Math 666 - Homework 5
SOLUTIONS
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1. **Exercise 8.9:1.** Let $\mathcal{D}$ and $\mathcal{E}$ each be the category with object set $\{0,1\}$ and no morphisms other than the identity morphisms.

   (i) Suppose L is a lattice, $L_{\rm pos}$ is the underlying partially ordered set, and let $\mathcal{C} = (L_{\rm pos})_{\rm cat}$ be the corresponding category. Describe what it means to give a bifunctor $B\colon \mathcal{D}\times\mathcal{E}\to\mathcal{C}$ as in Lemma 8.9.1, verify that the indicated limits and colimits exist, and identify the comparison morphism c_B. (Since $\mathcal{C}$ is a category in which $\mathcal{C}(X,Y)$ has at most one element, you should specifically describe the domain and codomain, and prove that an arrow exists between them).

   Answer. A bifunctor $B\colon \mathcal{D}\times\mathcal{E}$ means giving a family of four elements of L, which we can label x_{00}, x_{01}, x_{10}, and x_{11}, corresponding to the objects of $\mathcal{D}\times\mathcal{E}$, which are $(0,0)$, $(0,1)$, $(1,0)$, and $(1,1)$. The only arrows in $\mathcal{D}\times\mathcal{E}$ are the identity arrows, so we do not need to worry about comparability of the x_{ij} in L.

   Here, because $\mathcal{D}\times\mathcal{E}$ has no arrows except the identity arrows, the limits of the functors $B(i,-)\colon \mathcal{E}\to\mathcal{C}$ are products, and the colimits of the functors $B(-,j)\colon\mathcal{D}\to\mathcal{C}$ are coproducts. In $(L_{\rm pos})_{\rm cat}$, a product of a family of elements is an element with maps into every element of the family (that is, less than or equal to each element of the family), such that any element with maps into each element of the family maps to the product (that is, larger than or equal to every element that is a lower bound to the family). That is, the product of a family is the meet of the family. Dually, the coproduct of a family is the join of the family. That means that the corresponding limits and colimits exist, since L has meets and joins of pairs of elements (by definition of a lattice).

   The objects in question are:
$$\varinjlim_{\mathcal{D}}\varprojlim_{\mathcal{E}} B(D,E) = (x_{00}\wedge x_{01})\vee(x_{10}\wedge x_{11}),$$
$$\varprojlim_{\mathcal{E}}\varinjlim_{\mathcal{D}} B(D,E) = (x_{00}\vee x_{10})\wedge(x_{01}\vee x_{11}).$$

   To verify that the comparison morphism c_B exists, we need to show that the former element is less than or equal to the latter element.

   Indeed, note that:
$$\begin{aligned} x_{00}\wedge x_{01} &\leq x_{00}\leq x_{00}\vee x_{10}\\ x_{00}\wedge x_{01} &\leq x_{01}\leq x_{01}\vee x_{11}, \end{aligned}$$

   That means that $x_{00}\wedge x_{01}$ is a lower bound for both $x_{00}\vee x_{10}$ and $x_{01}\vee x_{11}$, and therefore that
$$x_{00}\wedge x_{01}\leq (x_{00}\vee x_{10})\wedge(x_{01}\vee x_{11}).$$

   Similarly,
$$\begin{aligned} x_{10}\wedge x_{11} &\leq x_{10}\leq x_{00}\vee x_{10}\\ x_{10}\wedge x_{11} &\leq x_{11}\leq x_{01}\vee x_{11}\\ x_{10}\wedge x_{11} &\leq (x_{00}\vee x_{10})\wedge(x_{01}\vee x_{11}). \end{aligned}$$

   Since $(x_{00}\vee x_{10})\wedge(x_{01}\vee x_{11})$ is an upper bound for both $x_{00}\wedge x_{01}$ and $x_{10}\wedge x_{11}$, it is also an upper bound for their join. That is,
$$(x_{00}\wedge x_{01})\vee(x_{10}\wedge x_{11})\leq (x_{00}\vee x_{10})\wedge(x_{01}\vee x_{11}),$$

   which proves that c_B is the unique arrow between these two elements of L. $\square$

(ii) Let $\mathcal{C}$ be the two element lattice, and show that it is possible for the comparison morphism c_B to fail to be an isomorphism.

