Math 666 - Homework 1
Solutions
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1. For each of the following universal constructions, describe explicity an "auxiliary category" that we can use to define them, and whether they would be an initial or a terminal object in that category. Your description of the category should include an explicit description of both objects and arrows in the category.

   (i) The kernel of a given group homomorphism $f\colon G\to H$.

   Answer. Let $\mathcal{C}$ be the category whose objects are pairs, (K,g), where K is a group, and $g\colon K\to G$ is a homomorphism such that $f\circ g=\mathbf{z}$, where $\mathbf{z}$ is the trivial morphism sending everything to e. Morphisms in $\mathcal{C}$ are $\phi\colon (K,g)\to(L,h)$, where $\phi\colon K\to L$ is a group homomorphism such that $h\circ\phi=g$.

   In this category, $(\ker(f),\iota)$, where $\iota\colon\ker(f)\hookrightarrow G$ is the inclusion, is a terminal object.

   Although you do not need to prove that this is the case, I include a proof. First, note that $(\ker(f),\iota)$ is in the category. Next, if $(K,g)\in\mathcal{C}$, then we know that $g(K)\subseteq\ker(f)$, so g induces, via co-restriction, a morphism $\overline{g}\colon G\to\ker(f)$. The morphism $\overline{g}\colon(K,g)\to(\ker(f),\iota)$ is in the category, since $\iota\circ\overline{g}=g$. Moreover, the morphism is unique: if $\phi\colon(K,g)\to(\ker(f),\iota)$ is any morphism, then $\iota\circ\phi=g=\iota\circ\overline{g}$; and since ι is an embedding, it is a monomorphism and therefore left cancellable, so $\phi=\overline{g}$. Thus, $(\ker(f),\iota)$ is a terminal object. □

   (ii) The cokernel of a given abelian group homomorphism $\phi\colon A\to B$.

   Answer. Let $\mathcal{C}$ be the category whose objects are pairs (C,g), where C is an abelian group, and $g\colon B\to C$ is a morphism such that $g\circ\phi=\mathbf{z}$, where $\mathbf{z}$ is the zero morphism that sends everything to the identity. Morphisms in $\mathcal{C}$ are $f\colon(C,g)\to(D,h)$ where $f\colon C\to D$ is an abelian group morphsm such that $f\circ g=h$.

   In this category, the cokernel $(\mathrm{Coker}(\phi),\pi)$, where $\pi\colon B\to B/\phi(A)=\mathrm{Coker}(\phi)$ is the canonical projection, is an initial object.

   Note that this is indeed an object of the category $\mathcal{C}$. To verify it is initial, let (C,g) be an object in $\mathcal{C}$. Then since $g\circ\phi=\mathbf{z}$, it follows that $\phi(A)\subseteq\ker(g)$, so g factors through $\mathrm{Coker}(\phi)$; hence there exists $\overline{g}\colon\mathrm{Coker}(\phi)\to C$ such that $g=\overline{g}\circ\pi$. To verify uniqueness, if $h\colon\mathrm{Coker}(\phi)\to C$ also satisfies $h\circ\pi=g$, then $h\circ\pi=\overline{g}\circ\pi$, and since π is surjective it is necessarily an epimorphism, hence right-cancellable. Thus, $h=\overline{g}$, proving uniqueness. □

   (iii) The equalizer of two given morphisms $f,g\colon X\to Y$ in some category $\mathcal{C}$.

   Answer. Let $\mathcal{D}$ be the category whose objects are pairs (Z,h), where Z is an object of $\mathcal{C}$, and $h\colon Z\to X$ is a morphism in $\mathcal{C}$ such that $f\circ h=g\circ h$. The morphisms of $\mathcal{D}$ are $\phi\colon(Z,h)\to(W,k)$, where $\phi\colon Z\to W$ is a morphism in $\mathcal{C}$ such that $k\circ\phi=h$. In this category, the equalizer (E,u) (where $u\colon E\to X$ is the structure map of the equalizer) is the terminal object.

