Amalgamation bases for nil-2 groups of odd order

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We study amalgams and the strong, weak, and special amalgamation bases in the varieties of nilpotent groups of class two and exponent n, where n is odd. The main result is the characterization of the special amalgamation bases for these varieties. We also characterize the weak and strong bases. For special amalgamation bases we show that there are groups which are special bases in varieties of finite exponent but not in the variety of all nil-2 groups, whereas for weak and strong bases we show this is not the case. We also show that in these varieties, as well as the variety of all nil-2 groups, a group has an absolute closure (in the sense of Isbell) if and only if it is already absolutely closed, i.e. if and only if it is a special amalgamation base.

Key Words: nilpotent, amalgam, amalgamation base, dominion, absolute closure, absolutely closed

0. INTRODUCTION

The main result in this paper is the characterization of the special amalgamation bases in the variety $\mathcal{N}_2 \cap \mathcal{B}_n$ of nilpotent groups of class at most two and exponent n, with n odd. We do this by first reducing to the case where n is a prime power, and then solving that case. We then also extend the results to characterize the strong and weak amalgamation bases. We also use the characterization of the special amalgamation bases to show that in these varieties, as well as the variety $\mathcal{N}_2$ of nilpotent groups of class at most two, the only groups with an absolute closure are the absolutely closed groups (we will recall the definitions below).

In this section we will recall the basic definitions, and in Section 1 we will prove some technical lemmas. Since the case of special amalgams turns out to be significantly more complicated than the general case, we will start by

2000 MSC: Primary 20E06, 20F18, 20D15, 08B25; Secondary 20E10.
The author was supported in part by CONACyT grant I 29922-E.

studying them. In Section 2 we will look at special amalgamation bases. In Section 3 we will use this characterization to establish that there are special amalgamation bases in the variety $\mathcal{N}_2\cap\mathcal{B}_{p^n}$ which are not special amalgamation bases in $\mathcal{N}_2\cap\mathcal{B}_{p^{n+1}}$. We will also look at absolute closures in these varieties, and address a question of Higgins about absolutely closed algebras. In Section 4 we will look at the strong and weak amalgamation bases. The characterization will yield that in fact the only strong and weak amalgamation bases in $\mathcal{N}_2\cap\mathcal{B}_{p^n}$ are those which are amalgamation bases in $\mathcal{N}_2$. Finally, in Section 5, we will discuss some related questions and some of the difficulties in the case of n even, as well as some partial results in that case.

Groups will be written multiplicatively, unless otherwise specified. We will use Z to denote the infinite cyclic group, which we also write multiplicatively. All maps are assumed to be group morphisms unless we explicitly note otherwise. The multiplicative identity of a group G will be denoted by e, and we will use e_G if there is danger of ambiguity. For a group G and elements $x,y\in G$, the commutator of x and y is $[x,y]=x^{-1}y^{-1}xy$; note that $[x,y]^{-1}=[y,x]$. Given subsets A,B of G, not necessarily subgroups, $[A,B]$ denotes the subgroup of G generated by all commutators $[a,b]$ with $a\in A$ and $b\in B$. The commutator subgroup of G is the subgroup $[G,G]$, which is sometimes denoted by G'. The center of G is denoted by $Z(G)$. All commutators will be written left-normed, so $[x,y,z]=[[x,y],z]$.

Let $\mathcal{N}_2$ denote the variety of all nilpotent groups of class at most two; that is, groups G such that $[G,G]\subseteq Z(G)$, or equivalently, for which the identity $[x,y,z]=e$ holds. It is easy to verify that for any nil-2 group (i.e. any nilpotent group of class at most two) the following identities hold, so we will use them without comment throughout the paper.

Proposition 0.1. *Let $G\in\mathcal{N}_2$. For all $x,y,z\in G$ and all integers n:*

(a) $[xy,z]=[x,z][y,z]$; $[x,yz]=[x,y][x,z]$.
(b) $[x^n,y]=[x,y]^n=[x,y^n]$.
(c) $(xy)^n=x^ny^n[y,x]^{n(n-1)/2}$.
(d) The value of $[x,y]$ depends only on the congruence classes of x and y modulo G' (in fact, modulo $Z(G)$).

From these properties, it is easy to see that if $G\in\mathcal{N}_2$ is generated by elements of exponent n with n odd, then G itself is of exponent n, and, if n is even, of exponent $2n$. Also, if K is generated by a subgroup G of exponent $2n$ and elements of exponent n, then K is also of exponent $2n$, whether n is even or not.

We use $\mathcal{B}_n$ to denote the variety of all groups of exponent $n > 0$. Thus, the variety of all nilpotent groups of class at most two and exponent n is denoted by $\mathcal{N}_2 \cap \mathcal{B}_n$.

For the remainder of this paper, all groups will be assumed to lie in $\mathcal{N}_2$ unless otherwise specified. Any presentation of a group will be understood to be a presentation in $\mathcal{N}_2$; that is, the identities on $\mathcal{N}_2$ will be imposed on the group, as well as all the relations specified in the presentation.

Given $A, B \in \mathcal{N}_2$, every element of their coproduct $A \coprod^{\mathcal{N}_2} B$ has a unique expression of the form $\alpha\beta\gamma$, where $\alpha \in A$, $\beta \in B$, and $\gamma \in [A, B]$. A theorem of T. MacHenry [6] states that the subgroup $[A, B]$ of $A \coprod^{\mathcal{N}_2} B$ is isomorphic to the tensor product $A^{\mathrm{ab}} \otimes B^{\mathrm{ab}}$.

Recall that an $\mathcal{N}_2$-amalgam of two groups $A, C \in \mathcal{N}_2$ with core B consists of groups A, B, and C, equipped with one to one group morphisms

$$\begin{aligned} \Phi_A\colon B &\rightarrow A \\ \Phi_C\colon B &\rightarrow C. \end{aligned}$$

To simplify notation, we denote this situation by $(A, C; B)$. To say the amalgam $(A, C; B)$ is *(weakly) embeddable in $\mathcal{N}_2$* means that there exists a group M in $\mathcal{N}_2$ and one to one group morphisms

$$\lambda_A\colon A \to M, \qquad \lambda_C\colon C \to M, \qquad \lambda\colon B \to M,$$

such that

$$\lambda_A \circ \Phi_A = \lambda \qquad \text{and} \qquad \lambda_C \circ \Phi_C = \lambda.$$

When we examine whether or not the amalgam $(A, C; B)$ is embeddable, the obvious candidate for M is the coproduct with amalgamation of A and C over B, denoted by $A \coprod_B^{\mathcal{N}_2} C$. This coproduct is sometimes called the $\mathcal{N}_2$-free product with amalgamation. We say that $(A, C; B)$ is *weakly embeddable* (in $\mathcal{N}_2$) if no two distinct elements of A are identified with each other in the coproduct with amalgamation, and similarly with two distinct elements of C. Note that weak embeddability does not preclude the possibility that an element x of $A \setminus B$ be identified with an element y of $C \setminus B$ in $A \coprod_B^{\mathcal{N}_2} C$. We say that $(A, B; C)$ is *strongly embeddable* (in $\mathcal{N}_2$) if there is also no identification between elements of $A \setminus B$ and elements of $C \setminus B$. By *special amalgam* we mean an amalgam $(A, A'; B)$, where there is an isomorphism $\psi\colon A \to A'$ such that $\psi \circ \Phi_A = \Phi_{A'}$. In this case, we usually write $(A, A; B)$ with $\psi = \mathrm{id}_A$ being understood. Since special amalgams are always weakly embeddable (say, by mapping to A), with special amalgams we are mostly interested in whether or not they can be strongly embedded.

