Math 666 - Homework 7
SOLUTIONS
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1. **Exercise 9.3:3** (partial)

 (i) Show that a free Ω-algebra is free on a unique set of generators. That is, if (F, u) is a free Ω-algebra, then the image in $|F|$ of the set map u is determined by the Ω-algebra structure of F.

 Proof. Suppose that (F, u) is a free Ω-algebra on X, and that $u'\colon X \to |F|$ is a set map such that (F, u') is also a free Ω-algebra on X. We wish to show that $u(X) = u'(X)$ as sets.

 We know that F is an Ω-term algebra on X using u, and also using u'. In particular, it satisfies the definition of "Ω-term algebra" given in Definition 9.3.1, relative to both u and u'. Explicitly:

 (1) The map u and all the maps s_F are one-to-one; they have pairwise disjoint images; and F is generated as an Ω-algebra by $u(X)$.
 (2) The map u' is also one-to-one; it has pairwise disjoint image with all s_F; and F is generated as an Ω-algebra by $u(X)$.

 Let $S^{(\alpha)}$ be the sets defined as in Section 9.2 with $S^{(0)} = u(X)$; and let $T^{(\alpha)}$ be the corresponding sets with $T^{(0)} = u'(X)$. We know that, taking γ to be the least infinite regular cardinal that is strictly larger than the arities of all $s \in |\Omega|$, we have $|F| = S^{(\gamma)} = T^{(\gamma)}$.

 Let $x \in X$. Then $u'(x)$ lies is $|F| = \cup_{\beta<\gamma} S^{(\beta)}$, so let α be the least ordinal such that $u'(x) \in S^{\alpha}$. Note that α cannot be a nonzero limit ordinal, because if δ is a nonzero limit ordinal, then $S^{(\delta)} = \cup_{\beta<\delta} S^{(\beta)}$, so there exists $\beta < \delta$ such that $u'(x) \in S^{(\beta)}$. Thus, if $\alpha \neq 0$, then $\alpha = \beta + 1$ for some ordinal β, hence

$$u'(x) \in S^{(\beta+1)} = S^{(\beta)} \cup \{s_F((y_i)) \mid s \in |\Omega|, y_i \in S^{(\beta)} \text{ for all } i \in \mathrm{ari}(s)\}.$$

 By choice of β, it must be the case that $u'(x) = s_F((y_i))$ for some suitable $s \in \Omega$ and tuple (y_i). But because (F, u') is an Ω-term algebra on X, we know that the images of u' and s_F are disjoint, so we cannot have $u'(x) = s_F((y_i))$. Thus, we have a contradiction.

 The contradiction arose from assuming that $\alpha \neq 0$. Thus, $\alpha = 0$, so $u'(x) \in S^{(0)} = u(X)$. This proves that $u'(X) \subseteq u(X)$. A symmetric argument shows that $u(X) \subseteq u'(X)$, proving that $u(X) = u'(X)$, as desired. $\Box$

 (ii) Is the analogous result true for free groups? Free monoids? Free rings?

 Answer. The analogous result is not true for free groups. For example, let $X = \{x\}$, and let $F = \mathbb{Z}$. Let $u, u'\colon X \to |F|$ be given by $u(x) = 1$ and $u'(x) = -1$. Then (F, u) and (F, u') are both free on X, but $u(X) \neq u(X')$.

 Similarly, the result is not true for free rings. If $R = \mathbb{Z}[y]$ is the polynomial ring in y over the integers, and $u\colon X \to |R|$ is given by $u(x) = y$, then (R, u) is a free ring on X. If we define $u'\colon X \to |R|$ by $u'(x) = -y$, then (R, u') is also a free ring on X (the unique morphism $F\colon R \to S$ induced by $f\colon X \to |S|$ with $Fu' = f$ is the unique morphism $R \to S$ such that $y \mapsto -f(x)$). Here again we have $u(X) \neq u(X')$.

 The result is *true* for monoids, however. Recall that the free monoid on a set X can be constructed as the collection of all reduced (monoid) words on X; that is, almost null sequences $(x_0, x_1, \ldots)$ of elements of $X \cup \{e\}$, with the property that if $x_i = e$ for some i, then $x_j = e$ for all $j \geq i$. The expression is unique, and the operation is concatenation (as usual).

