Math 666 - Homework 3

Solutions

Prof Arturo Magidin

1. Recall that a subset J of a partially ordered set I is *cofinal in* I if and only if for every $i\in I$ there exists $j\in J$ such that $i\leq j$. Let $\mathcal{C}$ be a category, I a directed set, and J cofinal in I.

   (i) Show that J is directed.

   Proof. Let $x,y\in J$. We know that there exists $z\in I$ such that $x\leq z$ and $y\leq z$, since I is directed. Because J is cofinal in I, there exists $w\in J$ such that $z\leq w$. Then $x\leq w$ and $y\leq w$, proving that J is directed. □

   (ii) Show that if $(X_i,(f_{ij}))_I$ is a directed system in $\mathcal{C}$, then $(X_j,(f_{jk}))_J$ (the collection of objects and maps where all indices lie in J) is also a directed system.

   Proof. The index set is a directed set; and if $j\leq k\leq m$ with all indices in J, then $f_{km}\circ f_{jk}=f_{jm}$, because this equality holds in $(X_i,f_{ij})_I$. In addition, $f_{jj}=\mathrm{id}_{X_j}$. So we have a directed system. □

   (iii) Show that $\varinjlim_I X_i$ "equals" $\varinjlim_J X_j$, in the sense that if one exists then so does the other, and they are isomorphic via a unique isomorphism that respects the coprojections.

   Proof. We will denote the direct limit over I, if it exists, by $L=\varinjlim_I X_i$, with coprojections $q_i\colon X_i\to L$. We will denote the direct limit over J, if it exists, by $M=\varinjlim_J X_j$ with coprojections $p_j\colon X_j\to M$.

   We will show that if L exists, then it has the universal property of M; and that if M exists, then it has the universal property of L.

   Assume L exists; then the coprojections $(q_i)_{i\in J}$ give a cone from the X_j to L. Now let W be an object, and let $w_j\colon X_j\to W$ be morphisms such that for all $j,k\in J$, if $j\leq k$ then $w_j=w_k\circ f_{jk}$. Given $i\in I$, $i\notin J$, let $j\in J$ be such that $i\leq j$. Define $w_i\colon X_i\to W$ by $w_i=w_j\circ f_{ij}$. I claim that this is well defined. Indeed, if $j'\in J$ is another index with $i\leq j'$, let $k\in J$ be such that $j,j'\leq k$. Then

   $$w_j\circ f_{ij}=(w_k\circ f_{jk})\circ f_{ij}=w_k\circ f_{ik}$$

   and

   $$w_{j'}\circ f_{ij'}=(w_k\circ f_{j'k})\circ f_{ij'}=w_k\circ f_{ik},$$

   so defining w_i using j or j' gives the same end result.

   Next, if $r,s\in I$, $r\leq s$, then let $j\in J$ be such that $s\leq j$. We need to show that $w_r=w_s\circ f_{rs}$. But indeed, we have that

   $$w_r=w_j\circ f_{rj}=w_j\circ f_{sj}\circ f_{rs}=w_s\circ f_{rs},$$

   as required. Thus, we have a cone from the direct family $(X_i,f_{ij})_I$ to W, and so there exists a unique morphism $w\colon L\to M$ such that $w_r=w\circ q_r$ for each $r\in I$. In particular, $w_j=w\circ q_j$ for all $j\in J$. Hence, L and the coprojections q_j satisfy the universal property of $\varinjlim_J X_j$, so $L\cong\varinjlim_J X_j$.

   Conversely, assume M exists. We define functions $q_i\colon X_i\to M$ as follows: if $i\in J$, then $q_i=p_i$. If $i\notin J$, then let $j\in J$ be such that $i\leq j$, and define $q_i=p_j\circ f_{ij}$.

   This is well-defined: if $j,j'\in J$ are such that $i\leq j$ and $i\leq j'$, let $k\in J$ be such that $j,j'\leq k$. Then

   $$p_j\circ f_{ij}=(p_k\circ f_{jk})\circ f_{ij}=p_k\circ f_{ik}$$

   and

   $$p_{j'}\circ f_{ij'}=(p_k\circ f_{j'k})\circ f_{ij'}=p_kf_{ik},$$

so q_i is well-defined.