Finally, recall that a group G, which belongs to a class of groups $\mathcal{C}$, is said to be a *weak amalgamation base* for $\mathcal{C}$ if and only if every amalgam of $\mathcal{C}$-groups with core G is weakly embeddable in $\mathcal{C}$; it is a *strong amalgamation base* if every such amalgam is strongly embeddable; and it is a *special amalgamation base* if every special amalgam of $\mathcal{C}$-groups with core G is strongly embeddable.

Special amalgams are closely related to the concept of *dominion.* Recall that Isbell [3] defines for a category $\mathcal{C}$ of algebras (in the sense of Universal Algebra) of a fixed type Ω, and an algebra $A \in \mathcal{C}$ and subalgebra B of A, the *dominion of B in A* (in the category $\mathcal{C}$) to be the intersection of all equalizers containing B. Explicitly,

$$\mathrm{dom}_A^{\mathcal{C}}(B) = \{a \in A | \forall f, g\colon A \to C,\ \text{if } f|_B = g|_B \text{ then } f(a) = g(a)\}$$

where C ranges over all algebras $C \in \mathcal{C}$, and f, g are morphisms. The connection between amalgams and dominions when working in a variety is through special amalgams: letting A' be an isomorphic copy of A, and $M = A \coprod_B^{\mathcal{C}} A'$, we have that

$$\mathrm{dom}_A^{\mathcal{C}}(B) = A \cap A' \subseteq M$$

where we have identified B with its common image in A and A'. In other words, $\mathrm{dom}_A^{\mathcal{C}}(B)$ is the smallest subalgebra D of A such that $B \subseteq D$ and the amalgam $(A, A; D)$ is strongly embeddable. If $\mathrm{dom}_A^{\mathcal{C}}(B) = B$, we say that the dominion of B is "trivial" (meaning it is as small as possible), and we say it is "nontrivial" otherwise.

In general, $\mathrm{dom}_A^{\mathcal{C}}(-)$ is a closure operator on the lattice of subalgebras of A. If we are working in a variety of groups (i.e., a full subcategory of $\mathcal{G}roup$ which is closed under taking subgroups, quotients, and arbitrary direct products), then the dominion construction respects finite direct products (that is, if $H_1 < G_1$ and $H_2 < G_2$, then the dominion of $H_1 \times H_2$ in $G_1 \times G_2$ is the product of the dominions of H_1 in G_1 and of H_2 in G_2), and also respects quotients: if $N \triangleleft G$ is contained in H, then

$$\mathrm{dom}_{G/N}^{\mathcal{V}}(H/N) = \left(\mathrm{dom}_G^{\mathcal{V}}(H)\right) \,/\, N.$$

For a proof of these assertions we direct the reader to [8].

Because of the connection with dominions, special amalgamation bases are also said to be *absolutely closed.* We direct the reader to the survey paper by Higgins [2] for a more complete discussion of amalgams and their connection with dominions.

1. PRELIMINARY RESULTS

In the variety $\mathcal{N}_2$ not every amalgam is weakly embeddable, and not every dominion is trivial; see for example [8] and [12].

The following easy observations will be key:

PROPOSITION 1.1. *Let $(G, K; H)$ be an amalgam of $\mathcal{N}_2$-groups. Assume both G and K are of exponent n. If n is odd, then $(G, K; H)$ is weakly (resp. strongly) embeddable into an $\mathcal{N}_2$-group if and only if $(G, K; H)$ is weakly (resp. strongly) embeddable into an $\mathcal{N}_2 \cap \mathcal{B}_n$-group. If n is even, then $(G, K; H)$ is weakly (resp. strongly) embeddable into an $\mathcal{N}_2$-group if and only if $(G, K; H)$ is weakly (resp. strongly) embeddable into an $\mathcal{N}_2 \cap \mathcal{B}_{2n}$-group.*

Proof. Clearly, we may assume that a group into which we embedd the amalgam $(G, K; H)$ is generated by G and K. For n odd this means the group itself is of exponent n, and for n even of exponent $2n$. ∎

PROPOSITION 1.2. *Let $(G, K; H)$ be an amalgam of $\mathcal{N}_2$-groups. If both G and K are of exponent n, then $(G, K; H)$ is weakly (resp. strongly) embeddable into an $\mathcal{N}_2$-group if and only if for every prime p dividing n, the amalgam $(G_p, K_p; H_p)$ is weakly (resp. strongly) embeddable, where G_p is the p-part of G, and likewise for K_p and H_p.*

Proof. Since G, K, and H are of finite exponent, they are the direct product of their p-parts, i.e. $G = \prod G_p$, $K = \prod K_p$, and $H = \prod H_p$. Clearly, an embedding of $(G, K; H)$ provides embeddings for $(G_p, K_p; H_p)$ for each p; conversely, if we have embeddings into M_p for each $(G_p, K_p; H_p)$, then $\prod M_p$ will give an embedding for $(G, K; H)$. ∎

In view of the above two propositions, we may restrict ourselves to the varieties $\mathcal{N}_2 \cap \mathcal{B}_{p^n}$ with p a prime. In this paper we will further restrict ourselves to the case of p an odd prime, except for some comments near the end.

Note that the varieties $\mathcal{N}_2 \cap \mathcal{B}_{p^n}$ are smaller than the variety $\mathcal{N}_2$. Clearly if a group $G \in \mathcal{N}_2 \cap \mathcal{B}_{p^n}$ is an amalgamation base for $\mathcal{N}_2$ (weak, strong, or special), then it is also an amalgamation base (of the same type) for $\mathcal{N}_2 \cap \mathcal{B}_{p^n}$, by Proposition 1.1; the converse, however, is not immediate and could fail. So a characterization of such bases is an interesting problem, whether it turns out that the converse does hold or that it does not.

It will be helpful to recall a theorem on adjunction of roots to $\mathcal{N}_2$.

THEOREM 1.3 (Saracino, Theorem 2.1 in [14]). *Let G be a nilpotent group of class at most two, let $m > 0$, let $\mathbf{n}$ be an m-tuple of positive* *integers, and let* $\mathbf{g}$ *be an* m*-tuple of elements of* G*. Then there exists a nilpotent group* K *of class two, containing* G*, and which contains an* n_i*-th root for* g_i $(1 \leq i \leq m)$ *if and only if for every* $m \times m$ *array* $\{c_{ij}\}$ *of integers such that* $n_i c_{ij} = n_j c_{ji}$ *for all* i *and* j*, and for all* $y_1, \ldots, y_m \in G$*,*

$$\text{if } y_j^{n_j} \equiv \prod_{i=1}^{m} g_i^{c_{ij}} \pmod{G'} \qquad \text{then } \prod_{j=1}^{m} [y_j, g_j] = e.$$

Note that the theorem implies that we can always adjoin n_i-th roots to a family of central elements, and so to a family of commutators. Therefore, if the g_i are n_i-th powers modulo the commutator in G, then there is an extension with n_i-th roots for the g_i.