 If we define the length of such an element to be

$$\mathrm{length}(x_0, x_1, \ldots) = \min\{k \mid x_k = e\}.$$

Note that here we have that if w and w' are words, then $\text{length}(ww') = \text{length}(w) + \text{length}(w')$, because in a free monoid no identity allows us to "cancel" letters in the free words. In particular, no element of length 1 can be obtained as a product of two nontrivial words.

Let X be a set, let M be the free monoid described above, and let $u\colon X \to |M|$ be the embedding sending $x \in X$ to $(x, e, e, e, \ldots)$, which makes (M, u) into a free monoid in X. If $u'\colon X \to |M|$ is a map such that (M, u') is also free monoid in X, then $u(X) \subseteq u'(X)$, because no element of $u(X)$ can be obtained as a nontrivial word in the elements of $u'(X)$ that has length greater than 1. And if $u'(y)$ has length greater than 1 for any $y \in X$, then y can be obtained in terms of the elements of $u(X)$, which would mean that (M, u') is not free on X (since there is a monoid relation between $u'(y)$ and $u'(X) - \{u(y)\}$). Thus, every element of $u'(X)$ has length 1, and thus $u'(X) \subseteq u(X)$, giving equality. □

2. **Exercise 9.3:4**

(i) Show that every subalgebra of a free Ω-algebra F is free.

Proof. Let X be a set and $u\colon X \to |F|$ be a set map such that (F, u) is a free Ω-algebra on X; note that this means that F is an Ω-term algebra on X. Let γ be an infinite regular cardinal that is strictly larger than the arities of all $s \in |\Omega$, and let $S^{(\alpha)}$, $\alpha \leq \gamma$, be the sets defined as in Section 9.2 with $S^{(0)} = u(X)$. Let A be a subalgebra of F.

We recursively define set $Y^{(\alpha)}$ and $T^{(\alpha)}$ as follows:

(1) $Y^{(0)} = A \cap S^{(0)}$; $T^{(0)} = Y^{(0)}$.

(2) If $Y^{(\alpha)}$ and $T^{(\alpha)}$ have been defined, we define

$$T^{(\alpha+1)} = T^{(\alpha)} \cup Y^{(\alpha)} \cup \{s_F((x_i)) \mid s \in |\Omega|, x_i \in T^{(\alpha)} \cup Y^{(\alpha)} \text{ for all } i \in \text{ari}(s)\},$$
$$Y^{(\alpha+1)} = \left(A \cap S^{(\alpha+1)}\right) - T^{(\alpha+1)}.$$

(3) If δ is a nonzero limit ordinal, then let $T^{(\delta)} = \cup_{\beta<\delta} T^{(\beta)}$ and $Y^{(\delta)} = \cup_{\beta<\delta} Y^{(\beta)}$.

Note that $T^{(\gamma)} = A \cap S^{(\gamma)} = A$. Let $Y = Y^{(\gamma)}$, and let $i\colon Y \hookrightarrow |A|$ be the embedding of Y into A.

I claim that (A, i) is a free Ω-Algebra. To prove this, we can equivalently prove that (A, i) is an Ω-term algebra on Y.

Because i is an embedding, it is one-to-one. And because A is a subalgebra of F, we also know that its operations s_A (which are the restrictions of the operations s_F) are one-to-one. The images of the s_A are pairwise disjoint, because the image of s_A is contained in the image of s_F. And the construction of Y ensures that if $a \in Y$, then a is not equal to s_A of any tuple of elements of A (again invoking the fact that the maps s_F are one-to-one with disjoint images). So, the image of u is disjoint from the image of any s_A.

Finally, we must show that A is generated as an Ω-algebra by Y. Let $B = \langle Y \rangle_\Omega$. We show that $T^{(\alpha)} \subseteq B$ for all $\alpha \leq \gamma$. Indeed, $T^{(0)} = Y^{(0)} \subseteq Y \subseteq B$.

Assuming $T^{(\alpha)} \subseteq B$, let $a \in T^{(\alpha+1)}$. If $a \in T^{(\alpha)}$, theN $a \in B$ by assumption. If $a \in Y^{(\alpha)}$, then $a \in Y \subseteq B$ by construction. And if $a = s_F((x_i))$ with each x_i in either $T^{(\alpha)}$ (hence in B) or in $Y^{(\alpha)}$ (hence also in B), then $a \in B$ because B is a subalgebra of F. Thus, $T^{(\alpha+1)} \subseteq B$. And if δ is a nonzero limit ordinal and $T^{(\beta)} \subseteq B$ for all $\beta < \delta$, then trivially $T^{(\delta)} \subseteq B$. Thus, inductively, $A = T^{(\gamma)} \subseteq B \subseteq A$, proving that A is generated by Y.