Answer. We need to make a choice of x_{ij} such that the two elements described above are distinct.

We can achieve this by letting $x_{00} = x_{11} = 0$ and $x_{01} = x_{10} = 1$. Then we have

$$\varinjlim_{\mathcal{D}} \varprojlim_{\mathcal{E}} B(D,E) = (0 \wedge 1) \vee (1 \wedge 0) = 0 \vee 0 = 0$$
$$\varprojlim_{\mathcal{E}} \varinjlim_{\mathcal{D}} B(D,E) = (0 \vee 1) \wedge (1 \vee 0) = (1 \wedge 1) = 1.$$

This, c_B is the unique arrow between 0 and 1, which is not an isomorphism. □

(iii) Do the same with $\mathcal{C} = \mathsf{Set}$, and $\mathcal{D}$ and $\mathcal{E}$ as above.

Answer. In the case of Set, the colimits will correspond again to coproducts, i.e. disjoint unions; and the limits will correspond to products, i.e. cartesian products. Thus, here we will have four sets, A_{ij}, and the limits and colimits will be:

$$\varinjlim_{\mathcal{D}} \varprojlim_{\mathcal{E}} B(D,E) = (A_{00} \times A_{01}) \amalg (A_{10} \times A_{11})$$
$$\varprojlim_{\mathcal{E}} \varinjlim_{\mathcal{D}} B(D,E) = (A_{00} \amalg A_{10}) \times (A_{01} \amalg A_{11}).$$

For ease, denote elements of A_{ij} with a superscript label, e.g. $a^{(ij)}$ for an element of A_{ij}.

An element of $(A_{00} \times A_{01}) \amalg (A_{10} \times A_{11})$ will be either of the form $(a^{(00)}, a^{(01)})$ or of the form $(a^{(10)}, a^{(11)})$. An element of $(A_{00} \amalg A_{10}) \times (A_{01} \amalg A_{11})$ will be of the form $(a^{(ij)}, b^{(rs)})$, where $(i,j) \in \{(0,0),(1,0)\}$ and $(r,s) \in \{(0,1),(1,1)\}$.

Note that this means that $(A_{00} \times A_{01}) \amalg (A_{10} \times A_{11})$ is a subset of $(A_{00} \amalg A_{10}) \times (A_{10} \amalg A_{11})$, and the comparison map c_B is the inclusion map. To show that it is need not be an isomorphism it suffices to show that it is not always surjective.

This is straightforward; note that the cardinality of the domain is

$$\mathrm{card}(A_{00})\mathrm{car}(A_{01}) + \mathrm{card}(A_{10})\mathrm{card}(A_{11}),$$

whereas the cardinality of the codomain is:

$$\begin{aligned}\mathrm{card}\big((A_{00} \amalg A_{10}) \times (A_{01} \amalg A_{11})\big) &= (\mathrm{card}(A_{00}) + \mathrm{card}(A_{10}))(\mathrm{card}(A_{01}) + \mathrm{card}(A_{11})) \\ &= \mathrm{card}(A_{00})\mathrm{card}(A_{01}) + \mathrm{card}(A_{00})\mathrm{card}(A_{11}) \\ &\quad + \mathrm{card}(A_{10})\mathrm{card}(A_{01}) + \mathrm{card}(A_{10})\mathrm{card}(A_{11}).\end{aligned}$$

For finite sets, the two sets will have equal cardinality if and only if

$$\mathrm{card}(A_{00})\mathrm{card}(A_{11}) + \mathrm{card}(A_{10})\mathrm{card}(A_{01}) = 0.$$

Thus, taking any nonempty sets will provide an example where c_B is not a bijection. Explicitly, taking each $A_{ij} = \{a_{ij}\}$, we have

$$\varinjlim_{\mathcal{D}} \varprojlim_{\mathcal{E}} B(D,E) = \{(a_{00}, a_{01}), (a_{10}, a_{11})\}$$
$$\varprojlim_{\mathcal{E}} \varinjlim_{\mathcal{D}} B(D,E) = \{(a_{00}, a_{01}), (a_{00}, a_{11}), (a_{10}, a_{01}), (a_{10}, a_{11})\},$$

so c_B is not a bijection. □

2. Let $\{G_{1n}, G_{2n}\}_{n=1}^{\infty}$ be a family of groups. We can view the family as the image of a bifunctor $B \colon \mathcal{D} \times \mathcal{E} \to \mathsf{Group}$, $\mathcal{D} = \mathbf{P}_{\mathrm{cat}}$ where $\mathbf{P}$ is the two element set with the discrete order (each element is only comparable to itself), and $\mathcal{E} = \mathbf{Q}_{\mathrm{cat}}$ where $\mathbf{Q}$ is the positive integers with the discrete order.

