Comments, suggestions, corrections, and further references are most welcomed!

CAPABLE GROUPS OF PRIME EXPONENT AND CLASS TWO

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Abstract. A group is called capable if it is a central factor group. We consider the capability of finite groups of class two and exponent p, p an odd prime. We restate the problem of capability as a problem about linear transformations, which may be checked explicitly for any specific instance of the problem. We use this restatement to derive some known results, and prove new ones. Among them, we reduce the general problem to an oft-considered special case, and prove that a 3-generated group of class 2 and exponent p is either cyclic or capable.

1. Introduction

In his landmark paper [10] on the classification of finite p-groups, P. Hall remarked:

> The question of what conditions a group G must fulfill in order that it may be the central quotient group of another group H, $G \cong H/Z(H)$, is an interesting one. But while it is easy to write down a number of necessary conditions, it is not so easy to be sure that they are sufficient.

Following M. Hall and Senior [9], we make the following definition:

Definition 1.1. A group G is said to be *capable* if and only if there exists a group H such that $G \cong H/Z(H)$.

Capability of groups was first studied by R. Baer in [2], where, as a corollary of deeper investigations, he characterised the capable groups that are a direct sum of cyclic groups. Capability of groups has received renewed attention in recent years, thanks to results of Beyl, Felgner, and Schmid [3] characterising the capability of a group in terms of its epicenter; and more recently to work of Graham Ellis [5] that describes the epicenter in terms of the nonabelian tensor square of the group.

We will consider here the special case of nilpotent groups of class two and exponent an odd prime p. This case was studied in [13], and also addressed elsewhere (e.g., Prop. 9 in [5]). As noted in the final paragraphs of [1], currently available techniques seem insufficient for a characterization of the capable finite p-groups of class 2, but a characterization of the capable finite groups of class 2 and exponent p seems like a more modest and possibly attainable goal. The present work is a contribution towards achieving that goal.

Throughout the paper, p will be an odd prime. All groups will be written multiplicatively, and the identity element will be denoted by e; if there is danger of ambiguity or confusion, we will use e_G to denote the identity of the group G. The center of G is denoted by $Z(G)$. Recall that if G is a group, and $x, y \in G$,

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we let the commutator of x and y be $[x,y]=x^{-1}y^{-1}xy$. We write commutators left-normed, so that $[x,y,z]=[[x,y],z]$. Given subsets A and B of G we define $[A,B]$ to be the subgroup of G generated by all elements of the form $[a,b]$ with $a\in A$, $b\in B$. The terms of the lower central series of G are defined recursively by letting $G_1=G$, and $G_{n+1}=[G_n,G]$. A group is *nilpotent of class at most* k if and only if $G_{k+1}=\{e\}$, if and only if $G_k\subset Z(G)$. We usually drop the "at most" clause, it being understood. The class of all nilpotent groups of class at most k is denoted by $\mathfrak{N}_k$.

The following commutator identities are well known, and may be verified by direct calculation:

Proposition 1.2. *Let G be any group. Then for all $x,y,z\in G$,*

(a) $[xy,z]=[x,z][x,z,y][y,z]$.
(b) $[x,yz]=[x,z][z,[y,x]][x,y]$.
(c) $[x,y,z][y,z,x][z,x,y]\equiv e \pmod{G_4}$.
(d) $[x^r,y^s]\equiv[x,y]^{rs}[x,y,x]^{s\binom{r}{2}}[x,y,y]^{r\binom{s}{2}} \pmod{G_4}$.
(e) $[y^r,x^s]\equiv[x,y]^{-rs}[x,y,x]^{-r\binom{s}{2}}[x,y,y]^{-s\binom{r}{2}} \pmod{G_4}$.

Here, $\binom{n}{2}=\frac{n(n-1)}{2}$ *for all integers n.*

As in [17], our main tool will be the nilpotent product of groups, specifically the 2-nilpotent and 3-nilpotent products. We restrict Golovin's original definition [7] to the situation we will consider:

Definition 1.3. Let $A_1,\ldots,A_n$ be cyclic groups. The k-nilpotent product of $A_1,\ldots,A_n$, denoted by $A_1\amalg^{\mathfrak{N}_k}\cdots\amalg^{\mathfrak{N}_k}A_n$, is defined to be the group $G=F/F_{k+1}$, where F is the free product of the A_i, $F=A_1*\cdots*A_n$, and F_{k+1} is the $(k+1)$-st term of the lower central series of F.

Note that if G is the k-nilpotent product of the A_i, then $G\in\mathfrak{N}_k$, and G/G_k is the $(k-1)$-nilpotent product of the A_i.

When we have the k-nilpotent product of cyclic p-groups, with $p\geq k$, we may write each element uniquely as a product of basic commutators of weight at most k on the generators, as shown in Theorem 3 in [19]; see § 12.3 of [8] for the definition of basic commutators which we will use. In our applications, where each cyclic group is of order p, the order of each basic commutator is likewise equal to p.

From the definition, it is clear that the k-nilpotent product is the coproduct in the variety $\mathfrak{N}_k$, so it will have the usual universal property.

Finally, when we say that a group is *k-generated* we mean that it can be generated by k elements, but may in fact need less. If we want to say that it can be generated by k elements, but not by m elements for some $m<k$, we will say that it is *minimally k-generated*, or *minimally generated by k elements.*

2. Some group theory

Using the 2- and 3-nilpotent product of cyclic groups, we can produce a natural "candidate for witness" to the capability of a given finite nilpotent groups of class 2 and exponent p, which we can then investigate directly. This will be the theme throughout this article. We begin with an easy observation:

Lemma 2.1 (cf. Lemma 2.1 in [14])**.** *Let G be a capable group, generated by $g_1,\ldots,g_n$, $n>1$. Then there exists a group H, and elements $h_1,\ldots,h_n\in H$*

such that H is generated by $h_1,\ldots,h_n$, and $H/Z(H)\cong G$, where the isomorphism is induced by the map sending h_i to the corresponding g_i.

Proof. Let K be any group such that $K/Z(K)\cong G$. Let $h_i\in K$ be any element mapping to g_i. Let $H=\langle h_1,\ldots,h_n\rangle$. We then have $HZ(K)=K$, so it follows that $Z(H)=Z(K)\cap H$ and hence $H/Z(H)\cong H/(Z(K)\cap H)\cong K/Z(K)\cong G$. □

Let G be a finite nilpotent group of class at most 2 and exponent p, minimally generated by $g_1,\ldots,g_n$, $n>1$. Then G is a quotient of the 2-nilpotent product of n cyclic p-groups,

$$G\cong \left(\langle x_1\rangle \amalg^{\mathfrak{N}_2}\cdots\amalg^{\mathfrak{N}_2}\langle x_n\rangle\right)/N$$

where N is the kernel of the map induced by mapping x_i to g_i (using the universal property of the coproduct), and is contained in the commutator subgroup of the 2-nilpotent product. Let $y_1,\ldots,y_n$ generate infinite cyclic groups, let $\mathfrak{K}$ be the 3-nilpotent product of the cyclic groups,

$$\mathfrak{K}=\langle y_1\rangle \amalg^{\mathfrak{N}_3}\cdots\amalg^{\mathfrak{N}_3}\langle y_n\rangle,$$

and let $\mathcal{K}=\mathfrak{K}/[\mathfrak{K},\mathfrak{K}]^p$. By abuse of notation, we denote the images of the y_i in $\mathcal{K}$ by y_i as well.

Note that the commutator subgroup of $\langle x_1\rangle\amalg^{\mathfrak{N}_2}\cdots\amalg^{\mathfrak{N}_2}\langle x_r\rangle$ is isomorphic to the subgroup of $[\mathcal{K},\mathcal{K}]$ generated by the commutators of the form $[y_j,y_i]$, $1\leq i<j\leq n$, by mapping $[x_j,x_i]$ to $[y_j,y_i]$. Let $\mathcal{N}$ be the subgroup of $\mathcal{K}$ that corresponds to N under this identification.

Let $\mathcal{M}=[\mathcal{N},\mathcal{K}]$. Finally, let $K=\mathcal{K}/\mathcal{M}$. Again, we also denote the images of the y_i in K by y_i.

Theorem 2.2. *Let G, $\mathfrak{K}$, $\mathcal{K}$, and K be as in the previous three paragraphs. Then G is capable if and only if $G\cong K/Z(K)$.*

Proof. First, note that $K/\langle K^p,\mathcal{N}\mathcal{M},K_3\rangle$ is isomorphic to G. Since each of K^p, $\mathcal{N}\mathcal{M}$, and K_3 are central in K, it is always be the case that $K/Z(K)$ is a quotient of G.

