Math 666 - Homework 5
Due Wednesday March 12

1. **Exercise 8.9:1.** Let $\mathcal{D}$ and $\mathcal{E}$ each be the category with object set $\{0, 1\}$ and no morphisms other than the identity morphisms.

   (i) Suppose L is a lattice, L_{pos} is the underlying partially ordered set, and let $\mathcal{C} = (L_{\text{pos}})_{\text{cat}}$ be the corresponding category. Describe what it means to give a bifunctor $B\colon \mathcal{D} \times \mathcal{E} \to \mathcal{C}$ as in Lemma 8.9.1, verify that the indicated limits and colimits exist, and identify the comparison morphism c_B. (Since $\mathcal{C}$ is a category in which $\mathcal{C}(X, Y)$ has at most one element, you should specifically describe the domain and codomain, and prove that an arrow exists between them).

   (ii) Let $\mathcal{C}$ be the two element lattice, and show that it is possible for the comparison morphism c_B to fail to be an isomorphism.

   (iii) Do the same with $\mathcal{C} = \mathsf{Set}$, and $\mathcal{D}$ and $\mathcal{E}$ as above.

2. Let $\{G_{1n}, G_{2n}\}_{n=1}^{\infty}$ be a family of groups. We can view the family as the image of a bifunctor $B\colon \mathcal{D} \times \mathcal{E} \to \mathsf{Group}$, $\mathcal{D} = \mathbf{P}_{\text{cat}}$ where $\mathbf{P}$ is the two element set with the discrete order (each element is only comparable to itself), and $\mathcal{E} = \mathbf{Q}_{\text{cat}}$ where $\mathbf{Q}$ is the positive integers with the discrete order.

   (i) Describe explicitly the comparison morphism

   $$c_B\colon \varinjlim_m \varprojlim_n G_{mn} \longrightarrow \varprojlim_n \varinjlim_m G_{mn}.$$

   (ii) Determine whether the map is always an isomorphism. If the map is always an isomorphism, prove this is the case. If not, then give an explicit example where it is not, and prove that it is not.

3. **Exercise 8.9.10.** Determine whether the abelianization functor $\mathsf{Group} \to \mathsf{Ab}$ respects

   (i) Inverse limits;

   (ii) Products;

   (iii) Equalizers.

   In each case where the answer is negative, determine whether it is injectivity, surjectivity, or both proeprties of the comparison morphism that may fail.