Math 666 - Homework 8
Due Wednesday April 2

1. Recall that an Ω-algebra A is *locally finite* if and only if whenever $X \subseteq |A|$ is finite, the subalgebra it generates, $|\langle X \rangle|$ is finite. Let $\mathbf{V}$ be a variety of Ω-algebras. Let $\mathcal{C}$ be the collection of all $\mathbf{V}$-algebras that are locally finite.

   (i) Show that $\mathcal{C}$ is closed under taking subalgebras, homomorphic images, and direct limits. That is, subalgebras of locally finite $\mathbf{V}$-algebras are locally finite; homomorphic images of locally finite $\mathbf{V}$-algebras are locally finite, etc.

   (ii) Show that the collection of all locally finite groups is not a variety.

2. Let Ω be a type, $\mathbf{V}$ a variety of Ω-algebras. We prove Noether's Isomorphism Theorems for $\mathbf{V}$:

   (i) **The First Isomorphism Theorem.** Let $A, B \in \mathrm{Ob}(\mathbf{V})$, and $f\colon A \to B$ a surjective morphism. Then $B \cong A/\sim_f$, where $\sim_f$ is the congruence on A associated to f, via the map that sends $[a]_{\sim_f}$ to $f(a)$.

   (ii) Let $A \in \mathrm{Ob}(\mathbf{V})$, B a subalgebra of A, and R a congruence on A. Let $R[B]$ be the set of elements of A that are R-equivalent to some element of B. Then $R[B]$ is also a subalgebra of A.

   (iii) Let $A \in \mathrm{Ob}(\mathbf{V})$, B a subalgebra of A, and R a congruence on A. Then $R_B = R \cap (B \times B)$ is a congruence on B.

   (iv) **Second Isomorphism Theorem**. Let $A \in \mathrm{Ob}(\mathbf{V})$, B a subalgebra of A, and R a congruence on A. Then B/R_B is isomorphic to $R[B]/R_{R[B]}$, via the map that takes the equivalence class $[b]_{R_B}$ to the class of b in $R[B]/R_{R[B]}$.

   (v) **Third Isomorphism Theorem.** Let $A \in \mathrm{Ob}(\mathbf{V})$, let R and S be congruences on A with $R \subseteq S$, and let T be a congruence on A/R. Define

   $$\begin{aligned} S/R &= \{([a]_R, [a']_R) \in (A/R) \times (A/R) \mid (a, a') \in S\} \\ T^{-1}[A] &= \{(a, a') \in A \times A \mid ([a]_R, [a']_R) \in T\}. \end{aligned}$$

   Then S/R is a congruence on A/R, $T^{-1}[A]$ is a congruence on A, and we have isomorphisms

   $$\begin{aligned} (A/R)/(S/R) &\cong A/S & &\text{via } [[a]_R]_{S/R} \longmapsto [a]_S \\ (A/R)/T &\cong A/T^{-1}[A] & &\text{via } [[a]_R]_T \longmapsto [a]_{T^{-1}[A]}. \end{aligned}$$

   (vi) **Fourth Isomorphism Theorem.** Let $A \in \mathrm{Ob}(\mathbf{V})$, and let R be a congruence on A. There is a one-to-one, inclusion preserving correspondence between congruences on A that contain R and congruences on A/R.