Assume G is capable, and let $H\in\mathfrak{N}_3$ be a group with $H/Z(H)\cong G$. We may assume H is generated by elements $h_1,\ldots,h_n$ mapping to $g_1,\ldots,g_n$, respectively. Then $H^p\subset Z(H)$. We claim this implies that $H_2^p=\{e\}$.

Indeed, first note that $H_3^p=\{e\}$. For if $c\in H_2$, $h\in H$, then using Proposition 1.2 we have that $[c,h]^p=[c,h^p]=e$. Since H_3 is abelian, and generated by all elements $[c,h]$ as above, this shows that H_3 is of exponent p. Now let $a,b\in H$; we have (once again using Proposition 1.2):

$$\begin{aligned}[a,b]^p &= [a^p,b][a,b,a]^{-\binom{p}{2}}[a,b,b]^{-p\binom{1}{2}}\\ &= [a^p,b][a,b,a]^{-p\left(\frac{p-1}{2}\right)}\\ &= e.\end{aligned}$$

Since H_2 is abelian and generated by all such $[a,b]$, which are of exponent p, we conclude that $H_2^p=\{e\}$, as claimed.

The natural surjection $\mathfrak{K} \to G$, mapping y_i to g_i therefore factors through H and $\mathcal{K}$, so we have the following exact diagram:

$$\begin{array}{ccccccccc} & & & & \mathcal{K} & & & & \\ & & & & \downarrow & & & & \\ 1 & \to & Z(H) & \to & H & \to & G & \to & 1. \\ & & & & \downarrow & & & & \\ & & & & 1 & & & & \end{array}$$

If $y \in \mathcal{N}$, then its image in H must map to the trivial element in G, so its image is in the center of H. It follows that $\mathcal{M} = [\mathcal{N}, \mathcal{K}]$ is in the kernel of the map from $\mathcal{K}$ to H, so the map factors through K giving a diagram with exact row and column:

$$\begin{array}{ccccccccc} & & & & K & & & & \\ & & & & \downarrow & & & & \\ 1 & \to & Z(H) & \to & H & \to & G & \to & 1. \\ & & & & \downarrow & & & & \\ & & & & 1 & & & & \end{array}$$

Since the center of K maps into the center of H, it follows that $K/Z(K)$ has G as a quotient. Since we already know that $K/Z(K)$ is a quotient of G, it follows that $K/Z(K) \cong G$, as desired. $\square$

Next, note that $Z(K/K^p) = Z(K)/K^p$, so we have:

Theorem 2.3 (cf. Theorem 8.6 in [17]). *Let G be a nilpotent group of class at most 2 and exponent an odd prime p, minimally generated by $g_1, \ldots, g_n$, with $n > 1$. Let $A_1, \ldots, A_n$ be cyclic groups of order p, generated by $x_1, \ldots, x_n$, respectively, and let $\mathcal{G}$ be the 2-nilpotent product of the A_i, $\mathcal{G} = A_1 \amalg^{\mathfrak{N}_2} \cdots \amalg^{\mathfrak{N}_2} A_n$. Let $N \triangleleft \mathcal{G}$ be the kernel of the homomorphism $\mathcal{G} \to G$ induced by mapping x_i to g_i. Let K be the 3-nilpotent product of the A_i, $K = A_1 \amalg^{\mathfrak{N}_3} \cdots \amalg^{\mathfrak{N}_3} A_n$, and identify N with the corresponding subgroup of K_2. Then G is capable if and only if*

$$G \cong (K/[N,K])/Z(K/[N,K]).$$

What are the elements of $Z(K/[N,K])$? An element $k[N,K]$ will be central in $K/[N,K]$ if and only if $[k,K] \subseteq [N,K]$. This includes the center of K (which is equal to K_3, see Theorem 5.1 in [17]), as well as N. Since $K/\langle N, K_3\rangle \cong G$, it follows that G is capable if and only if

$$Z(K/[N,K]) = \langle K_3, N\rangle/[N,K].$$

The left hand side always contains the right hand side. However, the problem is that there could be more elements in $Z(K/[N,K])$. Let $k \in K$, and write k in normal form:

$$\begin{aligned} k \quad = \quad & x_1^{a_1} \cdots x_n^{a_n} \prod_{1 \leq i < j \leq n} [x_j, x_i]^{a_{ji}} \prod_{1 \leq i < j \leq n} [x_j, x_i, x_i]^{a_{jii}} [x_j, x_i, x_j]^{a_{jij}} \\ & \prod_{1 \leq i < j < k \leq n} [x_j, x_i, x_k]^{a_{jik}} [x_k, x_i, x_j]^{a_{kij}}. \end{aligned}$$

Suppose that there exists $i \in \{1, \ldots, n\}$ with $a_i \neq 0$. If $i < n$, consider $[x_n, k]$. Using the formulas in Proposition 1.2, it is easy to verify that the normal form for $[x_n, k]$ will have a nonzero exponent for $[x_n, x_i]$; since $N \subset K_2$, $[N,K] \subset K_3$, so

$[x_n, k] \notin [N, K]$. If $i = n$, then consider $[k, x_{n-1}]$, and once again we find that this element has nonzero exponent in $[x_n, x_{n-1}]$, so $[k, x_{n-1}] \notin [N, K]$. That is, we have shown that if $k \in K$ satisfies $[k, K] \subseteq [N, K]$, then $k \in K_2$. With this observation, we obtain the following:

Corollary 2.4. *Let G, K, and N be as in Theorem 2.3. Then G is capable if and only if the following condition holds: For every element $k \in K$ of the form*

$$k = \prod_{1 \leq i < j \leq n} [x_j, x_i]^{a_{ji}} \tag{2.5}$$

with $0 \leq a_{ji} < p$, if $[k, x_i] \in [N, K]$ for $i = 1, \ldots, n$, then $k \in N$.

Proof. We know that G is capable if and only if

$$\{k \in K_2 \mid [k, K] \subseteq [N, K]\} = \langle N, K_3 \rangle.$$

If $k \in K_2$, then $[k, K] = \langle [k, x_1], \ldots, [k, x_n] \rangle$ by Proposition 1.2, so if G is capable then all k such that $[k, x_i] \in [N, K]$, $i = 1, \ldots, n$, must satisfy $k \in \langle N, K_3 \rangle$. Conversely, assume the condition holds and let $k \in K_2$ be such that $[k, K] \subset [N, K]$; we want to show that $k \in \langle N, K_3 \rangle$. Multiplying by suitable elements of K_3, we may assume that k is of the form (2.5). Since $[k, K] \subset [N, K]$, it follows that $[k, x_i] \in [N, K]$ for each i; so $k \in N$, by the condition given. This proves the corollary. □

3. Some linear algebra

The main advantage of Corollary 2.4, beyond giving a very precise and explicit condition to check, is that the condition can also be recast as a statement about vector spaces and linear transformations. At that point, we have a whole array of tools that can be brought to bear upon the problem. For example, Theorem 4.5 below uses algebraic geometry to settle several cases.

We will now translate the condition into the promised statement about vector spaces and linear transformations. The key observation is the simple fact that if $H \in \mathfrak{N}_3$, then for each $h \in H$ the map $\varphi_h \colon H_2 \to H_3$ given by $\varphi_h(c) = [c, h]$ is an abelian group homomorphism; if H_2 is of exponent p, then the map becomes a linear transformation of spaces over $\mathbb{F}_p$, the Galois field of p elements.

Let $n > 1$ be a fixed integer. Let $K = \langle x_1 \rangle \amalg^{\mathfrak{N}_3} \cdots \amalg^{\mathfrak{N}_3} \langle x_n \rangle$, where x_i is of order p, p an odd prime. Then K_2 is a vector space over $\mathbb{F}_p$, with basis given by all elements of the form $[x_j, x_i]$ and $[x_s, x_r, x_t]$, where $1 \leq i < j \leq n$, $1 \leq r < s \leq n$, and $1 \leq r \leq t \leq n$.