   Indeed, by definition of the qualizer we have $f\circ u=g\circ u$, so (E,u) is in the category. And if (Z,h) is an object in $\mathcal{C}$, then $f\circ h=g\circ h$, so by the definition of the equalizer there exists a unique morphism $k\colon Z\to E$ such that $u\circ k=h$; this gives the unique morphism $k\colon(Z,h)\to(E,u)$, showing that (E,u) is a terminal object in this category. □

   (iv) The coequalizer of two given morphisms $\phi,\theta\colon Z\to W$ in some category $\mathcal{C}$.

   Answer. Let $\mathcal{D}$ be the category whose objects are pairs (X,f), where X is an object in $\mathcal{C}$, and $f\colon W\to X$ is a morphism in $\mathcal{C}$ such that $f\circ\phi=f\circ\theta$. The morphisms in $\mathcal{D}$ are $\psi\colon(X,f)\to(Y,g)$, where $\psi\colon X\to Y$ is a morphism in $\mathcal{C}$ such that $\psi\circ f=g$. In this category, the coequalizer is an initial object.

   Indeed, if (C,q) is the coequalizer of ϕ,θ, then it is an object of $\mathcal{D}$. And given any object (X,f) in $\mathcal{D}$, the universal property of the coequalizer guarantees the existence of a unique morphism $h\colon C \to X$ such that $f = h \circ q$; this morphism is the unique morphism in $\mathcal{D}$ from (C, q) to (X, f). □

(v) The product of a given family of objects $\{X_i\}_{i\in I}$ in some category $\mathcal{C}$.

Answer. Let $\mathcal{D}$ be the category whose objects are pairs, $(Y, \{f_i\}_{i\in I})$, where Y is an object in $\mathcal{C}$, and for each i, $f_i\colon Y \to X_i$ is a morphism in $\mathcal{C}$. The morphisms in $\mathcal{D}$ are $\phi\colon (Y, \{f_i\}_{i\in I}) \to (Z, \{g_i\}_{i\in i})$ where $\phi\colon Y \to Z$ is a morphism in $\mathcal{C}$ such that $f_i = g_i \circ \phi$ for each $i \in I$.

The product of the family $\{X_i\}_{i\in I}$ is a terminal object in $\mathcal{D}$. Indeed, if $(P, \{p_i\}_{i\in I})$ is the product and the projections, this is an object in $\mathcal{D}$. And if $(Y, \{f_i\}_{i\in I})$ is any object in $\mathcal{D}$, then the universal property of the product tells us that there exists a unique $\phi\colon Y \to P$ such that $f_i = p_i \circ \phi$ for each $i \in I$. This is the unique morphism $\phi(Y, \{f_i\}_{i\in I}) \to (P, \{p_i\}_{i\in I})$ in $\mathcal{D}$. □

(vi) The coproduct of a given family of objects $\{Y_j\}_{j\in J}$ in some category $\mathcal{C}$.

Answer. Let $\mathcal{D}$ be the category whose objects are pairs $(X, \{r_j\}_{j\in J})$, where X is an object of $\mathcal{C}$, and for each $j \in J$, $r_j\colon Y_j \to X$ is a morphism in $\mathcal{C}$. The morphisms of $\mathcal{D}$ are $\phi\colon (X, \{r_j\}_{j\in J}) \to (W, \{s_j\}_{j\in J})$, where $\phi\colon X \to W$ is a morphism in $\mathcal{C}$ such that for each $j \in J$, we have $s_j \circ \phi = r_j$.

The coproduct $(Q, \{q_j\}_{j\in J})$, where the q_j are the structure maps, is the initial object in this category. Indeed, if $(X, \{r_j\}_{j\in J})$ is any object in $\mathcal{D}$, then the universal property of the coproduct says that there exists a unique morphism $\phi\colon Q \to X$ such that for each $j \in J$, $\phi \circ q_j = r_j$, and this gives a morphism $\phi\colon (Q, \{q_j\}_{j\in J}) \to (X, \{r_j\}_{j\in J})$. □

(vii) The pushout of a given pair of morphisms, $f_1\colon X_0 \to X_1$ and $f_2\colon X_0 \to X_2$, in some category $\mathcal{C}$.