(i) Describe explicitly the comparison morphism

$$c_B\colon \varinjlim_m \varprojlim_n G_{mn} \longrightarrow \varprojlim_n \varinjlim_m G_{mn}.$$

Answer. Again, the colimit corresponds to a coproduct (free product of groups), and the limit to a product of groups. So we have:

$$\begin{aligned}
\varinjlim_m \varprojlim_n G_{mn} &= \left(\prod_{n=1}^{\infty} G_{1n}\right) * \left(\prod_{n=1}^{\infty} G_{2n}\right)\\
\varprojlim_n \varinjlim_m G_{mn} &= \prod_{n=1}^{\infty} (G_{1n} * G_{2n}).
\end{aligned}$$

In order to describe the comparison morphism, let us use superscript of the form $^{(ij)}$ to indicate that a given element lies in the group G_{ij}.

The elements of $\varinjlim \varprojlim G_{mn}$ are words in the alphabet of tuples $(g^{(1n)})$ and $(g^{(2n)})$. That is, a general element will look like

$$\left(g_1^{(1n)}\right)\left(h_1^{(2n)}\right)\left(g_2^{(1n)}\right)\left(h_2^{(2n)}\right)\cdots\left(g_k^{(1n)}\right)\left(h_k^{(2n)}\right),$$

where none of the tuples, except perhaps for $(g_1^{(1n)})$ and $(h_k^{(2n)})$, are the trivial element.

The elements of $\varprojlim \varinjlim G_{mn}$ are tuples of the form (w_n), where $w_n \in G_{1n} * G_{2n}$ is a word of the form

$$x_1^{(1n)} y_1^{(2n)} x_2^{(1n)} y_2^{(2n)} \cdots x_\ell^{(1n)} y_\ell^{(2n)},$$

where none of the elements, except perhaps for $x_1^{(1n)}$ and $y_\ell^{(2n)}$, are the identity.

The map c_B is given by

$$\left(g_1^{(1n)}\right)\left(h_1^{(2n)}\right)\left(g_2^{(1n)}\right)\left(h_2^{(2n)}\right)\cdots\left(g_k^{(1n)}\right)\left(h_k^{(2n)}\right) \xmapsto{c_B} \left(g_1^{(1n)} h_1^{(2n)} \cdots g_k^{(1n)} h_k^{(2n)}\right). \quad \square$$

(ii) Determine whether the map is always an isomorphism. If the map is always an isomorphism, prove this is the case. If not, then give an explicit example where it is not, and prove that it is not.

Answer. If there exist two indices $j \neq k$ such that G_{1j} and G_{2k} are both nontrivial, then the map is not injective. If there are infintely many indices j for which both G_{1j} and G_{2j} are nontrivial, then the map is not surjective.

To verify that the map is not injective when we have indices $j \neq k$ with G_{1j} and G_{2k} both nontrivial, let $x \in G_{1j}$, $y \in G_{2k}$ both be nontrivial. Let $\mathbf{x} \in \prod G_{1n}$ and $\mathbf{y} \in \prod G_{2n}$ be the elements with

$$\pi_r(\mathbf{x}) = \begin{cases} x & \text{if } r = j,\\ e & \text{otherwise;}\end{cases} \qquad \text{and} \qquad \pi_s(\mathbf{y}) = \begin{cases} y & \text{if } s = k,\\ e & \text{otherwise.}\end{cases}$$

Now consider the element $\mathbf{x}\mathbf{y}\mathbf{x}^{-1}\mathbf{y}^{-1} \in \varinjlim_m \varprojlim_n G_{mn}$. This is a nontrivial element, but we have:

$$\pi_t\left(c_B(\mathbf{x}\mathbf{y}\mathbf{x}^{-1}\mathbf{y}^{-1})\right) = \left\{\begin{array}{ll} xex^{-1}e^{-1} & \text{if } t = j,\\ eye^{-1}y^{-1} & \text{if } t = k,\\ e & \text{otherwise}\end{array}\right\} = \mathbf{e}.$$

Thus, the map is not injective.

