$ in $\mathbb{Z}[x]$ is not principal: it consists of all polynomials with integer coefficients that have even constant term.

2. Let $R_1, \ldots, R_n$ be rings with unity. Show that if I is an ideal of $R_1 \times \cdots \times R_n$, then there exist ideals $J_i \triangleleft R_i$ for $i = 1, \ldots, n$, such that $I = J_1 \times \cdots \times J_n$. *(10 points)*

   Proof. For each j, let $\pi_j \colon R_1 \times \cdots \times R_n \to R_j$ be the projection onto the jth coordinate.

   Let $I \triangleleft R_1 \times \cdots \times R_n$. Let $J_i = \pi_i(I)$ for $i = 1, \ldots, n$. We will prove that $I = J_1 \times \cdots \times J_n$.

   Note that because π_j is a surjective homomorphism, the image of an ideal of $R_1 \times \cdots \times R_n$ is an ideal of R_j, so $J_j \triangleleft R_j$ for each j. Also, since $J_j = \pi_j(I)$, it follows that $I \subseteq J_1 \times \cdots \times J_n$.

   To prove that $J_1 \times \cdots \times J_n \subseteq I$, let $(a_1, \ldots, a_n) \in J_1 \times \cdots \times J_n$. Fix i, $1 \leq i \leq n$. Since $a_i \in J_i$, there exists $x \in I$ such that $\pi_i(x) = a_i$. So $x = (b_1, \ldots, b_{i-1}, a_i, b_{i+1}, \ldots b_n)$ for some $b_j \in R_j$, $j = 1, \ldots, i-1, i+1, \ldots, n$.

   Let $e = (0, \ldots, 0, 1_{R_i}, 0, \ldots, 0)$ be the element of $R_1 \times \cdots \times R_n$ that has the unity of R_i in the ith coordinate, the 0 of R_j in the jth coordinate for $j \neq i$. Since I is an ideal and $e \in R$, then $ex \in I$. And of course $ex = (0, \ldots, 0, a_i, 0, \ldots, 0)$.

   We can do this for each i, $i = 1, \ldots, n$. Thus, we have

$$(a_1, 0, \ldots, 0), \quad (0, a_2, 0, \ldots, 0), \quad \ldots, \quad (0, 0, \ldots, 0, a_n) \in I.$$

   Since I is an ideal, it is closed under sums, so

$$(a_1, \ldots, a_n) = (a_1, 0, \ldots, 0) + (0, a_2, 0, \ldots, 0) + \cdots + (0, \ldots, 0, a_n) \in I.$$

   Thus, we have that if $(a_1, \ldots, a_n) \in J_1 \times \cdots \times J_n$, then $(a_1, \ldots, a_n) \in I$. This proves that $J_1 \times \cdots \times J_n$ is contained in I, and we obtain the desired equality. □

3. Let R be a Euclidean commutative ring with unity, with Euclidean function φ. Prove that $a \in R$ is a unit if and only if $\varphi(a) = \varphi(1_R)$. *(10 points)*

   Proof. First: if $r \neq 0$, theN $\varphi(1_R) \leq \varphi(r)$; indeed, we have that the first property of a Euclidean function yields $\varphi(1_R) \leq \varphi(1_R r) = \varphi(r)$.

   Now let $a \in R$. If a is a unit, then there exists $b \in R$ such that $ab = 1_R$. Then again using the first property of the Euclidean function we have $\varphi(a) \leq \varphi(ab) = \varphi(1_R)$. Thus, we have $\varphi(a) \leq \varphi(1_R)$, and $\varphi(1_R) \leq \varphi(a)$ always holds, so $\varphi(1_R) = \varphi(a)$, as desired.

   Conversely, if $\varphi(a) = \varphi(1_R)$, then using the second property of a Euclidean function to divide 1_R by a, we have that there exist $q, r \in R$ such that $1_R = qa+r$, and either $r = 0$ or $\varphi(r) < \varphi(a) = \varphi(1_R)$.