If $r, s \in I$ are such that $r \leq s$, then let $j \in J$ be such that $s \leq j$. Then

$$q_r = p_j \circ f_{rj} = p_j \circ (f_{sj} \circ f_{rs}) = (p_j \circ f_{sj}) \circ f_{rs} = q_s \circ f_{rs},$$

so the q_i form a cone to M.

If W is an object and $w_i \colon X_i \to W$ are morphisms such that for all $i, j \in I$ with $i \leq j$, $w_i = w_j \circ f_{ij}$, then restricting to w_j with $j \in J$ we get a cone from $(X_j, f_{ij})_J$, so there exists a unique morphism $w \colon M \to W$ such that for all $j \in J$, $w_j = w \circ p_j$. If $i \in I$, let $j \in J$ be such that $i \leq j$. Then

$$w \circ q_i = w \circ (p_j \circ f_{ij}) = (w \circ p_j) \circ f_{ij} = w_j \circ f_{ij} = w_i$$

so M has the universal property of $\varinjlim_I X_i$, as required. □

(iv) What can you say about $\varinjlim_I X_i$ if I has a maximal element?

Answer. Note that a maximal element m in a directed set I is necessarily a maximum. Indeed, if $i \in I$, then because I is directed we know that there exists $j \in I$ such that $i \leq j$ and $m \leq j$; but since m is maximal, this gives $m = j$, and hence $i \leq m$. Thus, m is a maximum of I.

So if I has a maximal element m, then $J = \{m\}$ is cofinal in I. Thus, (X_m, id_{X_m}) is a directed system, and X_m is a direct limit, with coprojection id_{X_m}. Thus, $\varinjlim_J X_j = X_m$. Therefore, $\varinjlim_I X_i = X_m$, with coprojections $q_i = f_{im}$ for all $i \in I$. □

2. Let I be a directed set, and let $(G_i, (f_{ij}))_I$ be a directed family of groups. Define an operation on the direct limit $\varinjlim_I |G_i|$ in Set as follows: given $[g, i]$ and $[h, j]$, let $k \in I$ be such that $i \leq k$ and $j \leq k$. Then define the product of $[g, i]$ and $[h, j]$ by:

$$[g, i] \cdot [h, j] = [f_{ik}(g) f_{jk}(h), k],$$

where the product on the right hand side occurs in G_k.

(i) Prove that the operation is well defined, and makes $\varinjlim_I |G_i|$ into a group, denoted $\varinjlim_I G_i$.

Proof. First, we verify that the definition does not depend on the specific k chosen. If $k, k' \in I$ are such that $i \leq k, k'$, $j \leq k, k'$, then let $m \in I$ be such that $k, k' \leq m$. Then we have that

$$\begin{aligned} f_{im}(g) f_{jm}(h) &= f_{km}(f_{ik}(g)) f_{km}(f_{jk}(h)) \\ &= f_{km}(f_{ik}(g) f_{jk}(g)), \\ f_{im}(g) f_{jm}(h) &= f_{k'm}(f_{ik'}(g)) f_{k'm}(f_{jk'}(h)) \\ &= f_{k'm}(f_{ik'}(g) f_{jk'}(h)). \end{aligned}$$

Therefore, we have that $[f_{ik}(g) f_{jk}(h), k] = [f_{ik'}(g) f_{jk'}(h), k']$. So the equivalence class of the result, for fixed pairs (g, i) and (h, j), does not depend on the choice of k.

Next we verify that it does not depend on the choice of representatives for $[g, i]$ and $[h, j]$. Suppose that $[g, i] = [a, s]$ and $[h, j] = [b, t]$.

Since $[g, i] = [a, s]$, there exists $m \in I$, $i, s \leq m$, such that $f_{im}(g) = f_{sm}(a)$. And from $[h, j] = [b, t]$, we know that there exists $n \in I$, $j, t \leq n$, such that $f_{jn}(h) = f_{tn}(b)$. Now let $p \in I$ be such that $m, n \leq p$. Then we also have $f_{ip}(g) = f_{sp}(a)$, and $f_{jp}(h) = f_{tp}(b)$. And we have

$$\begin{aligned} [g, i][h, j] &= [f_{ip}(g) f_{jp}(h), p] \\ &= [f_{sp}(a) f_{tp}(b), p] \\ &= [a, s][b, t], \end{aligned}$$

from what we proved before. Thus, the operation is well-defined.