We also have the following two results:

LEMMA 1.4. *Let* $G \in \mathcal{N}_2$ *and let* $g_1, \ldots, g_r \in Z(G)$*. For all* r*-tuple of postive integers* $\mathbf{n} = (n_1, \ldots, n_r)$ *there is an extension* K *of* G *with an* n_i*-th root* h_i *of* g_i*, and such that* h_i *is central in* K*, for* $1 \leq i \leq r$*.*

Proof. Let a_i be the order of g_i ($a_i = 0$ if g_i is not torsion); then consider the group $F = G \times (Z/n_1 a_1 Z) \times \cdots \times (Z/n_r a_r Z)$. Denote the generator of the cyclic groups by h_i, and mod out by the subgroup generated by $g_i h_i^{-n_i}$. Since the g_i are central, the subgroup is normal and this works out. ∎

LEMMA 1.5. *Let* $G \in \mathcal{N}_2$ *and let* $g_1, \ldots, g_r \in Z(G)$*. Then there exists an extension* K *of* G*, with* $K \in \mathcal{N}_2$*, and such that* $g_i \in [K, K]$ *for every* i*. If* G *is of odd exponent* n*, then we can choose* K *of exponent* n*. If* G *is of even exponent* $2n$*, then we can always choose* K *of exponent* $4n$*. If* G *is of even exponent* $2n$ *and the* g_i *are of exponent* n*, we may in fact choose* K *of exponent* $2n$*.*

Proof. Let a_i be the order of g_i, with $a_i = 0$ if g_i is not torsion. Consider the group

$$F = G \times \left((Z/a_1 Z) \coprod\nolimits^{\mathcal{N}_2} (Z/a_1 Z) \right) \times \cdots \times \left((Z/a_r Z) \coprod\nolimits^{\mathcal{N}_2} (Z/a_r Z) \right)$$

and denote the generators of the cyclic groups by t_{i1} and t_{i2}. Then mod out by the subgroup generated by $g_i[t_{i2}, t_{i1}]$. Again, since the g_i are central, this subgroup is normal. If G is of odd exponent n, then F is generated by elements of exponent n, and so it is again of exponent n, hence so is F/N.

If G is of exponent $2n$, then F is generated by elements of exponent $2n$ and so F/N is of exponent at most $4n$. If, furthermore, the g_i are of exponent n, then F is generated by a group of exponent $2n$ and elements of exponent n, so F is of exponent $2n$, and then so is F/N. ▮

The following lemma will be used several times:

LEMMA 1.6. *Let $G \in \mathcal{N}_2$, $x, y \in G$. Let K be any $\mathcal{N}_2$-extension of G, and assume that for some $n > 0$, $r, s \in K$, $r', s' \in K'$, we have that $x = r^n r'$ and $y = s^n s'$. For any integers a, b, c, and j, and all $g_1, g_2 \in G$,*

$$\textit{if} \quad \begin{array}{rcl} g_1^n & \equiv & x^a y^b \pmod{K'} \\ g_2^n & \equiv & x^{b+j} y^c \pmod{K'} \end{array} \quad \textit{then} \quad [r,s]^{jn} = [g_1,x][g_2,y].$$

In particular, $[r,s]^{jn}$ lies in G.

Proof. Since xr^{-n} and ys^{-n} are commutators in K, they are central. Therefore:

$$\begin{aligned} e &= [g_1 r^{-a} s^{-b-j}, xr^{-n}][g_2 r^{-b} s^{-c}, ys^{-n}] \\ &= [g_1,x][g_2,y][r^{-a}s^{-b-j},x][r^{-b}s^{-c},y][g_1,r^{-n}] \\ &\qquad [g_2,s^{-n}][s^{-b-j},r^{-n}][r^{-b},s^{-n}] \\ &= [g_1,x][g_2,y][r,x^{-a}y^{-b}][s,x^{-b-j}y^{-c}][g_1^{-n},r][g_2^{-n},s][r,s]^{-jn} \\ &= [g_1,x][g_2,y][r,g_1^{-n}][s,g_2^{-n}][g_1^{-n},r][g_2^{-n},s][r,s]^{-jn} \\ &= [g_1,x][g_2,y][r,s]^{-jn}. \end{aligned}$$

Therefore, $[r,s]^{jn} = [g_1,x][g_2,y]$, as claimed. ▮

We note that the condition depends only on the classes of x and y modulo G'; moreover, the congruences are actually symmetric on x and y. If you exchange x and y, then simply exchange a with c, and replace, for any $j \geq 0$, b with $-b-j$, g_1 with g_2^{-1}, and g_2 with g_1^{-1}.

Remark 1. 1. In many of our applications, the congruences in Lemma 1.6 will hold modulo G' rather than K'; however, since $G' \subseteq K'$, the Lemma will apply in that case as well.

Remark 1. 2. If x or y lie in G^nG', then we can always find integers a, b, and c, and elements g_1, $g_2 \in G$ satisfying the congruences in Lemma 1.6, so in that case $[r_1,r_2]^{jn} \in G$. For example, if $x \in G^nG'$, write $x = g^n g'$ and let $a = b = c = 0$, $g_1 = e$, and $g_2 = g^j$.

2. SPECIAL AMALGAMATION BASES

In this section we characterize the special amalgamation bases in the variety $\mathcal{N}_2 \cap \mathcal{B}_{p^n}$, with p an odd prime.

Note that if $K \in \mathcal{N}_2 \cap \mathcal{B}_{p^n}$, with p an odd prime, then, in view of Proposition 1.1, the dominion of a subgroup G of K in $\mathcal{N}_2 \cap \mathcal{B}_{p^n}$ is the same as the dominion in $\mathcal{N}_2$. The description of dominions is:

THEOREM 2.1 (Theorem 3.31 in [8]). *Let $K \in \mathcal{N}_2$, G a subgroup of K. Let D be the subgroup of K generated by G and all elements $[r,s]^q$, where $q > 0$ and $r^q, s^q \in G[K,K]$. Then $D = \mathrm{dom}_K^{\mathcal{N}_2}(G)$.*

Remark 2. 1. It is an easy exercise to verify that we can restrict q to prime powers.

Remark 2. 2. Also note that if $r^q, s^q \in G[K,K]$, and one of them has a q-th root in G modulo $[K,K]$, then $[r,s]^q \in G$. For if r^q has such a q-th root, then $r^q \equiv g^q \pmod{K'}$ for some $g \in G$, and

$$[r,s]^q = [r^q, s] = [g^q, s] = [g, s^q]$$

which lies in G since s^q is congruent to an element of G modulo K'. A similar calculation holds if s^q has a q-th root.

In particular, if K is of exponent p^n with p an odd prime, then in Theorem 2.1 we can restrict q to prime powers p^i, with $1 \leq i < n$.

In order to characterize special amalgamation bases, we will start with a group G, and we will be interested to know when, for any overgroup K, an element $[r,s]^q$ which lies in the dominion must also lie in G. The idea is to look at pairs of elements of G; call these x and y; we are thinking that $x = r^q r'$ and $y = s^q s'$, with $r, s \in K$ and $r', s' \in K'$. We want $[r,s]^q \in G$. Now, for any pair of elements, it is possible that no such extension exists (for example, if no $\mathcal{N}_2$-extension exists with q-th roots for both x and y); or else that the situation is possible, in which case we want a condition like that found in Lemma 1.6 to ensure that $[r,s]^q \in G$. So we will have a disjunction of conditions, where some of them are used to rule out the possiblity that $x = r^q r'$ and $y = s^q s'$ in some overgroup, and the rest are used to guarantee that, even if we *can* find such K, r, s, r', and s', we will nevertheless have $[r,s]^q \in G$.

When we work in $\mathcal{N}_2$, the only case in which there is no K with elements r, s, r', and s' with the given properties is when no $\mathcal{N}_2$-extension with q-th roots for x and y exists at all. This gives:

THEOREM 2.2 (Theorem 2.9 in [7]). *Let $G \in \mathcal{N}_2$. Then G is absolutely closed in $\mathcal{N}_2$ if and only if for all x, $y \in G$, and for all $n > 0$, one of the following holds:*

(i) There exist integers a, b, and c, and elements $g_1, g_2 \in G$ such that

$$\begin{aligned} g_1^n &\equiv x^a y^b \pmod{G'} \\ g_2^n &\equiv x^b y^c \pmod{G'} \end{aligned}$$

and $[g_1, x][g_2, y] \neq e$; or

(ii) There exist integers a, b, and c, and elements $g_1, g_2 \in G$ such that

$$\begin{aligned} g_1^n &\equiv x^a y^b \pmod{G'} \\ g_2^n &\equiv x^{b+1} y^c \pmod{G'}. \end{aligned}$$

Our main result follows. Its statement is more complicated than Theorem 2.2 because the conditions for the existence of an overgroup K in $\mathcal{N}_2 \cap \mathcal{B}_{p^n}$ with the property explained above are more complicated.