Answer. Let $\mathcal{D}$ be the category whose objects are pairs $(Y, \{\phi_1, \phi_2\})$, where Y is an object in $\mathcal{C}$, and $\phi_1\colon X_1 \to Y$ and $\phi_2\colon X_2 \to Y$ are morphisms in $\mathcal{C}$ such that $\phi_1 \circ f_1 = \phi_2 \circ f_2$. The morphisms in $\mathcal{C}$ are $g\colon (Y, \{\phi_1, \phi_2\}) \to (Z, \{\psi_1, \psi_2\})$ where $g\colon Y \to Z$ is a morphism in $\mathcal{C}$ such that $\psi_1 = g \circ \phi_1$ and $\psi_2 = g \circ \phi_2$.

The pushout is an initial object in this category. Indeed, if $(P, \{p_1, p_2\})$ is the pushout, then it is an object in the category. And if $(Y, \{\phi_1, \phi_2\})$ is an object in $\mathcal{D}$, then because $\phi_1 \circ f_1 = \phi_2 \circ f_2$, the universal property of the pushout gives a unique morphism $g\colon P \to Y$ in $\mathcal{C}$ such that $\phi_1 = g \circ p_1$ and $\phi_2 = g \circ p_2$; this gives the unique morphism g from $(P, \{p_1, p_2\})$ to $(Y, \{\phi_1, \phi_2\})$ in $\mathcal{D}$. □

(viii) The pullback of a given pair of morphisms $g_1\colon Y_1 \to Y_3$ and $g_2\colon Y_2 \to Y_3$ in some category $\mathcal{C}$.

Answer. Let $\mathcal{D}$ be the category whose objects are pairs $(X, \{f_1, f_2\})$, where X is an object in $\mathcal{C}$, and $f_1\colon X \to Y_1$, $f_2\colon X \to Y_2$ are morphisms in $\mathcal{C}$ such that $g_1 \circ f_1 = g_2 \circ f_2$. The morphisms are $\phi\colon (X, \{f_1, f_2\}) \to (W, \{h_1, h_2\})$ where $\phi\colon X \to W$ is a morphism in $\mathcal{C}$ such that $f_1 = h_1 \circ \phi$ and $f_2 = h_2 \circ \phi$.

The pullback of g_1 and g_2 in $\mathcal{C}$ is a terminal object in $\mathcal{D}$. Indeed, if P is the pullback, with structure morphisms $\phi_i\colon P \to Y_i$, then $(P, \{\phi_1, \phi_2\})$ is an object of $\mathcal{D}$; and if $(X, \{f_1, f_2\})$ is any object in $\mathcal{D}$, then because $g_1 \circ f_1 = g_2 \circ f_2$, there exists a unique morphism $\psi\colon X \to P$ in $\mathcal{C}$ such that $f_1 = \phi_1 \circ \psi$ and $f_2 = \phi_2 \circ \psi$; this gives the unique morphism ψ from $(X, \{f_1, f_2\})$ to $(P, \{\phi_1, \phi_2\})$ in $\mathcal{D}$. □

2. Let $F\colon \mathsf{Set} \to \mathsf{Set}$ be the functor associating to every set S the set S^ω of all sequences $(s_0, s_1, \ldots)$ of elements of S. Use Yoneda's Lemma to determine all morphisms from F to the identity functor of Set.

Answer. Note that F is equal to the functor $h_\omega = \mathsf{Set}(\omega, -)$. Thus, by Yoneda's Lemma, morphisms from $F = h_\omega$ to $\mathrm{Id}_{\mathsf{Set}}$ correspond to elements of $\mathrm{Id}_{\mathsf{Set}}(\omega) = \omega$.

Namely, let $n \in \omega$. According to Yoneda's Lemma, we obtain a morphism $\nu_n \colon F \to \mathrm{Id}_{\mathsf{set}}$ as follows: given $S \in \mathrm{Ob}(\mathsf{Set})$, we define the map $a(S) \colon \mathsf{Set}(\omega, S) \to \mathrm{Id}_{\mathsf{Set}}(S)$ by

$$f \longmapsto \mathrm{Id}_{\mathsf{Set}}(f)\big(n\big) = f(n).$$

That is, given $s \in S^{\omega}$, $\nu_n(s) = s_n$, the projection onto the nth term of the sequence.

