Math 666 - Homework 4

SOLUTIONS

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1. **Exercise 8.5:9.** For this exercise, assume the known facts that

   (i) Every subgroup of a free group is free; and

   (ii) In the free group on x and y, the subgroup generated by the commutator $x^{-1}y^{-1}xy$ and $x^{-2}y^{-1}x^2y$ is free on those two generators.

   Let F be the free group on x and y, and let $f\colon F\to F$ be the endomorphism of F determined by mapping x to $x^{-1}y^{-1}xy$, and y to $x^{-2}y^{-1}x^2y$. Let G be the direct limit of the system

   $$F \xrightarrow{f} F \xrightarrow{f} F \xrightarrow{f} F \xrightarrow{f} \cdots$$

   Show that G is nontrivial, and every finitely generated subgroup of G is free, but that G equals its own commutator subgroup; that is, that the abelianization of G is the trivial group.

   Conclude that G is "locally free" but not free. (A group is "locally X" if every finitely generated subgroup of G has property X).

   Proof. Note that each of the coprojections $q_i\colon F\to G$ given by $q_i(a)=[a,i]$ is one-to-one. This follows because if $q_i(a)=e_G=[e,1]$, then there exists an integer $k\geq i$ such that the image of a in the kth copy of F equals the image of e; but the image of a is obtained by applying f to a $k-i$ times, and since f is one-to-one, this implies that $a=e$.

   To show that G is locally free, let H be a finitely generated subgroup of G; we want to show that H is free. Let $[a_1,i_1],\ldots,[a_n,i_n]$ be generators for H. Letting $k=\max(i_1,\ldots,i_n)$, and letting b_i be the image of a_i in the kth copy of F obtained by applying f a suitable number of times, we have that H is generated by $[b_1,k],\ldots,[b_n,k]$. That means that H is the image under q_k of $\langle b_1,\ldots,b_n\rangle\leq F$. But this is a subgroup of a free group, hence free. Since q_k is one-to-one, it induces an isomorphism between $\langle b_1,\ldots,b_n\rangle$ and its image H in G. Therefore, H is free, as claimed.

   However, because the image of f lies in the commutator subgroup of F, and for any group morphism $\phi\colon K_1\to K_2$ we have $\phi([K_1,K_1])\subseteq[K_2,K_2]$, we have that for any $[a,i]\in G$,

   $$[a,i]=[f(a),i+1]=q_{i+1}(f(a))\in q_{i+1}([F,F])\subseteq[G,G].$$

   That is, $q_i(F)\subseteq[G,G]$ for all i; since G is generated by the images of the q_i, it follows that $G\subseteq[G,G]$, hence $G=[G,G]$.

   Thus, the abelianization of G is trivial, since $G^{\mathrm{ab}}=G/[G,G]$. Since the abelianization of the free group on X is the free abelian group on X, this means that for G to be free it would have to be trivial. But since it contains $q_1(F)\cong F$, this means that G cannot be free.

   Thus, G is locally free, but not free, as claimed. □

2. **Exercise 8.8:1**

   (i) Combine the fact that left adjoints respect colimits and right adjoints respect limits with the characterization of monomorphisms and epimorphisms in Lemma 7.8.11 (as being parts of certain pullback or pushout diagrams) to obtain results about how left and right adjoint functors behave with respect to epimorphisms and monomorphisms.

Answer. Lemma 7.8.11 shows that $f\colon X \to Y$ is a monomorphism if and only if

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{id}_X} & X \\ {\scriptstyle \mathrm{id}_X}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{f} & Y \end{array}$$

is a pullback diagram, and is an epimorphism if and only if

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_Y} \\ Y & \xrightarrow{\mathrm{id}_Y} & Y \end{array}$$

is a pushout diagram.

Let $\mathcal{C}, \mathcal{D}$ be categories, and let $U\colon \mathcal{D} \to \mathcal{C}$ and $F\colon \mathcal{C} \to \mathcal{D}$ be adjoint functors, with U the right adjoint and F the left adjoint.

Since left adjoints respect colimits and the pushout is a colimit, we have that if f is an epimorphism in $\mathcal{C}$, then

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle F(\mathrm{id}_Y)} \\ F(Y) & \xrightarrow{F(\mathrm{id}_Y)} & F(Y) \end{array}$$

is a pushout diagram; since $F(\mathrm{id}_Y) = \mathrm{id}_{F(Y)}$, that means that $F(f)$ is an epimorphism. Thus, left adjoints send epimorphisms to epimorphisms.

Dually, applying U to the pullback diagram (which is a limit, and thus is respected by U) it follows that if f is a monomorphism, then $U(f)$ is a monomorphism. That is, right adjoints send monomorphisms to monomorphism. □

(ii) The results you get in (i) will not state that both left and right adjoints respect both epimorphisms and monomorphisms. For each implication that was not proven by part (i) provide an example showing that the corresponding result need not hold in general.

Answer. We want an example of a monomorphism whose image under a left adjoint functor is not a monomorphism; and an epimorphism whose image under a right adjoint functor is not an epimorphism.

For the former, consider the adjoint pair $I\colon \mathsf{Ab} \to \mathsf{Group}$ and $F\colon \mathsf{Group} \to \mathsf{Ab}$, where I is the inclusion functor from abelian groups to groups, and F is the abelianization functor, $F(G) = G^{\mathrm{ab}}$. We know that $\mathsf{Group}(G, I(A)) \cong \mathsf{Ab}(G^{\mathrm{ab}}, A)$, so F is the left adjoint of I.

Now consider the embedding $f\colon C_3 \to S_3$ of the cyclic group of order 3 into the symmetric group of degree 3. This is one-to-one, hence a monomorphism in Group. But $(C_3)^{\mathrm{ab}} = C_3$ and $(S_3)^{\mathrm{ab}} \cong C_2$, so $F(f)\colon F(C_3) \to F(S_3)$ yields the trivial map from C_3 to C_2. This is not a monomorphism in Ab, so even though f is a monomorphism, it is not the case that $F(f)$ is a monomorphism.

For an example where a right adjoint does not send an epimorphism to an epimorphism, consider adjoint pair $U\colon \mathsf{Monoid} \to \mathsf{Set}$ and $F\colon \mathsf{Set} \to \mathsf{Monoid}$, where U sends a monoid to its underlying set, and F sends a set to the free monoid on the set. These are adjoint pairs, with U the right adjoint and F the left adjoint.

Let $f\colon \mathbb{N} \to \mathbb{Z}$ be the embedding of the nonnegative integers into the integers. We have seen (a few times) that this embedding is an epimorphism in Monoid. However, $U(f)\colon U(\mathbb{N}) \to U(\mathbb{Z})$ is not surjective, hence not an epimorphism in Set. □