THEOREM 2.3. *Let $G \in \mathcal{N}_2 \cap \mathcal{B}_{p^n}$, where p is an odd prime and $n \geq 1$. Then G is absolutely closed in $\mathcal{N}_2 \cap \mathcal{B}_{p^n}$ if and only if for every $x, y \in G$, and each each i, $1 \leq i \leq n-1$, one of the following holds:*

(a) There exist $g_1, g_2 \in G$, and integers a, b, c such that

$$\begin{aligned} g_1^{p^i} &\equiv x^a y^b \pmod{G'} \\ g_2^{p^i} &\equiv x^b y^c \pmod{G'} \end{aligned}$$

and $[g_1, x][g_2, y] \neq e$; or

(b) $[x, y]^{p^\alpha} \neq e$ where $\alpha = \max\{0, n-2i\}$; or

(c) $2i > n$ and there exist $g_1, g_2 \in G$, integers a, b, and c, and an integer $j \in \{p^{n-i}, 2p^{n-i}, \ldots, p^i\}$ such that

$$\begin{aligned} g_1^{p^i} &\equiv x^a y^b \pmod{G'} \\ g_2^{p^i} &\equiv x^{b+j} y^c \pmod{G'} \end{aligned}$$

and $[g_1, x][g_2, y] \neq e$; or

(d) There exist $g_1, g_2 \in G$ and integers a, b, c such that

$$\begin{aligned} g_1^{p^i} &\equiv x^a y^b \pmod{G'} \\ g_2^{p^i} &\equiv x^{b+1} y^c \pmod{G'}. \end{aligned}$$

Remark 2. 3. Note that the conditions on x and y depend only on their congruence class modulo G', and that they all are symmetric on x and y. Also note that in (a), (b), and (c), we can restrict a, b, and c to being integers between 0 and $p-1$.

Remark 2. 4. One should think of conditions (a), (b), and (c) as "pre-emptive," and condition (d) as being "reactive." Condition (a) is nothing more than the statement that there is no $\mathcal{N}_2$-extension of G with p^i-th roots for both x and y. The way in which condition (a) works to prevent the existence of an extension in which x and y are p^i-th powers modulo the commutator is clear; for condition (b), note that if x and y are both p^i-th powers modulo the commutator, then their commutator is really the p^{2i}-th power of a commutator in the overgroup, so $[x,y]^{p^\alpha}$ should be trivial. For i bigger than $n/2$, condition (c) ensures that x^{n-i} would not commute with the possible p^i-th root of y, which would be a problem if our groups are really groups of exponent p^n and x is a p^i-th power modulo the commutator. The conjunction of the negation of (a), (b), and (c) are thus necessary for x and y to be p^i-th powers modulo the commutator in some extension of exponent p^n, and the proof will show that they are also sufficient; condition (d) comes to the rescue in the situation when x and y *are* p^i-th powers modulo the commutator in some extension of exponent p^n.

Proof. Suppose that G satisfies the conditions, and let K be any overgroup of G with $K \in \mathcal{N}_2 \cap \mathcal{B}_{p^n}$. Let $r, s \in K$ and $r', s' \in K'$ be such that $r^{p^i}r', s^{p^i}s' \in G$ for some i, $1 \leq i \leq n-1$. Let $x = r^{p^i}r'$, $y = s^{p^i}s'$. We want to verify that $[r,s]^{p^i} \in G$.

Note that since x and y are p^i-th powers modulo the commutator in K, then there is an extension of K in $\mathcal{N}_2$ with p^i-th roots for both x and y; therefore condition (a) cannot hold for x, y, and i. Likewise, condition (b) does not hold: if $2i \leq n$, then $[x,y]^{p^{n-2i}} = [r,s]^{p^n} = e$, since K is of exponent p^n; and if $2i > n$, then $e = [r,s]^{p^{2i}} = [x,y]$.

We also have that x, y, and i cannot satisfy (c). For suppose that $2i > n$ and for some $j \in \{p^{n-i}, 2p^{n-i}, \ldots, p^i\}$, we have the congruences

$$\begin{aligned} g_1^{p^i} &\equiv x^a y^b \pmod{G'} \\ g_2^{p^i} &\equiv x^{b+j} y^c \pmod{G'} \end{aligned}$$

with $[g_1,x][g_2,y]\neq e$. Then by Lemma 1.6, we have $[r,s]^{jp^i}\neq e$. However, for all such j, jp^i is a multiple of p^n, hence $[r,s]^{jp^i}=e$ in K. Since we fail conditions (a)–(c), condition (d) must be satisfied, and then by Lemma 1.6 it follows that

$$[r,s]^{p^i}=[g_1,x][g_2,y]\in G.$$

So if the conditions hold, then G is absolutely closed, as claimed.

Necessity is more complicated. We assume that G, x, y, and i do not satisfy the conditions (a)–(d). We will proceed as follows: first we construct an extension where x and y have p^i-th roots r and s, respectively, and where $[r,s]^{p^i}\notin G$. Then, if necessary, we make $[r,s]^{p^n}$ trivial. Then we verify that $x^{p^{n-i}}$ and $y^{p^{n-i}}$ are central in this extension, so we can adjoin *central* p^n-th roots t and v, respectively, to each. Next, we make t^{p^i} and v^{p^i} into commutators. Then rt^{-1} and sv^{-1} are of exponent p^n, and their p^i-th powers are, modulo commutators, x and y. Finally, we select a suitable subgroup which is generated by elements of exponent p^n. If we ensure that throughout the process $[r,s]^{p^i}\notin G$, we will have an example where G is not equal to its dominion. We now proceed with this program:

STEP 1. To facilitate notation, we write x_1 and x_2 rather than x and y (respectively) in this step. Let $K_0=\left(G\coprod^{\mathcal{N}_2}(Z\coprod^{\mathcal{N}_2}Z)\right)/N$, where we denote the generators of the two copies of Z by r_1 and r_2, and N is the normal subgroup generated by $x_1r_1^{-p^i}$, $x_2r_2^{-p^i}$.

We need to verify that $N\cap G=\{e\}$ and that for every $g\in G$ we have $g[r_1,r_2]^{-p^i}\notin G$. The argument is the same as that found in the proof of Theorem 2.9 in [7], so we only sketch it here:

A general element of N can be written as

$$\prod_{j=1}^{2}\left(\prod_{k=1}^{s_j}(b_{jk}z_{jk})(x_jr_j^{-p^i})^{\epsilon_{jk}}(b_{jk}z_{jk})^{-1}\right)$$

where $b_{jk}\in G$, s_j is a positive integer, $\epsilon_{jk}=\pm1$, and $z_{jk}=r_1^{a_{jk1}}r_2^{a_{jk2}}$. By rewriting the element and expanding brackets, we get that the element equals:

$$\prod_{j=1}^{2}\left(\prod_{k=1}^{s_j}[b_{jk},x_j]^{\epsilon_{jk}}[b_{jk},r_j^{-p^i}]^{\epsilon_{jk}}[z_{jk},x_j]^{\epsilon_{jk}}[z_{jk},r_j^{-p^i}]^{\epsilon_{jk}}\right)(x_jr_j^{-p^i})^{t_j}$$

where $t_j=\sum_{k=1}^{s_j}\epsilon_{jk}$.

