Math 666 - Homework 6
Due Wednesday March 19

1. **Exercise 9.1:1.**

   (i) Show that if the empty algebra is excluded from Ω-Alg, then the resulting category may fail to have small limits.

   (ii) Can you give an example where the category over which the limit is taken is finite?

2. **Exercise 9.1:2.**

   (i) Show that in the context of Lemma 9.1.7 (equivalent conditions on an equivalence relation $E \subseteq |A| \times |A|$ that define a congruence on A), if the type Ω is finitary, then the three equivalent conditions are also equivalent to the condition:

   (c') For ever $s \in |\Omega|$ and every $\mathrm{ari}(s)$-tuple

   $$\big((a_0, b_0), \ldots, (a_{\mathrm{ari}(s)-1}, b_{\mathrm{ari}(s)-1})\big) \in E^{\mathrm{ari}(s)} \subseteq (|A| \times |A|)^{\mathrm{ari}(s)}$$

   such that $a_i \neq b_i$ for *at most one* $i \in \mathrm{ari}(s)$, one has

   $$s_{A\times A}\big((a_0, b_0), \ldots, (a_{\mathrm{ari}(s)-1}, b_{\mathrm{ari}(s)-1})\big) \in E.$$

   (ii) Show that the analog of the above is not true if the type Ω is not finitary.

3. **Exercise 9.1:3.** Complete the proof of Lemma 9.1.10 that if all operations in Ω have arities of cardinality less than α (with α infinite), then Ω-Alg has direct limits over all $<\alpha$-directed partially ordered sets, constructed by giving an Ω-algebra structure to the direct limit of underlying sets, by showing those operations are well-defined and this algebra has the universal property of $\varinjlim A$.