Next we verify it makes $\varinjlim_I |G_i|$ into a group. Note that $[g,i] = [f_{im}(g), m]$ for any $m \geq i$.

Associativity: Let $[g,r],[h,s],[k,t] \in \varinjlim_I |G_i|$. Let $m \in I$ be such that $r,s,t \leq m$. Then

$$\begin{aligned}
\Big([g,r][h,s]\Big)[k,t] &= [f_{rm}(g)f_{sm}(h), m][k,t] \\
&= [f_{mm}(f_{rm}(g)f_{sm}(h))f_{tm}(k), m] \\
&= [f_{rm}(g)f_{sm}(h)f_{tm}(k), m]. \\
[g,r]\Big([h,s][k,t]\Big) &= [g,r][f_{sm}(h)f_{tm}(k), m] \\
&= [f_{rm}(g)f_{mm}(f_{sm}(h)f_{tm}(k)), m] \\
&= [f_{rm}(g)f_{sm}(h)f_{tm}(k), m].
\end{aligned}$$

Identity element. Let $i \in I$. We claim that $[e_{G_i}, i]$ is the identity element of $\varinjlim_I |G_i|$ (we have used the fact that I is nonempty here). We note that $[e_{G_i}, i] = [e_{G_j}, j]$ for all $i,j \in I$. Indeed, if $m \in I$ is such that $i,j \leq m$, then $f_{im}(e_{G_i}) = e_{G_m}$ and $f_{jm}(e_{G_j}) = e_{G_m}$, since the maps f_{rs} are group homomorphisms. Thus, $[e_{G_i}, i] = [e_{G_j}, j]$ for all $i,j \in I$.

Let $[h,j]$ be any other element of the direct limit. Then

$$\begin{aligned}
[e_{G_i}, i][h,j] = [e_{G_j}, j][h,j] = [e_{G_j}h, j] = [h,j], \\
[h,j][e_{G_i}, i] = [h,j][e_{G_j}, j] = [he_{G_j}, j] = [h,j].
\end{aligned}$$

Thus, $[e_{G_i}, i]$ is the identity of $\varinjlim_I G_i$.

Inverses. Let $[g,i]$ be an element of the direct limit. Then $[g,i][g^{-1},i] = [gg^{-1},i] = [e_{G_i}, i]$ and $[g^{-1},i][g,i] = [g^{-1}g,i] = [e_{G_i}, i]$; so $[g^{-1},i] = [g,i]^{-1}$. □

(ii) Prove that this group is the direct limit of $(G_i,(f_{ij}))_I$ in Group.

Proof. Define the coprojection maps $q_i\colon G_i \to \varinjlim_I G_i$ by $q_i(g) = [g,i]$. These are group homomorphisms, since $q_i(gh) = [gh,i] = [g,i][h,i] = q_i(g)q_i(h)$.

These maps satisfy the defining property: if $i \leq j$ are in I, we want to show that $q_j \circ f_{ij} = q_i$. If $g \in G_i$, then

$$q_j \circ f_{ij}(g) = q_j(f_{ij}(g)) = [f_{ij}(g), j].$$

But $[f_{ij}(g), j] = [g,i]$. So $q_j \circ f_{ij} = q_i$, as required.

Finally, this group together with these maps satisfy the universal property. Let M be any group and let $m_i\colon G_i \to M$ be group homomorphisms such that $m_j \circ f_{ij} = m_i$ for all $i,j \in I$ with $i \leq j$. We want to show that there exists a unique morphism $m\colon \varinjlim_I G_i \to M$ such that $m_i = m \circ q_i$ for all i.

Because $\varinjlim_I |G_i|$ is the direct limit of the corresponding sets, we have a unique set function $m\colon \varinjlim_I |G_i| \to |M|$ such that $m_i = m \circ q_i$. This map is given by $m[g,i] = m_i(g)$. We just need to verify that this map is a group homomorphism.