Say that this is equal to an element of $G\coprod^{\mathcal{N}_2}(Z\coprod^{\mathcal{N}_2}Z)$ of the form $g[r_1,r_2]^{qp^i}$ for some integer q and some $g\in G$. Writing this as $\alpha\beta\gamma$ with

$\alpha \in G$, $\beta \in Z \coprod^{\mathcal{N}_2} Z$, $\gamma \in [G, Z \coprod^{\mathcal{N}_2} Z]$, we obtain equations:

$$\begin{aligned} g &= [h_1, x_1][h_2, x_2] \\ [r_2, r_2]^{qp^i} &= [r_1, r_2]^{p^i(c_{21}-c_{12})} \\ e &= [h_1^{-p^i} x_1^{c_{11}} x_2^{c_{21}}, r_1][h_2^{-p^i} x_1^{c_{12}} x_2^{c_{22}}, r_2] \end{aligned}$$

where $h_j = \prod_k b_{jk}^{\epsilon_{jk}}$, and $c_{ij} = -\sum \epsilon_{ik} a_{ikj}$. This means that

$$\begin{aligned} h_1^{p^i} &\equiv x_1^{c_{11}} x_2^{c_{21}} \pmod{G'} \\ h_2^{p^i} &\equiv x_1^{c_{12}} x_2^{c_{22}} \pmod{G'}. \end{aligned}$$

For $q = 0$, we have $c_{12} = c_{21}$, so the failure of condition (a) shows that $N \cap G = \{e\}$. By setting $q = -1$, we get that $c_{12} = c_{21} + 1$, so we are in the situation of (d); the fact that (d) fails in G means that no such solution exists, so no element of the form $g[r_1, r_2]^{-p^i}$ lies in N. Therefore, in K_0 we have $[r_1, r_2]^{p^i}$ lies in the dominion of G but not in G. We change the names of r_1 and r_2 to r and s respectively, and we again write x and y instead of x_1 and x_2, respectively, to reduce the number of indices.

So we have that K_0 is an extension of G, generated by G, r and s; that $r^{p^i} = x$, $s^{p^i} = y$, and that $[r, s]^{p^i} \notin G$. We also note that $[r, s]^{p^{2i+\alpha}} \in N$, with α as defined in (b). Indeed, $[xr^{-p^i}, ys^{-p^i}]^{p^\alpha} \in N$, and

$$[xr^{-p^i}, ys^{-p^i}]^{p^\alpha} = [x, y]^{p^\alpha}[x, s]^{-p^{i+\alpha}}[r, y]^{-p^{i+\alpha}}[r, s]^{p^{2i+\alpha}}.$$

However, $[x, y]^{p^\alpha} = e$ since G fails (b). And also

$$[x, s]^{-p^{i+\alpha}} = [x, y]^{p^\alpha}[x, s^{-p^i}]^{p^\alpha} = [x, ys^{-p^i}]^{p^\alpha} = (ys^{-p^i})^x ys^{-p^i} \in N,$$

and similarly with $[r, y]^{-p^{i+\alpha}}$. So $[r, s]^{p^{2i+\alpha}} \in N$.

We also claim that if $2i > n$ and $g[r, s]^{kp^n} \in N$ with $k \in \{1, 2, \ldots, p^{2i-n}\}$, then $g = e$. Indeed, if we write

$$g[r, s]^{kp^n} = g[r, s]^{p^i(kp^{n-i})}$$

we note that $kp^{n-i} \in \{p^{n-i}, 2p^{n-i}, \ldots, p^i\}$, and so by construction of K_0 we must have $g_1, g_2 \in G$, and a, b, c integers with

$$\begin{aligned} g_1 &\equiv x^a y^b \pmod{G'} \\ g_2 &\equiv x^{b+kp^{n-i}} y^c \pmod{G'} \end{aligned}$$

with $g = [g_1, x][g_2, y]$. However, the failure of condition (c) means that in this case, $g = e$, as claimed.

STEP 2. Let $K_1 = K_0/M$ where M is the normal subgroup generated by $[r,s]^{p^n}$. We claim that $M \cap G = \{e\}$, and that $g[r,s]^{p^i} \notin M$ for each $g \in G$.

Indeed, if $2i \leq n$, then M is actually trivial already, since $[r,s]^{p^{2i+\alpha}} \in N$; if $2i > n$, then we know that $[r,s]^{kp^n} \notin G$ for $k = 1, 2, \ldots, p^{2i-n}$ again by the observations above; thus $M \cap G = \{e\}$; and we can verify by inspection that $g[r,s]^{p^i} \notin M$ for each $g \in G$, since $[r,s]^{p^n}$ is central.

So $[r,s]^{p^i}$ is still in the dominion of G but not in G.

STEP 3. We note that $x^{p^{n-i}}$ and $y^{p^{n-i}}$ are central in K_1. That they are central in G follows from the fact that x and y are p^i-th powers in K_1 modulo K_1', and G is of exponent p^n. Thus, if $g \in G$, then

$$[x^{p^{n-i}}, g] = [x, g]^{p^{n-i}} = [r^{p^i}, g]^{p^{n-i}} = [r, g]^{p^n} = [r, g^{p^n}] = [r, e] = e$$

and analogous with y. Since K_1 is generated by G, r, and s, it suffices to show that $x^{p^{n-i}}$ and $y^{p^{n-i}}$ commute with both r and s. That $x^{p^{n-i}}$ commutes with r is obvious. To verify that it commutes with s we note that:

$$[x^{p^{n-i}}, s] = [(r^{p^i})^{p^{n-i}}, s] = [r^{p^n}, s] = [r, s]^{p^n} = e$$

since we moded out by $\langle [r,s]^{p^n} \rangle$. An analogous calculation holds for $y^{p^{n-i}}$.

STEP 4. We adjoin central p^n-th roots to both $x^{p^{n-i}}$ and $y^{p^{n-i}}$. We use Lemma 1.4 and adjoin roots, which we call t and v, respectively. Call the resulting group K_2. In K_2 we still have $[r,s]^{p^i}$ in the dominion of G but not in G, as can be easily verified.

STEP 5. We make t^{p^i} and v^{p^i} into commutators. We can use Lemma 1.5, since t and v are central. Denote the resulting group by K_3, and write $t^{p^i} = [q_1, q_2]$, $v^{p^i} = [q_3, q_4]$. Moreover, since $(t^{p^i})^{p^n} = (x^{p^{n-i}})^{p^i} = x^{p^n} = e$, and likewise $(s^{p^i})^{p^n} = e$, we can choose the q_i to be of exponent p^n each. Again, we still have $[r,s]^{p^i}$ in the dominion of G but not in G. Note, moreover, that t and v are still central in K_3.

STEP 6. Let K_4 be the subgroup of K_3 generated by G, rt^{-1}, sv^{-1}, q_1, q_2, q_3, and q_4.

Note that rt^{-1} and sv^{-1} are of exponent p^n. Indeed, since t and v are central, we have:

$$(rt^{-1})^{p^n} = x^{p^{n-i}} x^{-p^{n-i}} = e = y^{p^{n-i}} y^{-p^{n-i}} = (sv^{-1})^{p^n}.$$

Since each q_i and each element of G is also of exponent p^n, K_4 lies in $\mathcal{N}_2\cap\mathcal{B}_{p^n}$; moreover, it is an overgroup of G, and

$$\begin{aligned}(rt^{-1})^{p^i}[q_1,q_2] &= r^{p^i}t^{-p^i}[q_1,q_2]=r^{p^i}=x,\\ (sv^{-1})^{p^i}[q_3,q_4] &= s^{p^i}v^{-p^i}[q_3,q_4]=s^{p^i}=y.\end{aligned}$$

So $[rt^{-1},sv^{-1}]^{p^i}$ lies in $\mathrm{dom}_{K_4}^{\mathcal{N}_2}(G)$. Since t and v are central in K_4, $[rt^{-1},sv^{-1}]^{p^i}=[r,s]^{p^i}$, which we know does not lie in G; therefore, G cannot be absolutely closed in $\mathcal{N}_2\cap\mathcal{B}_{p^n}$.