Thus, A is an Ω-term algebra on Y, and hence is free, as required. □

(ii) Is the analogous statement true for monoids?

Answer. The analogous statement is not true for monoids. Let M be the free monoid on x, written multiplicatively, and consider the submonoid $\{x^2, x^3\}$. It is not free in a single generator, for that generator would have to be x^2, and $\langle x^2 \rangle$ does not contain x^3; and any subset that contains more than one nontrivial element cannot freely generate $\langle x^2, x^3 \rangle$ because there are nontrivial monoid relations between any two distinct nontrivial elements of M. So $\langle x^2, x^3 \rangle$ is not a free monoid. $\square$

3. **Exercise 9.3:6** (partial)

   (i) Show that every functor $A\colon \mathsf{Set} \to \mathsf{Set}$ sends surjective maps to surjective maps and injective maps with nonempty domain to injective maps.

   Proof. Recall that in Set, a function is surjective if and only if it has a right inverse (recall that we always assume the Axiom of Choice). Thus, if $f\colon X \to Y$ is surjective, and g is a right inverse of f, then

$$\mathrm{id}_{A(Y)} = A(\mathrm{id}_Y) = A(fg) = A(f)A(g),$$

   hence $A(f)$ has a right inverse, and thus is surjective.

   Likewise, in Set, a function $f\colon X \to Y$ with nonempty domain is injective if and only if it has a left inverse. Given such an f, if h is a left inverse to f, then

$$\mathrm{id}_{A(X)} = A(\mathrm{id}_X) = A(hf) = A(h)A(f),$$

   so $A(f)$ has a left inverse and therefore is injective. $\square$

   (ii) Show that the second clause of (i) becomes false if the qualification about nonempty domains is omitted.

   Proof. Let A be the functor that sends every nonempty set to $Y = \{\varnothing\}$; sends $\varnothing$ to $X = \{\varnothing, \{\varnothing\}\}$; sends every arrow between nonempty sets to id_Y; sends the unique element of $\mathsf{Set}(\varnothing, \varnothing)$ to id_X; and for any nonempty set S, sends the unique element of $\mathsf{Set}(\varnothing, S)$ to the unique function $f\colon X \to Y$. Note that if S is nonempty, then $\mathsf{Set}(S, \varnothing)$ is empty, so this describes the values of A at every object and every arrow. Then A is a functor, but it sends every injection with empty domain to the non-injective function f. $\square$

   (iii) Show that if A has the form UF, where $U\colon \mathcal{C} \to \mathsf{Set}$ is some functor from a category $\mathcal{C}$, and F is a left adjoint of U, then A carries maps with empty domain to injective maps.

   Proof. Note that because $\varnothing$ is initial in Set and F is a left adjoint, it follows that $F(\varnothing)$ is an initial object of $\mathcal{C}$. We will use the following Lemma:

   Lemma. Let $\mathcal{D}$ be a category, $I, X \in \mathrm{Ob}(\mathcal{D})$. If I is initial and $\mathcal{D}(X, I) \neq \varnothing$, then the unique arrow $u\colon I \to X$ is left invertible, and in particular is a monomorphism.

   Proof. Let $g\colon X \to I$. Then $gu \in \mathcal{D}(I, I) = \{\mathrm{id}_I\}$, so g is a left inverse for u, as desired. $\square$

   Let $f\colon \varnothing \to X$. We wish to show that $A(f)$ is one-to-one.

   If $\mathsf{Set}(X, A(\varnothing))$ is empty, then we must have $A(\varnothing) = \varnothing$, and there is nothing to do. If $\mathsf{Set}(X, A(\varnothing))$ is nonempty, then since

$$\mathsf{Set}(X, A(\varnothing)) = \mathsf{Set}(X, UF(\varnothing)) \cong \mathcal{D}(F(X), F(\varnothing)),$$

   it follows that the unique arrow $F(f)\colon F(\varnothing) \to F(X)$ is a monomorphism, by the Lemma. Since U is a right adjoint, it carries monomorphisms to monomorphisms (as it respects pullbacks), so $A(f) = UF(f)$ is a monomorphism in Set; that is, $A(f)$ is injective, as desired. $\square$