Let V be the vector space over $\mathbb{F}_p$ of dimension $\binom{n}{2}$, with basis

$$\{v_{ji} \mid 1 \leq i < j \leq n\}. \tag{3.1}$$

Let W be a vector space over $\mathbb{F}_p$ of dimension $2\left(\binom{n}{2} + \binom{n}{3}\right)$ with basis

$$\{w_{jik} \mid 1 \leq i < j \leq n,\ 1 \leq i \leq k \leq n\}. \tag{3.2}$$

We will refer to these bases as the *"standard bases"* for V and W, and we talk about *V and W corresponding to n* to mean the case where the indices of the coordinates range from 1 to n (subject to the conditions listed above). We can clearly identify K_3 with W, and K_2 with $V \oplus W$. To restate Corollary 2.4 in terms of V and W, we just need to describe what happens when we take the commutator of $[x_j, x_i]$ with x_r. We have two cases: if $i \leq r$, then $[x_j, x_i, x_r]$ is already in normal form, and there is nothing to do. If $i > r$, then the commutator $[x_j, x_i, x_r]$ must be rewritten in normal form. We use Proposition 1.2(c): since $[x_j, x_i, x_r][x_i, x_r, x_j][x_r, x_j, x_i] = e$, we have:

$$\begin{aligned}[x_j, x_i, x_r] &= [x_r, x_j, x_i]^{-1}[x_i, x_r, x_j]^{-1}\\ &= [[x_r, x_j]^{-1}, x_i][x_i, x_r, x_j]^{-1}\\ &= [x_j, x_r, x_i][x_i, x_r, x_j]^{-1}.\end{aligned}$$

So, for each $r = 1, \ldots, n$ we define a linear transformation $\varphi_r\colon V \to W$ to be given by:

$$\varphi_r(v_{ji}) = \begin{cases} w_{jir} & \text{if } r \geq i,\\ w_{jri} - w_{irj} & \text{if } r < i.\end{cases} \tag{3.3}$$

That is, φ_r codifies the map $c \mapsto [c, x_r]$. It is easy to verify that each φ_r is injective.

Let G be a non-cyclic group of class 2 and exponent p. Consider the situation in Corollary 2.4, and let X be the subspace of V corresponding to the subgroup N determined by G (we will refer to such X as *the subspace of V corresponding to G* or *determined by G*). We define Y_X to be the subspace of W spanned by the images of X; that is:

$$Y_X = \Big\langle \varphi_1(X), \ldots, \varphi_n(X) \Big\rangle. \tag{3.4}$$

Thus, Y_X corresponds to the subgroup $[N, K]$. Finally, let Z_X be the subspace of V given by:

$$Z_X = \bigcap_{i=1}^{n} \varphi_i^{-1}(Y_X). \tag{3.5}$$

Clearly, $X \subseteq Z_X$, and Z_X corresponds to the subgroup of all $k \in K_2$, written as in (2.5), such that $[k, x_i] \in [N, K]$ for $i = 1, \ldots, n$. We know that G is capable if and only if the subgroup of such k equals N, so we have:

Theorem 3.6. *Let G be a finite nilpotent group of class 2 and exponent p, minimally generated by $g_1, \ldots, g_n$, $n > 1$. Let V and W be the vector spaces over $\mathbb{F}_p$ defined in* (3.1) *and* (3.2)*, let $\varphi_1, \ldots, \varphi_n$ be as in* (3.3)*, and let X be the subspace of V determined by G (that is, corresponding to the kernel of the natural map $\langle x_1 \rangle \amalg^{\mathfrak{N}_2} \cdots \amalg^{\mathfrak{N}_2} \langle x_n \rangle \longrightarrow G$ given by $x_i \mapsto g_i$, where each x_i is of order p). Let Y_X and Z_X be defined by* (3.4) *and* (3.5)*. Then G is capable if and only if $X = Z_X$.*

Remark 3.7. We will feel free to drop the subscript X from Y_X and Z_X when it is clear from context; also, to avoid multiple subindices, if we have subspaces $X_1, \ldots, X_r$, we will denote Y_{X_i} and Z_{X_i} simply by Y_i and Z_i, respectively.

In discussing the situation in the linear algebra setting, we will refer to *"the (j, i) coordinate"* of vectors of V, or to *"the (j, i, k) coordinate"* of vectors of W. We refer, of course, to the coefficient of v_{ji} (resp. of w_{jik}) when the vector is expressed in terms of the standard bases given.

The following easy observations will be repeated several times in what follows, so we state them here as a lemma:

Lemma 3.8. *Let $n > 1$, and let V, W, and $\varphi_1, \ldots, \varphi_n$ be as in* (3.1)–(3.3)*.*

(a) *$\varphi_r(\mathbf{v})$ has nonzero (j, i, i) coordinate, $1 \leq i < j \leq n$, if and only if $\mathbf{v}$ has nonzero (j, i) coordinate, and $r = i$.*

(b) $\varphi_r(\mathbf{v})$ *has nonzero* (j,i,j) *coordinate,* $1 \leq i < j \leq n$*, if and only if* $\mathbf{v}$ *has nonzero* (j,i) *coordinate, and* $r = j$*.*

(c) $\varphi_r(\mathbf{v})$ *has nonzero* (k,i,j) *coordinate,* $1 \leq i < j \leq n$*,* $i < k \leq n$*, if and only if either* $r = j$ *and* $\mathbf{v}$ *has nonzero* (k,i) *coordinate, or else* $r = i$*, and either* $k > j$ *and* $\mathbf{v}$ *has nonzero* (k,j) *coordinate or* $j > k$ *and* $\mathbf{v}$ *has nonzero* (j,k) *coordinate. In the case where* $r = i$*, the* (k,i,j) *coordinate of* $\varphi_i(\mathbf{v})$ *is equal to minus the* (j,i,k) *coordinate.*

From these, we obtain:

Lemma 3.9. *Let* $n > 1$*, and let* V*,* W*, and* $\varphi_1,\ldots,\varphi_n\colon V \to W$ *be as in* (3.1)–(3.3)*. Let* X *be a subspace of* V*, and let* i *and* j *be fixed integers,* $1 \leq i < j \leq n$*. If all vectors in* X *have zero* (j,i) *coordinate, then all vectors in* Z_X *have zero* (j,i) *coordinate.*

Proof. Let $\mathbf{v} \in V$ be a vector with nonzero (j,i) coordinate. It suffices to show that $\mathbf{v} \notin \varphi_i^{-1}(Y_X)$. Indeed, since all vectors in X have zero (j,i) coordinate, it follows that all vectors in Y_X have zero (j,i,i) coordinate (applying part (a) of the lemma above). Since $\varphi_i(\mathbf{v})$ has nonzero (j,i,i) coordinate, it cannot lie in Y_X, and therefore $\mathbf{v} \notin \varphi_i^{-1}(Y_X)$. □

The extreme cases are easy to handle, since the φ_i are injective, and their images span W:

Lemma 3.10. *Let* $n > 1$*, and let* V*,* W*, and* $\varphi_1,\ldots,\varphi_n$ *be given as in* (3.1)–(3.3)*. If* $X = \{\mathbf{0}\}$*, then* $Z_X = X$*; if* $X = V$*, then* $Z_X = X$*.*

Theorem 3.11. *Let* $n > 1$*,* p *an odd prime. If* G *is a direct sum of* n *cyclic groups of order* p*, then* G *is capable. If* G *is the* 2*-nilpotent product of* n *cyclic groups of order* p*, then* G *is capable.*

Proof. The direct sum corresponds to the case $X = V$, while the 2-nilpotent product to $X = \{\mathbf{0}\}$. Of course, these two conclusions are simply special cases of Baer's theorem for abelian groups and its generalization to the k-nilpotent product of cyclic p-groups, $k < p$ (Theorem 6.4 in [17]). □

4. Some geometry

As we mentioned before, recasting our problem into a linear algebra setting allows us to bring other tools to bear on the problem. Specifically, in this section we give a geometric argument, due to David McKinnon (personal communication) which implies that if a group of exponent p and class two is "nonabelian enough", then it will necessarily be capable. Although we have only been able to apply the argument as-is to a limited number of cases, there is some hope that similar arguments may hold more generally. It also seems to present a striking example of why the linear algebra setting may be more amenable to a final solution than the group-theoretic setting by itself. I have taken much of the explanations that follow (sometimes verbatim) from personal communications with Prof. McKinnon, and I am very grateful for his permission to include them here.

For now we suspend the assumption that V and W are the vector spaces defined in (3.1) and (3.2). We will explicitly state when we reinstate that assumption.

Definition 4.1. Let V be a vector space of dimension n over a field F, and let k be an integer, $0 \leq k \leq n$. The *Grassmannian* $Gr(k,V)$ (of $Gr(k,n)$ if the field is understood from context) is defined to be the set of all k-dimensional subspaces of V.

For a description of the Grassmannian as a subset of $\mathbb{P}(\bigwedge^k(V))$ and the Plücker embedding, we direct the reader to Lecture 6 in [11]; the Plücker embedding endows $Gr(k,V)$ with the structure of an algebraic variety, so we may do algebraic geometry on it. Unfortunately, the main property we will exploit from this fact, state in Proposition 4.2 below, seems to be in that rather awkward situation of being either well-known or easy to figure out to those who work in algebraic geometry, yet requiring some technical machinery to prove and not being available anywhere explicitly for reference. I provide only a rough sketch of how one might go about proving it.