This establishes necessity, and proves the theorem. ▮

We note in particular that in the case $n=1$ the conditions are all vacuous. Indeed,

Corollary 2.4 (see Corollary 9.90 in [8]). *If p is an odd prime, then every $G\in\mathcal{N}_2\cap\mathcal{B}_p$ is absolutely closed (in that variety).*

Remark 2. 5. This corollary also follows from Corollary 1.3 in [11].

3. SPECIAL BASES AND ABSOLUTE CLOSURES

Given the similarity between the characterization of special amalgamation bases in $\mathcal{N}_2$ and in $\mathcal{N}_2\cap\mathcal{B}_{p^n}$, a natural question is whether there are any groups which are special amalgamation bases in the latter but not the former variety. Note that every special amalgamation base for $\mathcal{N}_2$ which lies in $\mathcal{N}_2\cap\mathcal{B}_{p^n}$ is necessarily also a special amalgamation base for $\mathcal{N}_2\cap\mathcal{B}_{p^n}$.

When we start looking for possible counterexamples to the converse implication, we might be tempted to try with easy groups, such as abelian groups or groups of exponent p. However, no such example will work. Recall that:

Theorem 3.1 (Theorems 3.7, 3.13 and 3.17 in [7]). *Let G be a nilpotent group of class two.*

(i) If G is abelian, then G is absolutely closed in $\mathcal{N}_2$ if and only if G/pG is cyclic or trivial for each prime p. Therefore, cyclic groups are absolutely closed in $\mathcal{N}_2$.

(ii) If G is of exponent p, for p a prime, then G is absolutely closed in $\mathcal{N}_2$ if and only if $Z(G)/G'$ is cyclic or trivial.

For $n>1$ we get the same result here:

THEOREM 3.2. *Let $G \in \mathcal{N}_2 \cap \mathcal{B}_{p^n}$, with p an odd prime and $n > 1$.*

(i) If G is abelian, then G is absolutely closed in $\mathcal{N}_2 \cap \mathcal{B}_{p^n}$ if and only if G/pG is cyclic or trivial. Therefore, G is absolutely closed if and only if it is cyclic or trivial.

(ii) If G is of exponent p, then G is absolutely closed in $\mathcal{N}_2 \cap \mathcal{B}_{p^n}$ if and only if $Z(G)/G'$ is cyclic or trivial.

Remark 3. 1. Note that if G is of exponent a power of p, then G is q-divisible for every integer q relatively prime to p; therefore, if q is a prime different from p, then G/qG is trivial, which means that condition *(i)* of Theorem 3.2 is the same as condition *(i)* in Theorem 3.1.

Proof. *(i)* As per the remark above, if G/pG is cyclic or trivial, then G is absolutely closed in $\mathcal{N}_2$, and hence also in $\mathcal{N}_2 \cap \mathcal{B}_{p^n}$. If G/pG is not cyclic, then let x and y be elements of G which project to distinct cyclic summands of G/pG. Then note that a product $x^a y^b$ is a p-th power in G if and only if $p|a$ and $p|b$. Suppose G is absolutely closed. Since G is abelian, G, x, y, and p fail conditions (a)-(c) of Theorem 2.3. Hence, it must satisfy (d), so there exist integers a, b, and c, and elements $g_1, g_2 \in G$ such that

$$\begin{aligned} g_1^p &= x^a y^b \\ g_2^p &= x^{b+1} y^c. \end{aligned}$$

But that means that $p|b$ and $p|(b+1)$ which is clearly impossible. So G is not absolutely closed. That this implies that G is cyclic follows because G/p^nG is a sum of cyclic groups, and has the same number of summands as G/pG, but since G is of exponent p^n the former group is equal to G itself.

(ii) Again, if $Z(G)/G'$ is cyclic or trivial, then G is absolutely closed in $\mathcal{N}_2$ and hence also in $\mathcal{N}_2 \cap \mathcal{B}_{p^n}$. If not, let x and y be elements of $Z(G)$ which project to distinct cyclic generators of $Z(G)/G'$ (which is an abelian group of exponent p). Again, G, x, y and p do not satisfy conditions (a)–(c) of Theorem 2.3, and the same argument as above (with congruences rather than equalities) yields a contradiction to the assumption that it satisfies (d). ∎

Fortunately, once we get past the urge to check those easy cases, we find the examples we were looking for. We need a technical lemma:

LEMMA 3.3 (Perturbation argument). *Let $G \in \mathcal{N}_2$, H a subgroup of G, and let $x^q, y^q \in H[G,G]$. If $h_1, h_1 \in H$ then*

$$[x,y]^q \in H \Longleftrightarrow [xh_1, yh_2]^q \in H.$$

Proof. Note that if x^q and y^q lie in $H[G,G]$, then so do $(xh_1)^q$ and $(yh_2)^q$, so both commutator brackets lie in the dominion of H.

Expanding the bracket bilinearly, we have

$$\begin{aligned}[xh_1, yh_2]^q &= [x,y]^q[h_1,y]^q[x,h_2]^q[h_1,h_2]^q\\ &= [x,y]^q[h_1,y^q][x^q,h_2][h_1,h_2]^q.\end{aligned}$$

Since $x^q, y^q \in H[G,G]$, and $h_i \in H$, the last three terms on the right hand side lie in H, so the left hand side lies in H if and only if $[x,y]^q$ lies in H, as claimed. ▮

THEOREM 3.4. *Let p be an odd prime. For every $n>0$ there exists a group $G\in\mathcal{N}_2\cap\mathcal{B}_{p^n}$, with p an odd prime, such that G is absolutely closed in $\mathcal{N}_2\cap\mathcal{B}_{p^n}$, but not absolutely closed in $\mathcal{N}_2\cap\mathcal{B}_{p^{n+1}}$. Namely, we let*

$$G = \left\langle x,y \;\middle|\; x^{p^n} = y^{p^n} = [x,y]^{p^{n-1}} = e\right\rangle.$$

Proof. If $n=1$, the result is easy: the group described is a special amalgamation base of $\mathcal{N}_2\cap\mathcal{B}_p$ (since everything there is a special amalgamation base), but not a special amalgamation base in $\mathcal{N}_2$ or $\mathcal{N}_2\cap\mathcal{B}_{p^m}$ for $m>1$, as it is abelian but not cyclic. So we may assume that $n>1$.

Consider the group

$$A = \left\langle a,b \mid a^{p^{n+1}} = b^{p^{n+1}} = [a,b]^{p^{n+1}} = e\right\rangle.$$

Then G is isomorphic to the subgroup generated by a^p and b^p; however, $[a,b]^p\in \mathrm{dom}_A^{\mathcal{N}_2}(G)\setminus G$, so G cannot be absolutely closed in $\mathcal{N}_2\cap\mathcal{B}_{p^{n+1}}$. We only need to show now that G is absolutely closed in $\mathcal{N}_2\cap\mathcal{B}_{p^n}$.

Let K be an $\mathcal{N}_2\cap\mathcal{B}_{p^n}$-overgroup of G, and suppose that $r^{p^i}, s^{p^i}\in G[K,K]$ for some $r,s,\in K$, $1\leq i\leq n-1$. We may assume that:

$$\begin{aligned} r^{p^i} &\equiv x^{\alpha}y^{\beta} \pmod{K'}\\ s^{p^i} &\equiv x^{\gamma}y^{\delta} \pmod{K'}.\end{aligned}$$

We want to show that $[r,s]^{p^i}$ lies in G. We write $\mathbf{x}=x^\alpha y^\beta$ and $\mathbf{y}=x^\gamma y^\delta$.