   Since no nonzero element r satisfies $\varphi(r) < \varphi(1_R)$, it follows that we must have $r = 0$. Thus, $1_R = qa$, which shows that a has a multiplicative inverse and therefore is a unit. □

4. Let R be a commutative ring, and let S be a multiplicative subset of R. Show that if R is a principal ideal ring, then $S^{-1}R$ is a principal ideal ring. *(10 points)*

   Proof. Let $s \in S$, and let $\varphi\colon R \to S^{-1}R$ be the canonical function $\varphi(a) = \frac{as}{s}$.

   Let J be an ideal of $S^{-1}R$, and let $I = \varphi^{-1}J$. We know from class that $J = S^{-1}I$. Since I is an ideal of R, there exists $a \in R$ such that $I = (a)$. We prove that $(\varphi(a)) = J$.

   Since $a \in I = \varphi^{-1}(J)$, we know $\varphi(a) \in J$, so $(\varphi(a)) \subseteq J$. Conversely, let $\frac{b}{t} \in J$. Since $J = S^{-1}I$, there exists $x \in I$ and $u \in S$ such that $\frac{b}{t} = \frac{x}{u}$. Therefore, there exists $v \in S$ such that $v(bu - xt) = 0$. Thus, $vbu = vxt \in I = (a)$.

   Therefore, there exist $n \in \mathbb{Z}$ and $r \in R$ such that $buv = na + ra$. Then

   $$\frac{bvus}{s} = \varphi(buv) = \varphi(na + ra) = n\left(\frac{as}{s}\right) + \left(\frac{rs}{s}\right)\left(\frac{as}{s}\right) \in \left(\frac{as}{s}\right).$$

   Now, since $\frac{bvus}{s} \in \left(\frac{as}{s}\right)$, and $vuss \in S$, we have that:

   $$\frac{b}{t} = \frac{b(vuss)}{t(vuss)} = \frac{s}{tvus}\left(\frac{bvus}{s}\right) \in \left(\frac{as}{s}\right) = (\varphi(a))$$

   Therefore, $\frac{b}{t} \in (\varphi(a))$. This proves that $J = S^{-1}I \subseteq (\varphi(a))$, and therefore we have the equality $(\varphi(a)) = J$, as desired. □

5. Let R be a commutative ring with unity, and $R[x]$ the ring of polynomials in one indeterminate with coefficients in R

   (i) Prove that $R[x]/(x) \cong R$, where (x) is the principal ideal generated by x. *(4 points)*

   Proof. Consider the identity map $\mathrm{id}_R\colon R \to R$ and the element $0 \in R$. By the Universal Property of the polynomial ring, there is a unique ring homomorphism $\varepsilon\colon R[x] \to R$ such that $\varepsilon(r) = r$ for each $r \in R$, and $\varepsilon(x) = 0$.

   Note that $\ker(\varepsilon) = (x)$. Indeed, x lies in the kernel, and if $f = a_0 + a_x + \cdots + a_n x^n \in \ker(\varepsilon)$, then $0 = \varepsilon(f) = a_0$. Thus, $f = x(a_1 + a_2 x + \cdots + a_n x^{n-1}) \in (x)$.

   Note also that ε is surjective, since for all $r \in R$, $\varepsilon(r) = r$.

   By the First Isomorphism Theorem, we have

   $$R \cong \frac{R[x]}{\ker(\varepsilon)} = \frac{R[x]}{(x)},$$

   as claimed. □

(ii) Prove that R is an integral domain if and only if (x) is a prime ideal of $R[x]$. *(3 points)*

Proof. Since R is a commutative ring with unity, so is $R[x]$. We know that if $I \triangleleft T$ where T is a commutative ring with unity, then I is a prime ideal of T if and only if T/I is an integral domain. Thus, (x) is a prime ideal of $R[x]$ if and only if $\frac{R[x]}{(x)} \cong R$ is an integral domain. $\square$

(iii) Prove that R is a field if and only if (x) is a maximal ideal of $R[x]$. *(3 points)*

Proof. And we know that T/I is a field if and only if I is a maximal ideal of T. Therefore, $R \cong \frac{R[x]}{(x)}$ is a field if and only if (x) is a maximal ideal of R. $\square$