Proposition 4.2. *Let $\mathfrak{V}$ be any variety (over the field F), and let $f\colon Gr(k,V) \to \mathfrak{V}$ be a regular map. If $\mathrm{char}(F) \neq 2$, then f is either constant, or has finite fibers.*

Sketch of proof. The proof of Proposition 4.2 proceeds in two steps. First, one proves that the Picard group of a Grassmannian is infinite cyclic, and then one uses a standard argument to show that an infinite cyclic Picard group implies finite fibers for regular maps in any variety. For the first part, one may invoke the Theorem in pp. 32 of [18] to obtain a description of the homogeneous coordinate ring of $Gr(k,V)$ and deduce from it that the Picard group is infinite cyclic, or else argue by covering $Gr(k,V)$ with large open affine sets (whose Picard group are shown to be trivial by examining the corresponding Chow groups) and then invoking Proposition 6.5 in [12] to show that the Picard group must be a quotient of the infinite cyclic group. Since $Gr(k,V)$ is a projective variety, it is known that the Picard group cannot be torsion, and therefore it must be infinite cyclic. For the second part, a nonconstant regular map $f\colon Gr(k,V) \to \mathfrak{V}$ from the Grassmannian to an arbitrary variety $\mathfrak{V}$ induces a map $f^*\colon \mathrm{Pic}(\mathfrak{V}) \to \mathrm{Pic}(Gr(k,V))$ given by $f^*(D) = f^{-1}(D)$. Since $\mathrm{Pic}(Gr(k,V))$ is infinite cyclic, then a generator must be ample by Theorem 7.6 in [12], and so by Prop. 7.5, also from [12], it follows that for every very ample $D \in \mathrm{Pic}(\mathfrak{V})$, either $f^*(D)$ or $-f^*(D)$ is ample. As a special case of Kleiman's Criterion (Prop. 1.27(a) in [4]) the support of an ample divisor intersects every curve in the variety, and the support of $f^*(D)$ is the same as the support of $-f^*(D)$. Thus, for every very ample $D \in \mathrm{Pic}(\mathfrak{V})$, the support of $f^*(D)$ intersects any curve on $Gr(k,V)$. Now assume that $f\colon Gr(k,V) \to \mathfrak{V}$ is a regular map which is not constant, and let y be any point in the image of f; we pick a very ample divisor D which intersects the image but does not contain y (which is always possible because f is not constant), and then note that the support of $f^*(D) = f^{-1}(D)$ is disjoint from $f^{-1}(y)$; so $f^{-1}(y)$ cannot contain any curves, and so must be 0-dimensional (i.e. a finite union of points). Therefore f has finite fibers. $\square$

Let $\varphi_1, \ldots, \varphi_n\colon V \to W$ be n linear transformations. For every element $X \in Gr(k,V)$, there is a subspace Y_X of W given by

$$Y_X = \Big\langle \varphi_1(X), \ldots, \varphi_n(X) \Big\rangle.$$

Fix X, let $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be a basis for X, and let $m = \dim(Y_X)$. Then we can identify Y_X as a point in $Gr(m, W)$ by setting it equal to

$$\sum_I \bigwedge_{(i,j)\in I} \varphi_i(\mathbf{v}_j) \tag{4.3}$$

where I ranges over all subsets of $\{1, \ldots, n\} \times \{1, \ldots, k\}$ of cardinality m. By construction, each pure wedge in the sum is a scalar multiple of a unique pure wedge associated to Y_X, and at least one of the summands is nonzero; by avoiding certain degenerate choices of bases for X, we can ensure that the sum itself is nonzero, and thus yields a nonzero pure wedge which is an element of $Gr(m, W)$.

Suppose further that there is a neighborhood U of X in $Gr(k, V)$ (in the Zariski topology) such that for all $X' \in U$, the corresponding subspace $Y_{X'}$ has dimension $m = \dim(Y_X)$. Then by choosing bases for elements of U in a suitably well-behaved manner (explicitly, by choosing, near X, a Zariski-local basis of sections of the tautological bundle on $Gr(k, V)$), we see that the correspondence $X \mapsto Y_X$ is in fact a rational map from $Gr(k, V)$ to $Gr(m, W)$. That is, $X \mapsto Y_X$ is defined on a dense open subset of X (in the Zariski topology), and is locally defined by quotients of polynomials, obtained by expanding the wedge product in the definition above. If $\dim(Y_X) = m$ for all X of dimension k, then we can take $U = Gr(k, V)$, and hence our rational map is defined everywhere, and is therefore a regular map from $Gr(k, V)$ to $Gr(m, W)$.

Remark 4.4. The discussion above is a bit more general than we will actually need for the limited cases in which we have been able to apply Theorem 4.5 below. In those cases, we have that $\dim(Y_X) = nk$ for all X of a given dimension k, and therefore, one need only choose a specific basis $\mathbf{v}_1, \ldots, \mathbf{v}_k$ of X, and define the morphism $Gr(k, V) \to Gr(nk, W)$ by mapping $\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_k$ to

$$\varphi_1(\mathbf{v}_1) \wedge \cdots \wedge \varphi_1(\mathbf{v}_k) \wedge \varphi_2(\mathbf{v}_1) \wedge \varphi_2(\mathbf{v}_2) \wedge \cdots \wedge \varphi_n(\mathbf{v}_k) \in Gr(nk, W).$$

Since each component of this wedge is determined by a linear transformation, this will be a regular map.

With these notions in hand, we now return to the situation we have associated to the group theoretic setting. Once again, we assume that V and W are defined by (3.1) and (3.2), respectively.

Theorem 4.5 (David McKinnon [16]). *Let $n \geq 2$, and let V, W, and $\varphi_1, \ldots, \varphi_n$ be as in* (3.1)–(3.3). *Let $\overline{V} = V \otimes_{\mathbb{F}_p} \overline{\mathbb{F}_p}$ and $\overline{W} = W \otimes_{\mathbb{F}_p} \overline{\mathbb{F}_p}$, where $\overline{\mathbb{F}_p}$ is the algebraic closure of $\mathbb{F}_p$. Extend φ_i to maps from $\overline{V}$ to $\overline{W}$ in the obvious way. Let k, m be integers, $0 < k < \dim(\overline{V})$, $0 < m < \dim(\overline{W})$. If $Y_X \in Gr(m, \overline{W})$ whenever $X \in Gr(k, \overline{V})$, then $X = Z_X$ for all $X \in Gr(k, V)$.*

Proof. By using the correspondence sketched above, since every subspace X of $\overline{V}$ of dimension k corresponds to a subspace of $\overline{W}$ of dimension m, we can define a regular map from $Gr(k, \overline{V})$ to $Gr(m, \overline{W})$.

Since our map is clearly not constant, it must have finite fibers by Proposition 4.2. We claim that in fact it is one-to-one. For, assume to the contrary, that there are two distinct subspaces X_1, X_2 of $\overline{V}$, both of dimension k, such that $Y_1 = Y_2$ in $\overline{W}$. Then for every subspace X' contained in $\langle X_1, X_2 \rangle$, we will necessarily have that $\varphi_r(X') \subset Y_1$. In particular, for any subspace X' contained in $\langle X_1, X_2 \rangle$, if $\dim(X') = k$ then $Y_{X'} = Y_1$. However, there are infinitely many such X', which would mean that Y_1 has an infinite fiber. As this is impossible, we conclude that the map $Gr(k,\overline{V}) \to Gr(m,\overline{W})$ is one-to-one. Therefore, the corresponding map $Gr(k,V) \to Gr(m,W)$ is also one-to-one.

Let X be a subspace of V of dimension k, and consider Z_X, defined as in (3.5). If X' is any subspace of dimension k contained in Z_X, then $Y_{X'} = Y_X$. Since the correspondence from $Gr(k,V)$ to $Gr(m,W)$ is one-to-one, there can be only one subspace of dimension k contained in Z_X. Therefore, $X = Z_X$, as claimed. $\square$

When do we have the condition given in Theorem 4.5? The following result gives a partial answer to that question (and, unfortunately, it is "seldom"):

Proposition 4.6. *Let $n > 2$, and let $\overline{V}$ and $\overline{W}$ be as in Theorem 4.5. Let X be a subspace of $\overline{V}$.*

(i) *If* $\dim(X) = 1$*, then* $\dim(Y_X) = n$*.*
(ii) *If* $\dim(X) = 2$*, then* $\dim(Y_X) = 2n$*.*
(iii) *If $2 < k < n$, then there exist subspaces X_1 and X_2 of $\overline{V}$ (in fact, subspaces that correspond to subspaces of V), with* $\dim(X_1) = \dim(X_2) = k$*, but* $\dim(Y_1) \neq \dim(Y_2)$*.*

Proof. For (iii), let

$$\begin{aligned} X_1 &= \langle v_{21}, \ldots, v_{(k+1)1} \rangle \\ X_2 &= \langle v_{21}, v_{31}, v_{32}, v_{51}, \ldots, v_{(k+1)1} \rangle. \end{aligned}$$

Then it is easy to verify that:

$$\begin{aligned} Y_1 &= \langle w_{r1s} \,|\, 1 \leq s \leq n, 2 \leq r \leq (k+1) \rangle \\ Y_2 &= \langle w_{r1s}, w_{32u} \,|\, 1 \leq s \leq n, r = 2, 3, 5, \ldots, (k+1), 2 \leq u \leq n \rangle. \end{aligned}$$

so $\dim(Y_1) = kn$ and $\dim(Y_2) = kn - 1$.