Since $r^{p^n}=s^{p^n}=e$, we must have that $x^{\alpha p^{n-i}}y^{\beta p^{n-i}}$ is central in G, and likewise with $x^{\gamma p^{n-i}}y^{\delta p^{n-i}}$. By calculating the commutators with x and y, we get that p^{n-1} divides αp^{n-i}, βp^{n-i}, γp^{n-i}, and δp^{n-i}. Therefore p^{i-1} divides α, β, γ, and δ.

So we rewrite the elements as $\mathbf{x} = x^{\zeta p^{i-1}} y^{\eta p^{i-1}}$, $\mathbf{y} = x^{\vartheta p^{i-1}} y^{\lambda p^{i-1}}$. Since we can perturb r and s with elements of G by the Perturbation Argument, we can perturb $\mathbf{x}$ and $\mathbf{y}$ by p^i-th powers of x and y, so we may assume that $0 \leq \zeta, \eta, \vartheta, \lambda < p$.

If $2i \leq n$, then we have that

$$[\mathbf{x}, \mathbf{y}]^{p^{n-2i}} = [r^{p^i}, s^{p^i}]^{p^{n-2i}} = [r, s]^{p^n} = e$$

since K is of exponent p^n. Substituting the values of $\mathbf{x}$ and $\mathbf{y}$, we have

$$\begin{aligned} [\mathbf{x}, \mathbf{y}]^{n-2i} &= [x^{\zeta p^{i-1}} y^{\eta p^{i-1}}, x^{\vartheta p^{i-1}} y^{\lambda p^{i-i}}]^{p^{n-2i}} \\ &= [x, y]^{(\zeta\lambda - \eta\vartheta)p^{n-2i+2i-2}} \\ &= [x, y]^{(\zeta\lambda - \eta\vartheta)p^{n-2}}. \end{aligned}$$

Therefore, $p|\zeta\lambda - \eta\vartheta$. If we consider the vectors (ζ, η) and (ϑ, λ) as being vectors over $\mathbf{Z}/p\mathbf{Z}$, this means that the two vectors are proportional. If one of them is the zero vector, then either $\mathbf{x}$ or $\mathbf{y}$ are the identity element, and there is nothing to do, e.g.:

$$[r, s]^{p^i} = [\mathbf{x}, s] = [e, s] = e \in G.$$

So by perturbing r and s by elements of G we may assume that $\mathbf{x}^k \equiv \mathbf{y} \pmod{K'}$ for some $k \geq 0$. But that means that $[r, s]^{p^i}$ actually lies in the dominion of the subgroup generated by $\mathbf{x}$, which is cyclic and hence absolutely closed; thus, $[r, s]^{p^i} \in \langle \mathbf{x} \rangle \subseteq G$, as desired. (In essence, since $\mathbf{y}$ is a power of $\mathbf{x}$, we can always find exponents to satisfy condition (d); see Theorem 3.7 in [7]).

If, on the other hand, $2i > n$, then by choosing $j = p^{n-i}$, $a = b = c = 0$, $g_1 = e$, and $g_2 = x^{\zeta p^{n-i-1}} y^{\eta p^{n-i-1}}$, we have:

$$\begin{aligned} e^{p^i} &\equiv \mathbf{x}^0 \mathbf{y}^0 \\ &\equiv \left(x^{\zeta p^{i-1}} y^{\eta p^{i-1}}\right)^0 \left(x^{\vartheta p^{i-1}} y^{\lambda p^{i-1}}\right)^0 \pmod{G'} \end{aligned}$$

$$\begin{aligned} \left(x^{\zeta p^{n-i-1}} y^{\eta p^{n-i-1}}\right)^{p^i} &\equiv \mathbf{x}^{0+p^{n-i}} \mathbf{y}^0 \\ &\equiv \left(x^{\zeta p^{i-1}} y^{\eta p^{i-1}}\right)^{p^{n-i}} \left(x^{\vartheta p^{i-1}} y^{\lambda p^{i-1}}\right)^0 \pmod{G'} \end{aligned}$$

so by Lemma 1.6, $[r, s]^{p^{n-i}p^i} = [e, \mathbf{x}][x^{\zeta p^{n-i-1}} y^{\eta p^{n-i-1}}, \mathbf{y}]$. However,

$$[r, s]^{p^{n-i}p^i} = [r, s]^{p^n} = e,$$

so we have:

$$\begin{aligned}
e &= [r,s]^{p^n}\\
&= [x^{\zeta p^{n-i-1}} y^{\eta p^{n-i-1}}, \mathbf{y}]\\
&= [x^{\zeta p^{n-i-1}} y^{\eta p^{n-i-1}}, x^{\vartheta p^{i-1}} y^{\lambda p^{i-1}}]\\
&= [x,y]^{(\zeta\lambda-\eta\vartheta)p^{n-i-1+i-1}}\\
&= [x,y]^{(\zeta\lambda-\eta\vartheta)p^{n-2}}
\end{aligned}$$

so again we have that $p|\zeta\lambda - \eta\vartheta$, and we proceed as we did in the case when $2i \leq n$.

Therefore, G is absolutely closed in $\mathcal{N}_2 \cap \mathcal{B}_{p^n}$, as claimed. ▮

We pause here to adress two questions concerning dominions.

In [1], P.M. Higgins asks whether the morphic image of an absolutely closed semigroup is necessarily absolutely closed. Higgins' question is in the context of the category of all semigroups, so we cannot answer his question, but we can answer the corresponding question for arbitrary categories of algebras, and even for some varieties of semigroups. The answer is negative.

Namely, the variety $\mathcal{N}_2 \cap \mathcal{B}_{p^n}$ can be defined entirely with semigroup identities; i.e. identities in which no negative number appears as an exponent. To do this, take the group identities that define the variety, replace x^{-1} with x^{p^n-1} wherever it occurs, and add identities that specify that x^{p^n} acts as a two sided identity. The semigroups which lie in this variety are just the groups in $\mathcal{N}_2 \cap \mathcal{B}_{p^n}$, so the description of dominions in this variety of semigroups is exactly the same as the description in the variety of groups. In particular, the description of which groups are absolutely closed is the same as the one given.

Now consider the $\mathcal{N}_2$-group presented by

$$G = \langle x,y,z | x^p = y^p = z^p = [x,y]^p = [x,z] = [y,z] = e\rangle.$$

This group is absolutely closed in $\mathcal{N}_2$ and so also in $\mathcal{N}_2 \cap \mathcal{B}_{p^n}$ for any n, since it is of exponent p and $Z(G)/G'$ is generated by the image of z. However, its abelianization is isomorphic to the product of three copies of Z/pZ, and therefore cannot be absolutely closed. Thus, G is absolutely closed and has a morphic image which is not absolutely closed.

It may be worth noting as well that the subgroup we are moding out by when considering the morphic image is the commutator subgroup, which is a fully invariant central subgroup of G, and will be normal (in fact, central) in any overgroup K of G (provided $K \in \mathcal{N}_2$). Since dominions respect quotients, any witness to the fact that G^{ab} is not absolutely closed cannot come from an overgroup K of G.

Note also that since dominions respect quotients, if G^{ab} is absolutely closed, then G is absolutely closed as well.

