Math 666 - Homework 3
Due Wednesday February 19

1. Recall that a subset J of a partially ordered set I is *cofinal in I* if and only if for every $i \in I$ there exists $j \in J$ such that $i \leq j$. Let $\mathcal{C}$ be a category, I a directed set, and J cofinal in I.

   (i) Show that J is directed.

   (ii) Show that if $(X_i, (f_{ij}))_I$ is a directed system in $\mathcal{C}$, then $(X_j, (f_{jk}))_J$ (the collection of objects and maps where all indices lie in J) is also a directed system.

   (iii) Show that $\varinjlim_I X_i$ "equals" $\varinjlim_J X_j$, in the sense that if one exists then so does the other, and they are isomorphic via a unique isomorphism that respects the coprojections.

   (iv) What can you say about $\varinjlim_I X_i$ if I has a maximal element?

2. Let I be a directed set, and let $(G_i, (f_{ij}))_I$ be a directed family of groups. Define an operation on the direct limit $\varinjlim_I |G_i|$ in Set as follows: given $[g, i]$ and $[h, j]$, let $k \in I$ be such that $i \leq k$ and $j \leq k$. Then define the product of $[g, i]$ and $[h, j]$ by:

$$[g, i] \cdot [h, j] = [f_{ik}(g) f_{jk}(h), k],$$

   where the product on the right hand side occurs in G_k.

   (i) Prove that the operation is well defined, and makes $\varinjlim_I |G_i|$ into a group, denoted $\varinjlim_I G_i$.

   (ii) Prove that this group is the direct limit of $(G_i, (f_{ij}))_I$ in Group.

3. Let I be a directed set, and let $(A_i, (f_{ij}))_I$ and $(B, (g_{ij}))_I$ be directed systems of abelian groups. By a HOMOMORPHISM $u \colon (A_i) \to (B_i)$ of directed system we mean a family of group homomorphism $u_i \colon A_i \to B_i$ such that for all $i, j \in I$, if $i \leq j$ then $u_j \circ f_{ij} = g_{ij} \circ u_i$.

   Suppose we are given three directed systems of abelian groups $(A_i, (f_{ij}))_I$, $(B_i, (g_{ij}))_I$, andf $(C_i, (h_{ij}))_I$, and homomorphisms $u \colon (A_i) \to (B_i)$ and $v \colon (B_i) \to (C_i)$, and that for each i, we have $\mathrm{Im}(u_i) = \ker(v_i)$.

   (i) Prove that u and v induce homomorphisms of direct limits $U \colon \varinjlim_I A_i \to \varinjlim_I B_i$ and $V \colon \varinjlim_I B_i \to \varinjlim_I C_i$.

   (ii) Show that $\mathrm{Im}(U) = \ker(V)$.

4. **Exercise 8.5:8.** Let $(X_i, (f_{ij}))_I$ be a directed system in Ab, where I is the set of positive integers ordered by divisiblity, each X_i is the additive group $\mathbb{Z}$, and for $j = ni$, the morphism $f_{ij} \colon \mathbb{Z} \to \mathbb{Z}$ is multiplication by n.

   (i) Show that $\varinjlim_I X_i$ may be identified with the additive group of rational numbers.

   (ii) Show that if you perform the same construction starting with an arbitrary abelian group A in place of $\mathbb{Z}$, the result is a $\mathbb{Q}$ vector space which can be characterized by a universal property relative to A.