For (i), assume that $X = \langle \mathbf{v} \rangle$, $\mathbf{v} \neq \mathbf{0}$. If the dimension is not equal to n, then the set $\{\varphi_1(\mathbf{v}), \ldots, \varphi_n(\mathbf{v})\}$ is linearly dependent. Let i_0 be the first index such that $\varphi_{i_0}(\mathbf{v})$ is a linear combination of the previous vectors. Note that $i_0 > 1$.

Let (r,s) be a nonzero coordinate of $\mathbf{v}$. Neither r nor s can equal i_0, because only $\varphi_{i_0}(\mathbf{v})$ would have a nonzero coordinate indexed by a triple that includes two i_0's.

If $s < i_0 < r$, then $\varphi_{i_0}(\mathbf{v})$ has nonzero (r,s,i_0) coordinate. Since the only other way to get a vector with nonzero (r,s,i_0) coordinate is by applying φ_s to a vector with nonzero (r,i_0) coordinate, then $\mathbf{v}$ must have nonzero (r,i_0) coordinate; but we have already noted this is impossible.

If $i_0 < s < r$, then $\varphi_{i_0}(\mathbf{v})$ has nonzero (r,i_0,s) and (s,i_0,r) coordinates; so to express it as a linear combination of other images of $\mathbf{v}$, we must be using the image under φ_r. Since $r > i_0$, this is impossible by choice of i_0.

Thus, we conclude that $r < i_0$, in which case $\varphi_{i_0}(\mathbf{v})$ involves a nonzero (r,s,i_0) coordinate. As above, that means that $\varphi_s(\mathbf{v})$ must have nonzero (r,s,i_0) coordinate, which means that $\mathbf{v}$ has nonzero (i_0,r) coordinate. But this is impossible as well, as we already noted. Therefore, Y_X must be of dimension n when $\dim(X) = 1$, as claimed.

For (ii), we proceed as above. Let $X = \langle \mathbf{v}_1, \mathbf{v}_2 \rangle$, of dimension 2. Assume that the set

$$\{\varphi_1(\mathbf{v}_1), \varphi_1(\mathbf{v}_2), \varphi_2(\mathbf{v}_1), \varphi_2(\mathbf{v}_2), \ldots, \varphi_n(\mathbf{v}_1), \varphi_n(\mathbf{v}_2)\}$$

is linearly dependent. By exchanging $\mathbf{v}_1$ and $\mathbf{v}_2$, and replacing $\mathbf{v}_1$ with a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ if necessary, we may assume that the first vector in the set which is a linear combination of the previous ones is $\varphi_{i_0}(\mathbf{v}_1)$ for some $i_0 > 1$.

Let (r,s) be a nonzero coordinate of $\mathbf{v}_1$. As above, neither r nor s may equal i_0. If $s < i_0 < r$, then we must have that either $\mathbf{v}_1$ or $\mathbf{v}_2$ have nonzero (r,i_0) coordinate, and it cannot be $\mathbf{v}_1$; so it must be $\mathbf{v}_2$. Then $\varphi_s(\mathbf{v}_2)$ must be used in expressing $\varphi_{i_0}(\mathbf{v}_1)$ as a linear combination of previous elements of the set, but this vector has nonzero (i_0,s,r) coordinate as well. To cancel that coordinate (which we must, since $\varphi_{i_0}(\mathbf{v}_1)$ has zero (i_0,s,r) coordinate), we must be using the image of a vector under φ_r, but this is impossible by choice of i_0. So we cannot have $s < i_0 < r$. A similar argument shows that we cannot have $i_0 < s < r$ either, so we must have $s < r < i_0$.

Then $\varphi_{i_0}(\mathbf{v}_1)$ has nonzero (r,s,i_0) coordinate. To obtain that, we must be using the image under φ_s of vector with nonzero (i_0,r) coordinate; that vector cannot be $\mathbf{v}_1$, so it is $\mathbf{v}_2$ that has a nonzero (i_0,r) coordinate. Since $\varphi_s(\mathbf{v}_2)$ has nonzero (i_0,s,r) coordinate as well, and $\varphi_{i_0}(\mathbf{v}_1)$ does not, we must also be using the image under φ_r of a vector with nonzero (i_0,s) coordinate; once again, this cannot be $\mathbf{v}_1$, so it is $\mathbf{v}_2$ which has a nonzero (i_0,s) coordinate as well. But then $\varphi_s(\mathbf{v}_2)$ has a nonzero (i_0,s,s) coordinate, which is not the case for $\varphi_{i_0}(\mathbf{v}_1)$ and which cannot be cancelled with any other available vector. This is a contradiction. Thus we conclude that $\dim(Y_X) = 2n$ when $\dim(X) = 2$. ☐

Corollary 4.7. *Let G be a finite nonabelian nilpotent group of class 2 and exponent p an odd prime, minimally generated by $x_1,\ldots,x_n$, $n \geq 2$. If*

$$\dim_{\mathbb{F}_p}\bigl([G,G]\bigr) \geq \binom{n}{2} - 2$$

then G is capable.

Proof. Since $\#[G,G] = p^{\binom{n}{2}}/\#N$, the condition guarantees that $\#N \leq p^2$, so if X corresponds to G, then $\dim(X) \leq 2$. The result now follows from Theorem 4.5, Proposition 4.6, and the trivial case of $X = \{\mathbf{0}\}$. ☐

Remark 4.8. Although Proposition 4.6 shows that the applicability of Theorem 4.5, as is, is limited, it may be possible to use similar ideas to obtain other results. It seems likely that "most" subspaces X will have $\dim(Y_X)$ equal to a fixed number, except for some degenerate exceptions where the dimension is smaller. If a map could be defined from a sufficiently nice subvariety of $\mathrm{Gr}(k,\overline{V})$, the conclusion may still follow.

5. Some consequences

In this section, we will use the linear algebra setting to prove several results regarding the capability of groups of class 2 and exponent p. It will be clear that many of the results could be proven by appealing directly to the normal forms in the groups in question, without having to refer to linear algebra, but at least the author found that the linear algebra setting was usually easier to think about.

We start with an example of how we can use the result to prove that a given group is not capable, and which also illustrates how to set up the linear algebra problem given a specific group.

Example 5.1. A group of class two and exponent p, which is not capable.

Let G be the semidirect product of two elementary abelian p-groups, that is $G = \langle x_1, x_2, c\rangle \rtimes \langle x_3, x_4\rangle$, satisfying $x_1^{x_3} = x_1c$, $x_2^{x_3} = x_2c$, $x_1^{x_4} = x_1^c$, $x_2^{x_4} = x^2$, $c^{x_3} = c$, and $c^{x_4} = c$. This is an extra-special group of order p^5; it is mentioned in Section 8 of [17], where the fact that it is not capable is deduced by invoking a theorem of Beyl, Felgner, and Schmid in [3]. We prove that fact here using our set-up.

The group is minimally generated by x_1, x_2, x_3, and x_4, and the defining relations (in addition to $x^p = 1$ for each element) may be written as $c = [x_1, x_3] = [x_2, x_3] = [x_1, x_4]$ and

$$[x_2, x_1] = [x_4, x_3] = [x_4, x_2] = [x_1, c] = [x_2, c] = [x_3, c] = [x_4, c] = 1.$$

Let V and W be the vector spaces defined in (3.1) and (3.2), corresponding to $n = 4$. The subspace X corresponding to G will be

$$X = \Big\langle v_{21}, v_{43}, v_{42}, v_{31} - v_{32}, v_{31} - v_{41}\Big\rangle.$$

For example, the fact that $v_{21} \in X$ corresponds to the fact that $[x_2, x_1]$ is trivia; $v_{31} - v_{32} \in X$ means that $[x_3, x_1][x_3, x_2]^{-1}$ is trivial, i.e., that $[x_3, x_1] = [x_3, x_2]$, etc.

