Math 666 - Homework 2

SOLUTIONS

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1. **Exercise 8.3:3.** Let $\mathcal{C}$ be a category with small coproducts (that is, any family of objects of $\mathcal{C}$ that is indexed by a small set has a coproduct in $\mathcal{C}$), and let $U\colon \mathcal{C} \to \mathsf{Set}$ be a functor. Prove that U has a left adjoint if and only if U is representable.

   Proof. Suppose first that U has a left adjoint, and call the adjoint F. Then for every object $C \in \mathrm{Ob}(\mathcal{C})$ and every set X we have a bijection

   $$\mathsf{Set}(X, U(C)) \cong \mathcal{C}(F(X), C),$$

   which give an isomorphism of functors $\mathsf{Set}^{\mathrm{op}} \times \mathcal{C} \to \mathsf{Set}$,

   $$\mathsf{Set}(-, U(-)) \cong \mathcal{C}(F(-), -).$$

   Let $X = \{\star\}$ be a singleton set, and consider $F(X)$. We claim that $U \cong h_{F(X)}$, that is, U is represented by $F(\{\star\})$.

   Indeed, property (ii) in the theorem defining adjoints tells us that $F(X)$ represents the functor

   $$\mathsf{Set}(X, U(-)) \cong \mathcal{C}(F(X), -) = h_{F(X)}.$$

   But we also know that h_X is isomorphic to the identity functor on Set. Restricting to the image of U, we obtain that since $h_X \cong \mathrm{Id}_{\mathsf{Set}}$, then

   $$\mathsf{Set}(X, U(-)) = h_X \circ U \cong \mathrm{Id}_{\mathsf{Set}} \circ U = U.$$

   Thus we obtain that $U \cong h_{F(X)}$, showing that U is representable.

   Conversely, assume that U is representable, and let C_0 be the representing object. Then we have a natural transformation $a\colon U \to h_{C_0}$ which is an isomorphism of functors. Thus, for every object C of $\mathcal{C}$, $a(C)\colon U(C) \to \mathcal{C}(C_0, C)$ is an isomorphism (that is, a bijection).

   Define a functor $F\colon \mathsf{Set} \to C$ by letting $F(X) = \amalg_{x\in X} C_0$; that is, the coproduct of a family of copies of C_0, indexed by X. We claim that F is the left adjoint of U.

   For simplicity, let us denote $F(X)$ by $\amalg_X C_0$, and let $q_x\colon C_0 \to \amalg_X C_0$ be the corresponding structure morphisms.

   First, F is a functor. Suppose that $f\colon X \to Y$ is a set map. From the universal property of the coproduct, we will obtain a map $\amalg_X C_0 \to \amalg_Y C_0$ if we have morphisms from each of $\{C_0\}_{x\in X}$ to $\amalg_Y C_0$. We obtain these maps by mapping the copy of C_0 indexed by x, via the identity map, to the copy indexed by $f(x)$, followed by the map $q_{f(x)}\colon C_0 \to \amalg_Y C_0$. Call this resulting map $\amalg_f \mathrm{id}$. Thus, $\amalg_f \mathrm{id}$ is the unique map such that $q_{f(x)} = (\amalg_f \mathrm{id}) \circ q_x$ for each $x \in X$.

   Therefore, $\amalg_{\mathrm{id}} \mathrm{id} = \mathrm{id}_{\amalg_X C_0}$, since this map satisfies $q_x = (\mathrm{id}_{\amalg_X C_0}) \circ q_x$ for all $x \in X$, and by the uniqueness clause of the universal property of the coproduct, that means this is the unique map that works.

   If $f\colon X \to Y$ and $g\colon Y \to Z$ are functions, then $\amalg_{g\circ f}\mathrm{id}$ is the unique map for which

   $$q_{(g\circ f)(x)} = (\amalg_{g\circ f}\mathrm{id}) \circ q_x;$$

   we know that $q_{f(x)} = (\amalg_f \mathrm{id}) \circ q_x$, and that $q_{g(y)} = (\amalg_g \mathrm{id}) \circ q_y$. Therefore,

   $$(\amalg_g \mathrm{id}) \circ ((\amalg_f \mathrm{id}) \circ q_x) = (\amalg_g \mathrm{id})\, q_{f(x)} = q_{g(f(x))} = q_{(g\circ f)(x)},$$

thus $(\amalg_g \mathrm{id}) \circ (\amalg_f \mathrm{id}) = \amalg_{g\circ f}\mathrm{id}$. Therefore, F sends compositions to compositions, hence F is a functor.

Before proceeding, note that the universal property of the coproduct tells us that

$$\mathcal{C}(\amalg_X C_0, C) \cong \prod_X \mathcal{C}(C_0, C),$$

since a map from the coproduct is equivalent to a family of maps from the factors, and similarly in any category.