Now assume that there are infinitely many indices $j_1, j_2, \ldots$ for which both G_{1j_k} and G_{2j_k} are nontrivial. Let $x_k \in G_{1j_k}$ and $y_k \in G_{2j_k}$ be nontrivial elements. Let $\mathbf{g}$ be the element of $\varprojlim \varinjlim G_{mn}$ with

$$\pi_{j_k}(\mathbf{g}) = (x_k y_k)^k.$$

Note that if $\mathbf{h}$ lies in the image of c_B, then there exists k such that for every t, $\pi_t(\mathbf{h})$ has at most k factors that alternate between elements of G_{1t} and G_{2t}; that is, the components have bounded "length" as elements of the corresponding free products. But $\mathbf{g}$ does not have this property: its components can be arbitrarily long. That means that $\mathbf{g}$ cannot lie in the image of c_B and thus that c_B is not surjective when there are infinitely many indices j for which both G_{1j} and G_{2j} are nontrivial. □

REMARK. In fact, it is not hard to verify that the conditions give above are "if and only if". Thus, c_B is one-to-one if and only if there is no pair $j \neq k$ of indices such that G_{1j} and G_{2k} are both nontrivial. And c_B is surjective if and only if there are only finite many indices j for which both G_{1j} and G_{2j} are nontrivial. That means that we will get an isomorphism if and only if either all G_{1n} are trivial, or all G_{2n} trivial (in which case both groups are just the product of the other family), or else there is an index n_0 such that G_{in} is nontrivial if and only if $n = n_0$ (in which case both groups in question are isomorphic to $G_{1n_0} * G_{2n_0}$).

3. **Exercise 8.9.10.** Determine whether the abelianization functor $\mathsf{Group} \to \mathsf{Ab}$ respects

   (i) Inverse limits;

   (ii) Products;

   (iii) Equalizers.

In each case where the answer is negative, determine whether it is injectivity, surjectivity, or both properties of the comparison morphism that may fail.

Answer. Note that the product of a countable family of groups can be realized as an inverse limit. Indeed, let $\{H_i\}_{i\in\omega}$ be a countable family. Let $G_n = H_1 \times \cdots H_n$, and for $n \leq m$, let $f_{mn} \colon G_m \to G_n$ be the projection map that eliminates the last $m-n$ components. Then $((G_n), f_{mn})$ is an inversely directed system of groups indexed by ω^{op}, and its inverse limit is isomorphic to $\prod_{i\in\omega} H_i$. Because of this, if we have a counterexample to the functor respecting *countable products*, then this will also be a counterexample for the functor respecting *countable inverse limits*. So I will do part (ii) first.

(ii) The comparison map

$$c\colon \left(\prod_{i\in I} G_i\right)^{\mathrm{ab}} \to \prod_{i\in I} G_i^{\mathrm{ab}},$$

is the map induced by mapping the tuple (g_i) to $(\overline{g_i})$, where $\overline{g_i}$ is the image of g_i in G_i^{ab} under the canonical morphism $G \to G^{\mathrm{ab}}$.

The kernel of this map is the image of $\prod_{i\in I}[G_i, G_i]$ in the quotient.

The comparison map need not be an isomorphism. It is always surjective, since it is the product of surjective maps: that is, given $(x_i) \in \prod G_i^{\mathrm{ab}}$ pick $g_i \in G_i$ such that $\overline{g_i} = x_i$; then $c((g_i)) = (x_i)$.