Since $\dim(V) = \binom{4}{2} = 6$ and $\dim(X) = 5$, $X \neq Z_X$ if and only if $Z_X = V$. To prove the latter, it is enough to show that $v_{41} \in Z_X$.

We need to show that each of w_{411}, w_{412}, w_{413}, and w_{414} are in

$$Y_X = \big\langle \varphi_1(X), \varphi_2(X), \varphi_3(X), \varphi_4(X)\big\rangle.$$

Indeed:

$$\begin{aligned}
w_{411} &= \varphi_1\big(v_{42} + (v_{31} - v_{32}) - (v_{31} - v_{41})\big)\\
&\qquad + \varphi_2\big(v_{31} - v_{41}\big) - \varphi_3\big(v_{21}\big) + \varphi_4\big(v_{21}\big).\\
w_{412} &= \varphi_1\big(v_{42}\big) + \varphi_4\big(v_{21}\big).\\
w_{413} &= \varphi_1\big(v_{43}\big) - \varphi_2\big(v_{43}\big) + \varphi_3\big(v_{42}\big) + \varphi_4\big(v_{31} - v_{32}\big).\\
w_{414} &= \varphi_3\big(v_{42}\big) - \varphi_2\big(v_{43}\big) + \varphi_4\big((v_{31} - v_{32}) - (v_{31} - v_{41})\big).
\end{aligned}$$

Therefore, G is not capable. $\square$

We will state most of our results as lemmas about the linear algebra situation, and deduce the corresponding conclusions for groups of class two and exponent p as theorems. Since some of our lemmas will appear contrived without the benefit of knowing the group theory result we are after, we will usually state the theorem first, and then the corresponding lemma about linear algebra.

A theorem of G. Ellis. The original impetus behind this work was my desire to obtain an alternative proof of the following result:

Theorem 5.2 (G. Ellis, Proposition 9 of [5])**.** *Let G be a finitely generated group of nilpotency class exactly two and of prime exponent. Let $\{x_1, \ldots, x_k\}$ be a subset of G corresponding to a basis of the vector space $G/Z(G)$, and suppose that those non-trivial commutators of the form $[x_j, x_i]$, $1 \leq i < j \leq k$ are distinct and constitute a basis for the vector space $[G, G]$. Then G is capable.*

Proof. Let $x_{k+1},\ldots,x_n$ be elements of $Z(G)$ that, together with $x_1,\ldots,x_k$, constitute a minimal generating set of G. The conditions given in the statement of the theorem imply that the subspace X of V corresponding to G will have a basis consisting of a subset of the standard basis of V; therefore, Ellis's Theorem will follow from Lemma 5.3 below. □

Lemma 5.3. *Let $n>1$, and let V, W, $\varphi_1,\ldots,\varphi_n$ be given as in* (3.1)–(3.3)*. If X has a basis consisting of a subset of $\{v_{ji}\,|\,1\leq i<j\leq n\}$ (that is, X is a "coordinate subspace" of V), then $Z_X = X$.*

Proof. Assume that X has a basis consisting of a subset $S\subset\{v_{ji}\,|\,1\leq i<j\leq n\}$. If $v_{rs}\notin S$, then all vectors in X have zero (r,s) coordinate, and therefore by Lemma 3.9, so do all vectors in Z_X. This means that $Z_X\subset\langle S\rangle = X$, giving equality. □

Capability of coproducts. Recall that we define the 2-nilpotent product of two $\mathfrak{N}_2$ groups (not necessarily cyclic) A and B as

$$A \amalg^{\mathfrak{N}_2} B = (A*B)/(A*B)_2,$$

where $A*B$ is their free product. A theorem of T. MacHenry in [15] shows that any element of $A\amalg^{\mathfrak{N}_2} B$ can be written uniquely as abc, where $a\in A$, $b\in B$, and $c\in[B,A]$, the 'cartesian', and that the cartesian is isomorphic to $B^{\mathrm{ab}}\otimes A^{\mathrm{ab}}$, via the map sending $[b,a]$ to $\overline{b}\otimes\overline{a}$.

Lemma 5.4. *Let G be a finite nontrivial group of class two and exponent p, p an odd prime, and let C_p be the cyclic group of order p. Then $G\amalg^{\mathfrak{N}_2} C_p$ is capable.*

Proof. The result follows from Theorem 3.11 if G is cyclic, so assume that G is minimally generated by $g_1,\ldots,g_n$, $n>1$, and denote the generator of C_p by x_{n+1}. Let V_n be the vector space of dimension $\binom{n}{2}$ with basis v_{ji}, $1\leq i<j\leq n$, and let V_{n+1} be the larger vector space that also includes the basis vectors $v_{(n+1)k}$, $1\leq k\leq n$. Identify V_n with the obvious subspace of V_{n+1}. If X is the subspace of V_n corresponding to G, then X is also the subspace of V_{n+1} corresponding to $G\amalg^{\mathfrak{N}_2} C_p$. This subspace does not contain any vectors with a nonzero $(n+1,k)$ coordinate, $1\leq k\leq n$. The result will now follow from Lemma 5.5 below. □

Lemma 5.5. *Let $n>1$, let V, W, $\varphi_1,\ldots,\varphi_n$ be as in* (3.1)–(3.3)*, and let X be a subspace of V. Suppose that there exists j, $1\leq j\leq n$, such that for all $\mathbf{v}\in X$ and all i, $1\leq i\leq n$, $i\neq j$, $\mathbf{v}$ has zero (i,j) or zero (j,i) coordinate (whichever makes sense). Then $X=Z_X$.*

Proof. We claim that

$$\varphi_j(V)\cap\Big\langle\varphi_1(X),\ldots,\varphi_{j-1}(X),\varphi_{j+1}(X),\ldots,\varphi_n(X)\Big\rangle = \big\{\mathbf{0}\big\}.$$

Indeed, every nonzero coordinate of a vector in $\varphi_j(V)$ has at least one index (either the second or the third) equal to j. But no vector in any of $\varphi_i(X)$, $i\neq j$, has a nonzero coordinate involving a j (in any position). Thus, neither do any of their linear combinations. This means that $\varphi_j^{-1}(Y_X) = X$ (since φ_j is injective), and therefore that $Z_X = X$. □

In fact, we can strengthen this result considerably:

Theorem 5.6. *Let G and H be any two nontrivial finite groups of class at most two and exponent p, p an odd prime. Then $G \amalg^{\mathfrak{N}_2} H$ is capable.*

Proof. If either G or H are cyclic then the result follows from Lemma 5.4. If they are both noncyclic, let G be minimally generated by $g_1, \ldots, g_a$, $a > 1$, and let H be minimally generated by $h_{a+1}, \ldots, h_{a+b}$, $b > 1$. Let X_1 be the subspace of $\langle v_{ji} \,|\, 1 \leq i < j \leq a\rangle$ corresponding to G under the obvious identification, and let X_2 be the subspace of $\langle v_{ji} \,|\, a+1 \leq i < j \leq a+b\rangle$ corresponding to H. Then $X_1 \oplus X_2$ is the subspace of $V = \langle v_{ji} \,|\, 1 \leq i < j \leq a+b\rangle$ that corresponds to $G \amalg^{\mathfrak{N}_2} H$. The result will follow from Lemma 5.7 below. □

We will need some notation prior to stating Lemma 5.7. Let $a, b > 1$. Let V, W, and $\varphi_1, \ldots, \varphi_n$ be as in (3.1)–(3.3), corresponding to $n = a+b$. We decompose V as $V = V_s \oplus V_m \oplus V_\ell$, where V_s is generated by the vectors v_{ji}, $1 \leq i < j \leq a$ (the "small index" vectors); V_ℓ is generated by the vectors v_{ji}, $a+1 \leq i < j \leq a+b$ (the "large index" vectors); and V_m is generated by the vectors v_{ji} with $1 \leq i \leq a < j \leq b$ (the "mixed index" vectors). For any vector $\mathbf{v} \in V$, we write

$$\mathbf{v} = \mathbf{v}_s + \mathbf{v}_m + \mathbf{v}_\ell,$$

where $\mathbf{v}_s \in V_s$, $\mathbf{v}_m \in V_m$, and $\mathbf{v}_\ell \in V_\ell$. The idea here is that the vectors in the "small index" part of V will correspond to the relations defining G, while the vectors in the "large index" part of V will correspond to relations defining H.