So, we obtain the natural transformations ν_n, given by

$$\begin{aligned} \nu_n \colon S^{\omega} &\longrightarrow S \\ s &\longmapsto s_n \end{aligned}$$

which is a natural transformation: given $f \colon S \to T$, the diagram

$$\begin{array}{ccc} S^{\omega} & \xrightarrow{\nu_n(S)} & S \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle f} \\ T^{\omega} & \xrightarrow{\nu_n(T)} & T \end{array}$$

commutes, because if $s \in S^{\omega}$, we have

$$\begin{aligned} f(\nu_n(S)(s)) &= f(s_n), \\ \nu_n(T)(F(f)(s)) &= \nu_n(f \circ s) = (f \circ s)(n) = f(s_n), \end{aligned}$$

as required.

While these morphisms of functors are morphisms we might have come up with without Yoneda's Lemma, the strength of Yoneda's Lemma here is that it guarantees that these are the *only* morphism $F \to \mathrm{Id}_{\mathsf{Set}}$. □

Remark. We can also construct these morphisms by noting that the identity functor on Set is *also* a representable/hom functor: it corresponds to $h_{\{\star\}}$, where $\{\star\}$ is a singleton set. Then the Yoneda Embedding tells us that morphism $F \to \mathrm{Id}_{\mathsf{Set}}$ (that is, morphisms $h_{\omega} \to h_{\{\star\}}$) correspond to set functions $\{\star\} \to \omega$; that is, elements of ω. □

3. Show that the functor $\mathsf{Monoid} \to \mathsf{Set}$ that sends a monoid M to the set of invertible elements in M is representable, and describe the representing object.

   Proof. Let M^* be the set of invertible elements of M.

   Since the functor is covariant, if it is representable it is isomorphic (as a functor) to a functor of the form h_N for some monoid N. As noted in class, that would mean a monoid N and an element $u \in N^*$ that is "universal", in the sense that for every element $x \in M^*$, there exists a unique monoid morphism $f \colon N \to M$ such that $f(u) = x$.

   That suggests a free object on one element, u; and in order for u to be invertible, we perhaps want to take the free group of rank u, $F(u)$, viewed as a monoid.

   Let us verify that this works: for every monoid M, and every element $x \in M^*$, since M^* is a group there exists a unique group homomorphism $f \colon F(u) \to M^*$ such that $f(u) = x$. If we compose this morphism with the embedding $M^* \hookrightarrow M$, we get a monoid homomorphism $f \colon F(u) \to M$ with $f(u) = x$. Moreover, since the image of $F(u)$ under any monoid morphism is completely determined by its value at u, this map is unique. That is, $(F(u), u)$ is a universal pair for this functor.

Thus, the functor sending M to M^* (and morphisms $f\colon M \to M'$ to the restriction of f to M^*, which necessarily has image inside $(M')^*$) is representable, isomorphic to $h_{F(u)}$, where $F(u)$ is the free group of rank 1, viewed as a monoid; technically, we should have the forgetful functor $U\colon \mathsf{Group} \to \mathsf{Monoid}$, and use $U(F(u))$ rather than U. $\square$

4. Show that the contravariant functor $\mathsf{Set} \to \mathsf{Set}$ that associates to every set X the set $\mathbf{P}(X)$ of all subsets of X, is representable.

   Proof. As this functor is contravariant, if it is representable it will be isomorphic to h^A for some A. That requires a set A and a universal object $u \in \mathbf{P}(A)$ such that for every subset Z of X, there exists a unique function $f\colon X \to A$ such that $\mathbf{P}(f)\colon \mathbf{P}(A) \to \mathbf{P}(X)$ has $\mathbf{P}(f)(u) = Z$. And recall that $\mathbf{P}(f)(u)$ is defined to be $\{x \in X \mid f(x) \in u\}$.

   Thus, we just need a set that can distinguish between "things in the subset" and "things not in the subset". So we can take $A = \{0,1\}$, and $u = \{1\}$. Then given Z, the unique function $f\colon X \to \{0,1\}$ with $\mathbf{P}(f)(\{1\}) = \{x \in X \mid f(x) \in \{1\}\} = Z$ is χ_Z, the characteristic function of Z.