Let $[g,i],[h,j] \in \varinjlim_I G_i$. Let $k \in I$ be such that $i,j \leq k$. Then $[g,i][h,j] = [f_{ik}(g)f_{jk}(h), k]$. But since $m_k(f_{ik}(g)) = m_i(g)$ and $m_k(f_{jk}(h)) = m_j(h)$, then

$$\begin{aligned}
m\Big([g,i][h,j]\Big) &= m\Big([f_{ik}(g)f_{jk}(h), k]\Big) \\
&= m_k(f_{ik}(g)f_{jk}(h)) \\
&= m_k(f_{ik}(g))m_k(f_{jk}(h)) \\
&= m_i(g)m_j(h) = m([g,i])m([h,j]).
\end{aligned}$$

This proves the result. □

3. Let I be a directed set, and let $(A_i, (f_{ij}))_I$ and $(B, (g_{ij}))_I$ be directed systems of abelian groups. By a HOMOMORPHISM $u\colon (A_i) \to (B_i)$ of directed system we mean a family of group homomorphism $u_i\colon A_i \to B_i$ such that for all $i, j \in I$, if $i \leq j$ then $u_j \circ f_{ij} = g_{ij} \circ u_i$.

   Suppose we are given three directed systems of abelian groups $(A_i, (f_{ij}))_I$, $(B_i, (g_{ij}))_I$, andf $(C_i, (h_{ij}))_I$, and homomorphisms $u\colon (A_i) \to (B_i)$ and $v\colon (B_i) \to (C_i)$, and that for each i, we have $\mathrm{Im}(u_i) = \ker(v_i)$.

   (i) Prove that u and v induce homomorphisms of direct limits $U\colon \varinjlim_I A_i \to \varinjlim_I B_i$ and $V\colon \varinjlim_I B_i \to \varinjlim_I C_i$.

   Proof. For simplicity, let $A = \varinjlim_I A_i$, $B = \varinjlim_I B_i$, and $C = \varinjlim_I C_i$; let $p_i\colon A_i \to A$, $q_i\colon B_i \to B$, and $r_i\colon C_i \to C$ be the corresponding coprojections.

   The maps $q_i \circ u_i\colon A_i \to B$ define maps from from the A_i to B, and if $i \leq j$, then

   $$(q_j \circ u_j) \circ f_{ij} = q_j \circ (u_j \circ f_{ij}) = q_j \circ (g_{ij} \circ u_i) = (q_j \circ g_{ij}) \circ u_i = q_i \circ u_i,$$

   so the maps form a cone from $(A_i, f_{ij})_I$ to B. Therefore, the universal property of A yields a morphism $U\colon A \to B$ such that for all i, $q_i \circ u_i = U \circ p_i$. In particular, $U([a, i]) = [u_i(a), i]$.

   The same argument shows that we get a map $V\colon B \to C$ such that for all i, $r_i \circ v_i = V \circ q_i$, and we have $V([b, j]) = [v_j(b), j]$. □

   (ii) Show that $\mathrm{Im}(U) = \ker(V)$.

   Proof. Let $[b, i] \in \ker(V)$. Then $[0, i] = V([b, i]) = [v_i(b), i] \in C$. Thus, there exists $j \in I$, $i \leq j$, such that $h_{ij}(v_i(b)) = 0$. But $h_{ij} \circ v_i = v_j \circ g_{ij}$, so we have $0 = v_j(g_{ij}(b))$ Thus, $g_{ij}(b) \in \ker(v_j)$, hence $g_{ij}(b) \in \mathrm{Im}(u_j)$.

   Therefore, there exists $a \in A_j$ such that $u_j(a) = g_{ij}(b)$. Hence

   $$U([a, j]) = [u_j(a), j] = [g_{ij}(b), j] = [b, i],$$

   and $[b, i] \in \mathrm{Im}(U)$. Thus, $\ker(V) \subseteq \mathrm{Im}(U)$.

   Conversely, let $[b, i] \in \mathrm{Im}(U)$. That means that there exists $[a, j] \in A$ such that $U([a, j]) = [b, i]$. Since $U([a, j]) = [u_j(a), j]$, we have $[u_j(a), j] = [b, i]$. Thus, there exists $k \in I$, $i, j \leq k$, such that $g_{jk}(u_j(a)) = g_{ik}(b)$. Thus we have

   $$V([b, i]) = V[g_{ik}(b), k] = V[g_{jk}(u_j(a)), k] = [v_k\big(g_{jk}(u_j(a))\big), k].$$