To establish that F is the left adjoint of U, we note that for every X and every object $C \in \mathrm{Ob}(\mathcal{C})$, we have:

$$\begin{aligned}\mathcal{C}(F(X), C) = \mathcal{C}(\amalg_X C_0, C) &\cong \prod_X \mathcal{C}(C_0, C) \cong \prod_X h_{C_0}(C) \cong \prod_X U(C)\\ &\cong \prod_X \mathsf{Set}(\{\star\}, U(C)) \cong \mathsf{Set}(\amalg_X\{\star\}, U(C)) \cong \mathsf{Set}(X, U(C)),\end{aligned}$$

and the latter because X is isomorphic to $\amalg_X\{\star\}$ in Set. Because both U and F are functors, we know from Lemma 8.2.10 that this will give us an isomorphism of bifunctors, proving that F is the left adjoint of U, as claimed. □

2. **Exercise 8.3:5.** Show that if $A\colon \mathcal{C} \to \mathcal{D}$ and $B\colon \mathcal{D} \to \mathcal{C}$ give an equivalence of categories, then B is both a right and a left adjoint of A.

   Proof. We know that A and B are both full and faithful; that every object $C \in \mathrm{Ob}(\mathcal{C})$ is isomorphic to $B(D_C)$ for some $D_C \in \mathrm{Ob}(\mathcal{D})$; and each object $D \in \mathrm{Ob}(\mathcal{D})$ is isomorphic to $A(C_D)$ for some $C_D \in \mathrm{Ob}(\mathcal{C})$. Also, we have isomorphisms of functors $a\colon BA \to \mathrm{Id}_{\mathcal{C}}$ and $b\colon AB \to \mathrm{Id}_{\mathcal{D}}$.

   Let C be an object of $\mathcal{C}$ and D an object of $\mathcal{D}$. Then: $\mathcal{C}(C, B(D)) \cong \mathcal{C}(BA(C), B(D)) \cong \mathcal{D}(A(C), D)$, with the first bijection induced by a^{-1}, and the last bijection because B is full and faithful; because A and B are known to be functors, Lemma 8.2.10 guarantees these identifications give an isomorphism of bifunctors, so A is the left adjoint of B.

   Symmetrically, $\mathcal{D}(D, A(C)) \cong \mathcal{D}(AB(D), A(C)) \cong \mathcal{C}(B(D), C)$, and Lemma 8.2.10 guarantees this gives an isomorphism of bifunctors, so A is the right adjoint of B. □

3. **Exercise 8.3:6.** Let $\mathcal{C}$ be the category with $\mathrm{Ob}(\mathcal{C}) = \mathrm{Ob}(\mathsf{Group})$, but with morphisms defined so that for any groups G and H, $\mathcal{C}(G, H) = \mathsf{Set}(|G|, |H|)$. Thus, Group is a subcategory of $\mathcal{C}$ with the same objects, but smaller morphism sets. Does the inclusion function $\mathsf{Group} \to \mathcal{C}$ have a left and/or a right adjoint?

   Answer. Note that we can also view $\mathcal{C}$ as a full subcategory of Set, where we only look at underlying sets of groups (assuming the axiom of choice, every nonempty set can be given a group structure, so this would just amount to "all nonempty sets", but with "repeats", since the same underlying set may have multiple group structures on it).

   Vieweing $\mathcal{C}$ as this full subcategory, we define $F\colon \mathcal{C} \to \mathsf{Group}$ to be the restriction of the free group functor $F\colon \mathsf{Set} \to \mathsf{Group}$ to the objects of $\mathcal{C}$. In particular, H is a functor, and for any groups G and H, we have $\mathsf{Group}(F(G), H) = \mathsf{Group}(F(|G|), H) \cong \mathsf{Set}(|G|, |H|) = \mathcal{C}(G, H) = \mathcal{C}(G, i(H))$, where $i\colon \mathsf{Group} \to \mathcal{C}$ is the inclusion functor. This shows that F is the left adjoint of the inclusion functor.

   On the other hand, if i has a right adjoint U, then we would necessarily have for every group G and H that

   $$\mathsf{Set}(|G|, |H|) = \mathcal{C}(G, H) = \mathcal{C}(i(G), H) \cong \mathsf{Group}(G, U(H)).$$

   But take G to be the trivial group; then $\mathsf{Group}(G, U(H))$ contains only the trivial homomorphism, hence $\mathcal{C}(i(G), H) = \mathsf{Set}(\{e\}, |H|)$ would necessarily be a singleton. But $\mathsf{Set}(\{e\}, |H|) \cong |H|$, so this would require H to be the trivial group. Thus, there can be no right adjoint U to i. □