Math 666 - Homework 1
Due Wednesday February 5

1. For each of the following universal constructions, describe explicity an "auxiliary category" that we can use to define them, and whether they would be an initial or a terminal object in that category. Your description of the category should include an explicit description of both objects and arrows in the category.

   (i) The kernel of a given group homomorphism $f\colon G \to H$.

   (ii) The cokernel of a given abelian group homomorphism $\phi\colon A \to B$.

   (iii) The equalizer of two given morphisms $f, g\colon X \to Y$ in some category $\mathcal{C}$.

   (iv) The coequalizer of two given morphisms $\phi, \theta\colon Z \to W$ in some category $\mathcal{C}$.

   (v) The product of a given family of objects $\{X_i\}_{i\in I}$ in some category $\mathcal{C}$.

   (vi) The coproduct of a given family of objects $\{Y_j\}_{j\in J}$ in some category $\mathcal{C}$.

   (vii) The pushout of a given pair of morphisms, $f_1\colon X_0 \to X_1$ and $f_2\colon X_0 \to X_2$, in some category $\mathcal{C}$.

   (viii) The pullback of a given pair of morphisms $g_1\colon Y_1 \to Y_3$ and $g_2\colon Y_2 \to Y_3$ in some category $\mathcal{C}$.

2. Let $F\colon \mathsf{Set} \to \mathsf{Set}$ be the functor associating to every set S the set S^ω of all sequences $(s_0, s_1, \ldots)$ of elements of S. Use Yoneda's Lemma to determine all morphisms from F to the identity functor of Set.

3. Show that the functor $\mathsf{Monoid} \to \mathsf{Set}$ that sends a monoid M to the set of invertible elements in M is representable, and describe the representing object.

4. Show that the contravariant functor $\mathsf{Set} \to \mathsf{Set}$ that associates to every set X the set $\mathbf{P}(X)$ of all subsets of X, is representable.

5. Let $(\mathcal{C}, U)$ be a concrete category (so $\mathcal{C}$ is a category, and $U\colon \mathcal{C} \to \mathsf{Set}$ is a faithful functor). Prove that the following are equivalent:

   (i) $\mathcal{C}$ has a free object on one generator with respect to U.

   (ii) The concretization functor U is representable.