   But $v_k \circ g_{jk} = h_{jk} \circ v_j$. So

   $$v_k(g_{jk}(u_j(a))) = h_{jk}(v_j(u_j(a))).$$

   By assumption, $\mathrm{Im}(u_j) = \ker(v_j)$, so $v_j(u_j(a)) = 0$, hence

   $$V([b, i]) = [v_k \circ g_{jk} \circ u_j(a), k] = [h_{jk} \circ v_j \circ u_j(a), k] = [h_{jk}(0), k] = [0, k],$$

   showing that $[b, i] \in \ker(V)$. This proves the equality. □

4. **Exercise 8.5:8.** Let $(X_i, (f_{ij}))_I$ be a directed system in Ab, where I is the set of positive integers ordered by divisiblity, each X_i is the additive group $\mathbb{Z}$, and for $j = ni$, the morphism $f_{ij}\colon \mathbb{Z} \to \mathbb{Z}$ is multiplication by n.

   (i) Show that $\varinjlim_I X_i$ may be identified with the additive group of rational numbers.

   Proof. For each positive integer k, define $q_k\colon \mathbb{Z} \to \mathbb{Q}$ by $q_k(a) = \frac{a}{k}$. This is an additive map. We claim that $\mathbb{Q}$ is has the universal property of the direct limit $\varinjlim_I X_i$ with the q_k as the coprojections.

First, we verify that these maps fit into the relevant commuting triangle. That is, if $j = ni$, we must show that $q_i(a) = q_j(f_{ij}(a))$. Indeed, since $j = in$, then $f_{ij}(a) = na$. Thus,

$$q_j(f_{ij}(a)) = q_j(na) = \frac{na}{j} = \frac{na}{ni} = \frac{a}{i} = q_i(a).$$

Now, suppose that B is an abelian group, and $g_i\colon \mathbb{Z} \to B$ are such that if $j = ni$, then $g_j \circ f_{ij} = g_i$. We want to show that there exists a unique abelian group morphism $G\colon \mathbb{Q} \to B$ such that $g_i = Gq_i$ for all i.

Define $G\colon \mathbb{Q} \to B$ as follows: given $\frac{a}{b} \in \mathbb{Q}$ with $b > 0$, let $G(\frac{a}{b}) = g_b(a)$.

This is well defined: if k is a positive integer, then

$$G\left(\frac{ak}{bk}\right) = g_{bk}(ak) = g_{bk}(f_{a,ak}(a)) = g_b(a) = G\left(\frac{a}{b}\right).$$

Thus, if $\frac{a}{b} = \frac{r}{t}$, then $\frac{at}{bt} = \frac{rb}{bt}$; so then

$$G\left(\frac{a}{b}\right) = G\left(\frac{at}{bt}\right) = G\left(\frac{rb}{bt}\right) = G\left(\frac{r}{t}\right).$$

Given $a \in \mathbb{Z}$, we have:

$$Gq_i(a) = G\left(\frac{a}{i}\right) = g_i(a),$$

so $Gq_i = g_i$, as desired.

Finally, G is unique: if H also has the property that $Hq_i = g_i$ for all i, then given $\frac{a}{b} \in \mathbb{Q}$, we have

$$H\left(\frac{a}{b}\right) = Hq_b(a) = g_b(a) = Gq_b(a) = G\left(\frac{a}{b}\right),$$

so $H = G$.

This proves that $\mathbb{Q}$ and the proejctions q_k have the universal property of $\varinjlim_I X_i$, which proves the desired result. □

(ii) Show that if you perform the same construction starting with an arbitrary abelian group A in place of $\mathbb{Z}$, the result is a $\mathbb{Q}$ vector space which can be characterized by a universal property relative to A.

Proof. Consider copies of A index by the positive integers, and for $j = ni$, the morphism $f_{ij}\colon A \to A$ by $f_{ij}(a) = na$. Let V_A be the direct limit of this directed system, with coprojections q_i, where as usual $q_i(a) = [a, i]$.

We claim that V_A has a $\mathbb{Q}$-vector space structure; and that if V is any $\mathbb{Q}$-vector space and there is an abelian group morphism $f\colon A \to V$, then there exists a unique linear transformation $F\colon V_A \to V$ such that $f = Fq_1$.

To show that V_A has a $\mathbb{Q}$-vector space structure, let $[a, i] \in V_A$. Given $\frac{r}{s} \in \mathbb{Q}$, define

$$\frac{r}{s}[a, i] = [ra, si].$$

To show this makes sense, we need to show it does not depend on the representation of the rational, or on the representative of the equivalence class $[a, i]$ chosen.