Similarly, we decompose W as $W = W_s \oplus W_{m_1} \oplus W_{m_2} \oplus W_\ell$; W_s is generated by the basis vectors which have all three indices smaller than or equal to a; W_{m_1} by the basis vectors which have exactly two indices smaller than or equal to a, one larger than a; W_{m_2} by the basis vectors which have exactly one index smaller than or equal to a, and two larger than a; and W_ℓ by the basis vectors in which all three indices are larger than a. As above, we can decompose any vector $\mathbf{w} \in W$ as

$$\mathbf{w} = \mathbf{w}_s + \mathbf{w}_{m_1} + \mathbf{w}_{m_2} + \mathbf{w}_\ell,$$

with $\mathbf{w}_s \in W_s$, etc.

Lemma 5.7. *Notation as in the previous paragraphs. Let X_1 be a subspace of V_s, X_2 a subspace of V_ℓ. If $X = X_1 \oplus X_2$, then $X = Z_X$.*

Proof. We prove this as a series of claims. Let $\mathbf{w} \in W$, $\mathbf{w} = \mathbf{w}_s + \mathbf{w}_{m_1} + \mathbf{w}_{m_2} + \mathbf{w}_\ell$.

CLAIM 1: $\mathbf{w} \in Y_X$ if and only if each of $\mathbf{w}_s$, $\mathbf{w}_{m_1}$, $\mathbf{w}_{m_2}$, and $\mathbf{w}_\ell$ are in Y_X.

Indeed, note that $\mathbf{v} \in X$ if and only if $\mathbf{v}_s \in X_1$, $\mathbf{v}_\ell \in X_2$, and $\mathbf{v}_m = \mathbf{0}$. Since $\varphi_i(X_1) \subset W_s$, $\varphi_i(X_2) \subset W_{m_2}$ for $1 \leq i \leq a$; and $\varphi_j(X_1) \subset W_{m_1}$, $\varphi_j(X_2) \subset W_\ell$ if $a+1 \leq j \leq a+b$, the claim follows.

CLAIM 2: $\mathbf{v} \in Z_X$ if and only if $\mathbf{v}_s, \mathbf{v}_\ell \in Z_X$ and $\mathbf{v}_m = \mathbf{0}$.

For, let $\mathbf{v} \in \varphi_i^{-1}(Y_X)$ for some fixed i, $1 \leq i \leq a+b$. Then $\mathbf{w} = \varphi_i(\mathbf{v}) \in Y_X$, so each of $\mathbf{w}_s$, $\mathbf{w}_{m_1}$, $\mathbf{w}_{m_2}$, and $\mathbf{w}_\ell$ are in Y_X. If $i \leq a$, we have that $\mathbf{w}_s = \varphi_i(\mathbf{v}_s)$, $\mathbf{w}_{m_1} = \varphi_i(\mathbf{v}_m)$, $\mathbf{w}_{m_2} = \varphi_i(\mathbf{v}_\ell)$, and $\mathbf{w}_\ell = \mathbf{0}$. On the other hand, if $a < i \leq a+b$, then $\mathbf{w}_s = \mathbf{0}$, $\mathbf{w}_{m_1} = \varphi_i(\mathbf{v}_s)$, $\mathbf{w}_{m_2} = \varphi_i(\mathbf{v}_m)$, and $\mathbf{w}_\ell = \varphi_i(\mathbf{v}_\ell)$. In either case, we see that each of $\mathbf{v}_s$, $\mathbf{v}_m$, and $\mathbf{v}_\ell$ lie in $\varphi_i^{-1}(Y_X)$. This proves that $\mathbf{v} \in Z_X$ if and only if each of $\mathbf{v}_s$, $\mathbf{v}_\ell$, and $\mathbf{v}_m$ are in Z_X. Since all vectors in X have zero (j,i) coordinate for all $1 \leq i \leq a < j \leq a+b$, so do all vectors in Z_X by Lemma 3.9, which gives the condition $\mathbf{v}_m = \mathbf{0}$.

CLAIM 3: For each i, $a+1 \leq i \leq a+b$,

$$\varphi_i(V_s) \cap \Big\langle \varphi_{a+1}(X_1), \ldots, \varphi_{i-1}(X_1), \varphi_{i+1}(X_1), \ldots, \varphi_{a+b}(X_1) \Big\rangle = \big\{\mathbf{0}\big\}.$$

Indeed, for any j, $a+1 \leq j \leq a+b$, the nonzero coordinates in any vector of $\varphi_j(V_s)$ have last index equal to j. Thus none of the nonzero coordinates of the vectors in the span of the $\varphi_j(X_1)$, $a+1 \leq j \leq a+b$, $j \neq i$, have a nonzero coordinate with last index equal to i, and hence cannot lie in $\varphi_i(V_s)$.

CLAIM 4: $Z_X \cap V_s = X_1$.

Let $\mathbf{v}_s \in Z_X \cap V_s$. Then

$$\mathbf{v}_s \in \bigcap_{i=a+1}^{a+b} \varphi_i^{-1}(Y_X).$$

and thus,

$$\mathbf{v}_s \in \bigcap_{i=a+1}^{a+b} \varphi_i^{-1}(Y_X \cap W_{m_1}).$$

But $Y_X \cap W_{m_1} = \langle \varphi_{a+1}(X_1), \ldots, \varphi_{a+b}(X_1)\rangle$, and from Claim 3 it follows that

$$\varphi_i^{-1}\left(\langle \varphi_{a+1}(X_1), \ldots, \varphi_{b+1}(X_1)\rangle\right) = X_1; \quad i = a+1, \ldots, a+b.$$

Therefore, $\mathbf{v}_s \in X_1$, as claimed.

CLAIM 5: For each i, $1 \leq i \leq a$,

$$\varphi_i(V_\ell) \cap \Big\langle \varphi_1(X_2), \ldots, \varphi_{i-1}(X_2), \varphi_{i+1}(X_2), \ldots, \varphi_a(X_2)\Big\rangle = \big\{\mathbf{0}\big\}.$$

This is analogous to Claim 3: the nonzero coordinates in any vector in $\varphi_j(V_\ell)$, $1 \leq j \leq a$, have middle index equal to j. So none of the nonzero vectors in the span of the $\varphi_j(X_2)$, $1 \leq j \leq a$, $j \neq i$ may lie in $\varphi_i(V_\ell)$.

CLAIM 6: $Z_X \cap V_\ell = X_2$.

This follows from Claim 5 in the same way that Claim 4 follows from Claim 3.

CLAIM 7: $Z_X = X_1 \oplus X_2 = X$.

From Claims 2, 4, and 6 we have that $Z_X = (Z_X \cap V_s) \oplus (Z_X \cap V_\ell) = X_1 \oplus X_2 = X$, as claimed. This proves the lemma. ☐

Reducing to a special case. A more interesting result is the following:

Theorem 5.8. *Let G be a finite noncyclic nilpotent group of class two and exponent p, p an odd prime; let C_p be the cyclic group of order p. Then*

$$G \text{ is capable} \iff G \oplus C_p \text{ is capable.}$$

Proof. Let G be minimally generated by n elements $g_1, \ldots, g_n$, $n > 1$. We think of C_p as generated by x_{n+1}. Let V, W be the spaces corresponding to $n+1$, and let X_1 be the subspace of $V_1 = \langle v_{ji} \,|\, 1 \leq i < j \leq n\rangle$ that corresponds to G. The subspace that corresponds to $G \oplus C_p$ is $X_1 \oplus \langle v_{(n+1)i} \,|\, 1 \leq i \leq n\rangle$. The statement that G is capable is the statement that $X_1 = Z_1$ (where we work only with V_1 and $\varphi_1, \ldots, \varphi_n$), and the statement that $G \oplus C_p$ is capable is equivalent to saying that $X = Z_X$ (this time working with V and $\varphi_1, \ldots, \varphi_{n+1}$). Therefore, the theorem will follow from Lemma 5.9 below. ☐

Lemma 5.9. *Let $n > 1$, and let V, W, $\varphi_1, \ldots, \varphi_{n+1}$ be as in* (3.1)–(3.3), *corresponding to $n+1$. Let*

$$\begin{aligned} V_1 &= \big\langle v_{ji} \;\big|\; 1 \leq i < j \leq n\big\rangle, \\ W_1 &= \big\langle w_{jik} \;\big|\; 1 \leq i < j \leq n,\; i \leq k \leq n\big\rangle. \end{aligned}$$

Let X_1 be a subspace of V_1, and let

$$\begin{aligned} Y_1 &= \Big\langle \varphi_1(X_1),\ldots,\varphi_n(X_1)\Big\rangle\\ Z_1 &= \bigcap_{i=1}^{n} \varphi_i^{-1}(Y_1). \end{aligned}$$

Let X be the subspace of V given by

$$X = X_1 \oplus \big\langle v_{(n+1)i}\,|\, 1\leq i\leq n\big\rangle,$$

and let Y_X and Z_X be given by

$$\begin{aligned} Y_X &= \Big\langle \varphi_1(X),\ldots,\varphi_n(X),\varphi_{n+1}(X)\Big\rangle\\ Z_X &= \bigcap_{i=1}^{n+1} \varphi_i^{-1}(Y_X). \end{aligned}$$

Then $X_1 = Z_1$ if and only if $X=Z$.