However, the map need not be one-to-one. The reason is that in general we have

$$\left[\prod_{i\in I} G_i, \prod_{i\in I} G_i\right] \subseteq \prod_{i\in I}[G_i, G_i],$$

but the inclusion may be proper. The issue is similar to the one in Problem 2(ii): the elements of $[G_i, G_i]$ may be arbitrarily long products of pure commutators, and the lengths of the components in $\prod[G_i, G_i]$ may be unbounded, whereas the lengths in elements of $[\prod G_i, \prod G_i]$ are always bounded. Explicitly, if we let G_i be the free group on $2i$ elements $x_1, \ldots, x_{2i}$, then the element of $\prod[G_i, G_i]$ whose nth coordinate is $[x_1, x_2] \cdots [x_{2n-1}, x_{2n}]$ is not an element of $[\prod G_i, \prod G_i]$. □

(i) The inverse limit L of the system $((G_i), f_{ij})$ is the subgroup of $\prod G_i$ consisting of tuples (x_i) for which $f_{ij}(x_i) = x_j$ for all $i \leq j$ in the index system. The inverse limit of $((G_i^{\text{ab}}), f_{ij}^{\text{ab}})$ is the subgroup of $\prod G_i^{\text{ab}}$ consisting of elements $(x_i[G_i, G_i])$ for which $f_{ij}^{\text{ab}}(x_i[G_i, G_i]) = x_j[G_j, G_j]$ whenever $i \leq j$; this means the tuples for which

$$f_{ij}(x_i)[G_j, G_j] = x_j[G_j, G_j] \text{ whenever } i \leq j.$$

The comparison maps $c\colon L^{\text{ab}} \to M$ is the map $c((x_i)[L, L]) = (x_i[G_i, G_i])$, which is well defined since $[L, L] \subseteq \prod[G_i, G_i]$.

As noted above, since we have the product of a countable family in which the comparison map is not injective, this also shows that the comparison map for inverse limits need not be injective.

Although the map need not be surjective either, I confess to not having an example on hand. This is because under relatively mild condition we *will* have a surjective map; see for example I. Barnea and S. Shelah, *The abelianization of inverse limits of groups*, *Israel J. Math.* **227** (2018), no. 1, 455–483; MR3846331, where it is shown that if the index set is countable and the system satisfies the *Mittag-Leffler condition*, then the comparison map is surjective. An inversely directed system satisfies the Mittag-Leffler condition if for every t there is an s, $s \leq t$, such that for every $r \leq s$, $f_{rt}(G_r) = f_{st}(G_s)$. This holds if the structure morphisms are surjective or if the groups are finite. I will investigate to see if I can find an accessible example. □

(iii) For equalizers, the map need not be either surjective nor injective.

Let $f, g\colon G \to K$ be two group homomorphisms, and let $H = \text{Eq}(f, g)$; that is,

$$H = \{x \in G \mid f(x) = g(x)\}.$$

The two maps f, g induce maps $f^{\text{ab}}, g^{\text{ab}}\colon G^{\text{ab}} \to K^{\text{ab}}$, given by

$$f^{\text{ab}}(x[G, G]) = f(x)[K, K] \quad \text{and} \quad g^{\text{ab}}(x[G, G]) = g(x)[K, K].$$

The equalizer of this map is $E = \{x[G, G] \mid f(x)[K, K] = g(x)[K, K]\}$. The comparison map is then $c\colon H^{\text{ab}} \to E$, given by $c(x[H, H]) = x[G, G]$.

For an example where the map is not surjective, consider the two embeddings of $\mathbb{Z}/3\mathbb{Z}$ into S_3, one mapping the generator to (123) and the other to (132). The equalizer of these two maps is trivial; that is, $H = \{e\}$.

The abelianization maps are between $\mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ and are both trivial, so their equalizer is all of $\mathbb{Z}/3\mathbb{Z}$; that is, $E = \mathbb{Z}/3\mathbb{Z}$. So the comparison map is not surjective.

For an example where the map is not injective, consider the following two maps $f, g\colon S_3 \to S_5$: f is the obvious embedding, sending a permutation σ of $\{1, 2, 3\}$ into the permutation of $\{1, 2, 3, 4, 5\}$ that acts on $\{1, 2, 3\}$ like σ and fixes 4 and 5; and g is the embedding that agrees with f on A_3, and sends elements ρ of $S_3 - A_3$ to the permutation that acts like ρ on $\{1, 2, 3\}$ and that exchanges 4 and 5. This embedds S_3 into A_5.

Both S_3^{ab} and S_5^{ab} are cyclic of order 2; the map f^{ab} is the identity of $\mathbf{Z}_2$, and the map g^{ab} is the trivial map. Thus, their equalizer is trivial.

But the equalizer of f and g is A_3, whose abelianization is itself. Thus, the comparison map is the trivial map $A_3 \to \{e\}$. □