If $\frac{r}{s} = \frac{x}{y}$, then $ry = sx$; thus we have

$$\frac{r}{s}[a, i] = [ra, si] = [rya, syi] = [sxa, syi] = [xa, yi] = \frac{x}{y}[a, i].$$

So the representation of the rational $\frac{r}{s}$.

To verify that the representative of $[a, i]$ does not affect the result, note that for any $n \geq 1$ we have:

$$[rna, sni] = [n(ra), n(si)] = q_{nsi}(nra) = q_{nsi} f_{si,nsi}(ra) = q_{si}(ra) = [ra, si].$$

Thus, we also have that $\frac{r}{s}[a, i] = \frac{r}{s}[na, ni]$. Now assume that $[a, i] = [b, j]$. Then there exists k with $im = k$ and $jn = k$ such that $f_{ik}(a) = f_{jk}(b)$; that is, $ma = nb$. Therefore,

$$\frac{r}{s}[a, i] = \frac{r}{s}[ma, mi] = \frac{r}{s}[nb, nj] = \frac{r}{s}[b, j].$$

And this corresponds to a "$\mathbb{Q}$-scalar multiplication" on V_A. Indeed, we have $\frac{1}{1}[a, i] = [a, i]$. In addition,

$$\frac{r}{s}\left(\frac{u}{v}[a, i]\right) = \frac{r}{s}[ua, vi] = [rua, svi] = \frac{ru}{sv}[a, i].$$

For left distributivity we have

$$\left(\frac{r}{s} + \frac{u}{v}\right)[a, i] = \frac{rv + su}{sv}[a, i] = [(rv + su)a, svi].$$
$$\frac{r}{s}[a, i] + \frac{u}{v}[a, i] = [ra, si] + [ua, vi] = [rva, svi] + [sua, svi]$$
$$= [rva + sua, svi] = [(rv + su)a, svi].$$

And for right distributivity, we have:

$$\frac{r}{s}\big([a, i] + [b, j]\big) = \frac{r}{s}\big([aj, ij] + [bi, ij]\big) = \frac{r}{s}[aj + bi, ij] = [r(aj + bi), sij].$$
$$\frac{r}{s}[a, i] + \frac{r}{s}[b, j] = [ra, si] + [rb, js] = [rja, sij] + [rib, jsi] = [rja + rib, isj] = [r(aj + bi), sij].$$

Thus, V_A is a $\mathbb{Q}$-vector space.

Now assume that V is a $\mathbb{Q}$-vector space and $f \colon A \to V$ an abelian group homomorphism. Since any additive map between $\mathbb{Q}$-vector spaces is in fact a linear transformation, we just need to show that there exists a unique abelian group morphism $F \colon V_A \to V$ such that $f = Fq_1$. Indeed, define $F \colon V_A \to V$ by $F([a, i]) = \frac{1}{i} f(a)$. This is well defined:

$$F([na, ni]) = \frac{1}{ni} f(na) = \frac{1}{ni}\left(nf(a)\right) = \frac{n}{ni} f(a) = \frac{1}{i} f(a) = F([a, i]).$$

Then, as above, this means that F is well defined. It is additive, since

$$F([a, i] + [b, j]) = F([aj, ij] + [ib, ij]) = F([aj + ib, ij]) = \frac{1}{ij} f(aj + ib)$$
$$= \frac{1}{ij} f(aj) + \frac{1}{ij} f(ib) = \frac{j}{ij} f(a) + \frac{i}{ij} f(b) = \frac{1}{i} f(a) + \frac{1}{j} f(b) = F([a, i]) + F([b, j]),$$

from which it follows that F is additive.

And we have that $Fq_1(a) = F[a, 1] = f(a)$.

Finally, uniqueness follows because if $G \colon V_A \to V$ also satisfies that $Gq_1 = f$, then

$$iG([a, i]) = G([ia, i]) = G([a, 1]) = Gq_1(a) = f(a) = F([a, 1]),$$

so $G([a, i]) = \frac{1}{i} F([a, 1])$. On the other hand,

$$F([a, i]) = \frac{1}{i} f(a) = \frac{1}{i} F([a, 1]) = G([a, i]),$$

so $F = G$. This proves the universal property. $\square$