   Thus, the contravariant functor is isomorphic to h^2 (where $2 = \{0,1\}$ in Set), and hence is representable. $\square$

5. Let $(\mathcal{C}, U)$ be a concrete category (so $\mathcal{C}$ is a category, and $U\colon \mathcal{C} \to \mathsf{Set}$ is a faithful functor). Prove that the following are equivalent:

   (i) $\mathcal{C}$ has a free object on one generator with respect to U.

   (ii) The concretization functor U is representable.

   Proof. (i) $\Longrightarrow$ (ii) Let F be the free object on one generator $u \in U(F)$. Then for every $X \in \mathrm{Ob}(\mathcal{C})$ and every $x \in U(X)$, there exists a unique $f \in \mathcal{C}(F, X)$ such that $U(f)(u) = x$.

   I claim that $h_F \cong U$. Indeed, let $a\colon h_F \to U$ be defined as follows: for each $X \in \mathrm{Ob}(\mathcal{C})$, define

$$a(X)\colon \mathcal{C}(F,X) = h_F(X) \longrightarrow U(X) \quad \text{by} \quad a(X)(f) = U(f)(u) \in U(X).$$

   The universal property of $(F, \{u\})$ guarantees that this is a bijection, so we just need to show that it is a natural transformation in order to show that it is an isomorphism of functors.

   Given $g\colon X \to Y \in \mathcal{C}(X,Y)$, the following diagram should commute:

$$\begin{array}{ccc} \mathbf{C}(F,X) & \xrightarrow{a(X)} & U(X) \\ {\scriptstyle g\circ -}\downarrow & & \downarrow{\scriptstyle U(g)} \\ \mathbf{C}(F,Y) & \xrightarrow{a(Y)} & U(Y) \end{array}$$

   Indeed, given $f \in \mathcal{C}(F,X)$, we have

$$\begin{aligned} U(g)\Big(a(X)(f)\Big) &= U(g)\Big(U(f)(u)\Big) \\ &= U(g) \circ U(f)(u) = U(g \circ f)(u) \\ &= a(Y)(g \circ f) \\ &= a(Y)\Big(g \circ -(f)\Big). \end{aligned}$$

   Thus, this is a natural transformation, and since the connecting maps $a(X)$ are bijections, it is an isomorphism of functors. Thus, $h_F \cong U$, so U is representable, as desired.

(ii) $\Longrightarrow$ (i) Assume that U is representable by some $G \in \mathrm{Ob}(\mathcal{C})$, and let $b\colon h_G \to U$ be the isomorphism. Let $u = b(G)(\mathrm{id}_G)$ (by Yoneda's Lemma, this is the element that determines b). The claim is that the object G is free on $\{u\}$.

To verify this claim, let $X \in \mathrm{Ob}(\mathcal{C})$ and let $x \in U(X)$. We want to show that there exists a unique $\varphi\colon \mathcal{C}(G, X)$ such that $U(\varphi)(u) = x$.

Since $b(X)\colon \mathcal{C}(G, X) \to U(X)$ is a bijection (because b is an isomorphism), and $x \in U(X)$, there exists a unique $\varphi \in \mathcal{C}(G, X)$ such that $b(X)(\varphi) = x$. And we obtain the following commutative diagram:

$$\begin{array}{ccc}
\mathbf{C}(G,G) & \xrightarrow{\;b(G)\;} & U(G) \\
{\scriptstyle \varphi\circ-}\big\downarrow & & \big\downarrow{\scriptstyle U(\varphi)} \\
\mathbf{C}(G,X) & \xrightarrow{\;b(X)\;} & U(X)
\end{array}$$

so we have

$$\begin{aligned}
U(\varphi)(u) &= U(\varphi)\Big(b(G)(\mathrm{id}_G)\Big) \\
&= U\Big(\varphi) \circ b(G)\Big)(\mathrm{id}_G) \\
&= b(X)\Big(\varphi \circ \mathrm{id}_G\Big) \\
&= b(X)(\varphi) \\
&= x.
\end{aligned}$$

The uniqueness of φ follows from the bijectivity of $b(X)$. $\square$