Math 666 - Homework 9
Due Wednesday April 16

1. Let Ω be a type, and let $\mathbf{V}$ be a variety of Ω-algebras. If k is a cardinal (finite or infinite), then we say that an Ω-algebra A is "k-generated" if and only if there exists a subset $X \subseteq |A|$ such that $\mathrm{card}(X) \leq k$ and $\langle X \rangle = A$.

   (i) Prove that the class of all Ω-algebras A with the property that every k-generated subalgebra of A lies in $\mathbf{V}$ is a variety. Denote this variety by $\mathbf{V}_{(k)}$.

   (ii) Show that if $\mathbf{W}$ and $\mathbf{V}$ are varieties, then the class of all algebras in $\mathbf{W}$ with the property that their k-generated subalgebras lie in $\mathbf{V}$ is a subvariety of $\mathbf{W}$.

2. Let A be a finitely generated abelian group. Prove that there exists a set X and a function $u\colon X \to |A|$ such that (A, u) is relatively free if and only if A is *homocyclic*; that is, a direct sum of finitely many (possibly zero) isomorphic cyclic groups.

3. Let $(P, \leq)$ be a partially ordered set. A subset F of P is a **filter** if and only if

   (a) F is nonempty;

   (b) F is downward directed: if $x, y \in F$, there exists $z \in F$ such that $z \leq x$ and $z \leq y$; and

   (c) F is upward closed: for all $x \in F$ and $p \in P$, if $x \leq p$ then $p \in F$.

   If $F \neq P$, then we say that F is a *proper filter*. An **ultrafilter** is a proper filter such that $F \subseteq F' \subseteq P$ for some filter F' implies either $F = F'$ or $F' = P$. If $x \in P$, the set $\uparrow\!(x)$ of all $p \in P$ with $x \leq p$ is a filter. We say a filter F is *principal* if and only if $F = \uparrow\!(x)$ for some $x \in P$.

   Let I be a set. We will consider the partially ordered set $(\mathcal{P}(I), \subseteq)$ of subsets of I, partially ordered by inclusion.

   (i) Show that a filter F is proper if and only if $\varnothing \notin F$.

   (ii) Show that a proper filter F satisfies the *finite intersection property*: the intersection of any finite family of elements of F is nonempty.

   (iii) Show that if F is an ultrafilter on $\mathcal{P}(I)$, and $J \subseteq I$, then either $J \in F$ or $I \setminus J \in F$, but not both.

   (iv) Prove that if I is finite, then every ultrafilter on $\mathcal{P}(I)$ is principal.

   (v) Show that the Axiom of Choice implies that every proper filter in $\mathcal{P}(I)$ is contained in an ultrafilter.

   (vi) Prove that if I is infinite and F is a nonprincipal ultrafilter, then every cofinite subset of I is an element of F.

(vii) Show that if I is an infinite set, then (assuming the Axiom of Choice) $\mathcal{P}(I)$ has nonprincipal ultrafilters.

4. Let $\mathbf{V}$ be a variety of Ω-algebras, where Ω is finitary, and let $\{A_i\}_{i\in I}$ be a family of $\mathbf{V}$-algebras. Let F be a filter on $\mathcal{P}(I)$. Define a relation on $P = \prod_{i\in I} A_i$ as follows: for $a, b \in P$,

$$a \sim_F b \iff \{j \in I \mid a_j = b_j\} \in F.$$

(i) Prove that $\sim_F$ is a congruence on P, and thus that we can form the quotient $P/{\sim_F}$, which will be an algebra in $\mathbf{V}$. If F is an ultrafilter, then the quotient $P/{\sim_F}$ is called an *ultraproduct*, and is denoted P/F.

(ii) Show that if F is a nonprincipal ultrafilter, then two elements $a, b \in P$ have the same image in $P/{\sim_F}$ if a and b are equal almost everywhere ($a_i = b_i$ for almost all $i \in I$).

(iii) Łoś's Theorem (*the Fundamental Theorem of Ultraproducts*) states that if F is an ultrafilter, then a first order formula ϕ in the language of Ω-algebras is true in the ultraproduct $P/{\sim_F}$ if and only if the set of indices i for which the formula ϕ is true in A_i is an element of F.

Consider the product $P = \prod_{n\in\omega} \mathbb{R}$ of the real numbers with itself countably many times, and let F be a nonprincipal ultrafilter in ω. Show that:

(a) P/F is a field that contains a copy of $\mathbb{R}$.

(b) There is an element of P/F that is larger than every natural number.

(c) The field P/F contains infinitesimals, that is, elements ϵ such that for every positive integer n, $0 < \epsilon < \frac{1}{n}$.

The field P/F is sometimes called the *hyperreals*, and is a setting for non-standard analysis.