Proof. For simplicitly, let $X_2 = \langle v_{(n_1)i}\,|\,1\leq i\leq n\rangle$.

Note that all vectors $w_{(n+1)ij}$ and $w_{ji(n+1)}$, with $1\leq i<j\leq n+1$ are in Y_X. For $w_{(n+1)ij}$ is the image under φ_j of $v_{(n+1)i}$; and the image under φ_i of $v_{(n+1)j}$ is $w_{(n+1)ij} - w_{ji(n+1)}$; the first summand is already in Y_X, hence so is the second summand. This means that $V_1\subset \varphi_{n+1}^{-1}(Y_X)$, since $v_{ji} = \varphi_{n+1}^{-1}(w_{ji(n+1)})$. Since all vectors $v_{(n+1)i}$ are in X, we conclude that in fact $V=\varphi_{n+1}^{-1}(Y_X)$; (in terms of the group-theoretic setting, all we are saying is that since every generator commutes with x_{n+1}, every commutator does as well). We have:

$$\begin{aligned} Y_X &= \big\langle \varphi_1(X),\ldots,\varphi_{n+1}(X)\big\rangle\\ &= \big\langle \varphi_1(X_1),\ldots,\varphi_n(X_1),\varphi_1(X_2),\ldots,\varphi_n(X_2),\varphi_{n+1}(X)\big\rangle\\ &= \big\langle Y_1,\varphi_1(X_2),\ldots,\varphi_n(X_2),\varphi_{n+1}(X)\big\rangle\\ &= Y_1\oplus\big\langle w_{(n+1)ij}, w_{ji(n+1)}\;\big|\; 1\leq i<j\leq n+1\big\rangle. \end{aligned}$$

We claim that $Z_X = Z_1\oplus X_2$. From this claim, $Z_X = X \iff Z_1 = X_1$ will follow.

Trivially, Z_X contains X_2. To show it contains Z_1, note that if $\mathbf{v}\in Z_1$, then we just need to show that $\mathbf{v}\in\varphi_{n+1}^{-1}(Y_X)$; this inverse image is all of V_1, so $Z_1\subset Z_X$ as claimed. Conversely, let $\mathbf{v}\in Z_X$. We want to show that it is in $Z_1\oplus X_2$. By adding the necessary multiples of $v_{(n+1)j}$ we may assume that $\mathbf{v}\in V_1$. Therefore, $\varphi_i(\mathbf{v})$ is in $Y_X\cap W_1 = Y_1$ for $i=1,2,\ldots,n$, which implies that $\mathbf{v}\in Z_1$. Thus, $Z_x = Z_1\oplus X_2$, and this proves the lemma. □

The interest of Theorem 5.8 is that it allows us to reduce the problem of capability to a special case that has been considered often in the past. We do this in the following two results:

Lemma 5.10. *Let G be a finite nilpotent group of class two and exponent p, with p an odd prime. Then G may be written as $G = K\oplus C_p^r$, where K satisfies $Z(K) = [K,K] = [G,G]$, $r = \dim_{\mathbb{F}_p}(Z(G)/[G,G])$, and C_p is the cyclic group of order p.*

Proof. Let $z_1,\ldots,z_r\in Z(G)$ be elements that project onto a basis of the vector space $Z(G)/[G,G]$. Let $g_1,\ldots,g_n$ be elements of G whose projection extends the images of $z_1,\ldots,z_r$ to a basis of $G/[G,G]$. Let $K=\langle g_1,\ldots,g_n\rangle$. It is well known that a set of elements of a nilpotent group that generates the abelianization must also generate the group, so G is generated by $z_1, \ldots, z_r, g_1, \ldots, g_n$.

We have that $Z(G) = \langle z_1, \ldots, z_r\rangle \oplus [G, G]$, and that $\langle z_1, \ldots, z_r\rangle$ is a direct summand of G. By construction, we also have $[K, K] = [G, G]$, so

$$G = K \oplus \langle z_1, \ldots, z_r\rangle \cong K \oplus C_p^r.$$

Since K is cocentral in G, $Z(K) = Z(G) \cap K$, and therefore, any element which is central in K must be a commutator in G, by choice of $z_1, \ldots, z_r$. That is, $Z(K) = [G, G] = [K, K]$, which proves the lemma. □

The following result is now immediate:

Theorem 5.11. *Let G be a noncyclic finite nilpotent group of class two and exponent p, p an odd prime. Write as $G = K \oplus C_p^r$, where C_p is cyclic of order p, $r \geq 0$, and K satisfies $Z(K) = [K, K]$. Then G is capable if and only if K is capable.* □

One reason why Theorem 5.11 is interesting is that the condition $Z(K) = [K, K]$ is fairly strong, and has been used in the past in the study of capability for nilpotent groups of class 2 and exponent p; for example, Theorem 1 in [13]. It also means that we may discard certain subspaces from consideration in the linear algebra setting: any subspace that corresponds to a group in which $Z(G) \neq [G, G]$ may be ignored. Any subspace X that contains all vectors v_{kj} and v_{ji}, for a fixed j and $1 \leq j \leq n$, $i < j < k$, for instance.

Corollary 5.12. *The capability of finite groups of class two and odd prime exponent p is completely determined by the capability of finite groups G of class two and exponent p which satisfy the condition $Z(G) = [G, G]$, plus the observation that a nontrivial abelian group of exponent p is capable if and only if it is not cyclic.*

Combining Corollary 4.7 with Theorem 5.11 yields:

Theorem 5.13. *Let G be a finite noncyclic group of class two and exponent p, and let $k = \dim_{\mathbb{F}_p}(G/Z(G))$. If*

$$\dim_{\mathbb{F}_p}\bigl([G, G]\bigr) \geq \binom{k}{2} - 2$$

then G is capable.

Proof. We may write $G = K \oplus C_p^r$, where $r = \dim_{\mathbb{F}_p}(Z(G)/[G, G])$, and K satisfies $Z(K) = [K, K] = [G, G]$. Then G is capable if and only if K is capable, and we apply Corollary 4.7 to K, noting that $\dim_{\mathbb{F}_p}(K^{\mathrm{ab}}) = \dim_{\mathbb{F}_p}(G/Z(G)) = k$. □

This is strong enough to settle the 3-generated case:

Theorem 5.14. *Let G be a 3-generated group of class 2 and exponent p. Then G is either cyclic or capable.*

Proof. Let G be a non-cyclic 3-generated group of class 2 and exponent p, and let $k = \dim_{\mathbb{F}_p}(G/Z(G))$. Then we must have $k = 0$, 2, or 3; since $\dim_{\mathbb{F}_p}([G, G]) = 0$ if and only if $k = 0$, it follows that $\dim_{\mathbb{F}_p}([G, G]) \geq \binom{k}{2} - 2$ in all cases, so G is capable. □

Note that we already have an example of a minimally 4-generated nilpotent group of class 2 and exponent p which is not capable.

6. Final comments

The condition $Z(G) = [G,G]$ was used By Heineken and Nikolova to give an upper bound for the rank of G^{ab} in terms of the rank of $Z(G)$ for a capable group G satisfying the condition. Their result, Theorem 1 in [13], is that if G satisfies $Z(G) = [G,G]$, and G is capable, then when the rank of $Z(G)$ is k, the rank of G^{ab} is at most $2k + \binom{k}{2}$. Using Theorem 5.11, we can drop the hypothesis $Z(G) = [G,G]$, provided we replace $\mathrm{rank}(Z(G))$ with $\mathrm{rank}([G,G])$, and $\mathrm{rank}(G^{\mathrm{ab}})$ with $\mathrm{rank}(G/Z(G))$. Turning the theorem "upside down" to give a lower bound on the rank of $[G,G]$ from the rank of $G/Z(G)$, we have that if G is a capable finite nilpotent group of class at most 2 and exponent an odd prime p, then

$$\dim_{\mathbb{F}_p}([G,G]) \geq \left\lceil \frac{-3+\sqrt{9+8n}}{2} \right\rceil.$$

where $n = \dim_{\mathbb{F}_p}(G/Z(G))$.

In view of these results, and particularly of Theorem 5.6, it seems that capability for groups of class exactly two depends mainly on there not being too many relations among the commutators (that is, the subspace X not being "too big"; or equivalently, the commutator subgroup being "big enough").

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