We also take this opportunity to address the notion of absolute closures. Recall that Isbell proves in [3] that in a right-closed category of algebras (in the sense of Universal Algebra), every algebra can be embedded in an absolutely closed algebra; since varieties are right-closed, this implies that any group $G \in \mathcal{N}_2 \cap \mathcal{B}_{p^n}$ can be embedded in a group which is absolutely closed in the same variety (the same holds for $\mathcal{N}_2$). Isbell then gives the following definition:

Definition 3.5. *An absolute closure for the algebra B (in the category $\mathcal{C}$) is an absolutely closed $\mathcal{C}$-algebra D containing B, and such that there exists some overalgebra A of B, with $A \in \mathcal{C}$ and $\mathrm{dom}_A^{\mathcal{C}}(B) = D$.*

In other words, an absolute closure for B is an absolutely closed algebra D which is dominated by B in some overalgebra. In [3], Isbell notes that he did not know if absolute closures existed in general. Soon after, however, examples of algebras with no absolute closure were constructed.

On the other hand, it is not hard to construct examples of algebras with an absolute closure, aside from the absolutely closed algebras themselves. If we work in the variety of all semigroups, then every group is absolutely closed (this is an easy consequence of the Zigzag Lemma [3]). If we take a subsemigroup S of a group G, with S not a group, then the dominion of S in G will be the subgroup generated by S; this is absolutely closed (since it is a group), and properly contains S. Thus the subgroup of G generated by S is an absolute closure for S. For example $\mathrm{dom}_{\mathbf{Z}}(\mathbf{N}) = \mathbf{Z}$, and $\mathbf{Z}$ is absolutely closed, hence an absolute closure for $\mathbf{N}$.

I was unaware that examples of algebras with no absolute closure had already been constructed, so I began to see if the characterization of absolutely closed groups in $\mathcal{N}_2$ given in [7] could be used to construct such groups and to describe absolute closures. It turns out, as we will prove below, that in this variety we have the worst possible case of a negative answer to Isbell's original question; namely, a group has an absolute closure in $\mathcal{N}_2$ if and only if it is already absolutely closed. The same result holds for $\mathcal{N}_2 \cap \mathcal{B}_{p^n}$.

Theorem 3.6. *Let $G \in \mathcal{N}_2$. Then G has an absolute closure in $\mathcal{N}_2$ if and only if G is already absolutely closed in $\mathcal{N}_2$. The same is true if we replace $\mathcal{N}_2$ with $\mathcal{N}_2 \cap \mathcal{B}_{p^k}$ for an odd prime p and $k > 0$.*

Proof. We prove the $\mathcal{N}_2$ case first. One implication is trivial. So suppose that G is not absolutely closed, and let K be any nil-2 overgroup of G. Let $D_K = \mathrm{dom}_K^{\mathcal{N}_2}(G)$. We want to show that D_K is also not absolutely closed.

If $G = D_K$, then there is nothing to prove. So we may assume that G is properly contained in D_K. By the description of dominions from Theorem 2.1, there exist elements $r, s \in K$, and $n > 0$ such that r^n and s^n lie in $G[K,K]$, and $[r,s]^n$ does not lie in G (it lies in D_K, however). We write $[r,s]^n = d_0$. Note that the description of dominions also implies that $[D_K, D_K] = [G,G]$.

Choose $r', s' \in [K,K]$ such that $r^n r', s^n s' \in G$, and write $r^n r' = x$, $s^n s' = y$. We claim that x, y, and n do not satisfy either condition *(i)* or condition *(ii)* from Theorem 2.2 with respect to D_K. This will show that D_K is not absolutely closed, as these conditions describe the nil-2 absolutely closed groups.

Condition *(i)* is the easiest to handle: condition *(i)* is equivalent to the statement that no nil-2 extension of D_K contains n-th roots for both x and y. However, x and y are both n-th powers modulo the commutator in K, so there is a nil-2 extension of K with n-th roots for both x and y. In particular, there is such an extension of D_K as well. So x, y, and n do not satisfy condition *(i)*.

Condition *(ii)* is somewhat more difficult, and we proceed by contradiction. Suppose that condition *(ii)* is satisfied in D_K. Then there exist integers a, b, and c, and elements $d_1, d_2 \in D_K$ such that

$$\begin{aligned} {d_1}^n &\equiv x^a y^b \pmod{{D_K}'} \\ {d_2}^n &\equiv x^{b+1} y^c \pmod{{D_K}'}. \end{aligned}$$

By Theorem 2.1, we can write d_i as

$$d_i = g_i[k_{i11}, k_{i12}]^{n_{i1}} \cdots [k_{is_i1}, k_{is_i2}]^{n_{is_i}}$$

for some $g_i \in G$, $s_i \geq 0$, $k_{ijk} \in K$, with $k_{ijk}^{n_{ij}} \in G[K,K]$. In particular, $d_i \equiv g_i \pmod{K'}$.

By Lemma 1.6, the congruences above imply that $[r,s]^n = [d_1, x][d_2, y]$. However, since $d_i \equiv g_i \pmod{K'}$, this means that $[r,s]^n = [g_1, x][g_2, y]$. But the right hand side clearly lies in G, and this contradicts the choice of r and s. So D_K, x, y, and n cannot satisfy (ii) either.

Therefore D_K is not absolutely closed. We conclude that G has no absolute closure, which proves the $\mathcal{N}_2$ case.

For the $\mathcal{N}_2 \cap \mathcal{B}_{p^k}$ case, we proceed exactly as above, replacing n with p^i, and restricting K to the variety in question. The argument above shows that G, x, y and p^i do not satisfy (a) nor (d) of Theorem 2.3. If $[x,y]^{p^\alpha} \neq e$, then $[r,s]^{p^{2i+\alpha}} \neq e$, which contradicts the fact that K is of exponent p^k, since $k \leq 2i + \alpha$. Thus, they cannot satisfy (b). For (c), if $2i > k$ and there is a solution to the congruences with $[d_1, x][d_2, y] \neq e$, then by Lemma 1.6 we have that $[r,s]^{jp^i} \neq e$. But $p^k | jp^i$, which again contradicts the fact that K is of exponent p^k. Therefore, D_K is also not absolutely closed in this case, and this proves the theorem in full. ▮

4. WEAK AND STRONG BASES

The case of weak and strong amalgamation bases turns out to be remarkably simpler than the special amalgamation case. The results are just simple applications of our observations at the beginning, plus some trivial checks on the known conditions for embeddability. We include them for completeness, and to note some differences between this case and the special amalgamation case above.

Necessary and sufficient conditions for strong embeddability of an amalgam of $\mathcal{N}_2$-groups were first given by Wiegold in [15]. However, they were hard to work with, as they involved checking the existence of maps with certain properties on some tensor products. Later, B. Maier gave easier conditions for both weak and strong embeddability. We recall those conditions now:

THEOREM 4.1 (B. Maier, Hauptsatz in [9]). *The amalgam $(G,K;H)$ of nil-2 groups is weakly embeddable in $\mathcal{N}_2$ if and only if*

(i) $G' \cap H \subseteq Z(K)$ and $K' \cap H \subseteq Z(G)$; and

(ii) For all $s > 0$, $q_i > 0$, $g_i \in G$, $g_i' \in G'$, $k_i \in K$, $k_i' \in K'$ with $g_i^{q_i} g_i', k_i^{q_i} k_i' \in H$, $1 \leq i \leq s$, for all $h \in H$ we have

$$\prod_{i=1}^{s} [g_i, k_i^{q_i} k_i'] = h \Longleftrightarrow \prod_{i=1}^{s} [g_i^{q_i} g_i', k_i] = h.$$

THEOREM 4.2 (B. Maier, Satz 3 in [10]). *The amalgam $(G,K;H)$ of nil-2 groups is strongly embeddable in $\mathcal{N}_2$ if and only if*

(i) $G' \cap H \subseteq Z(K)$ and $K' \cap H \subseteq Z(G)$;

(ii) For each prime power p^i, if $g \in G$, $g' \in G'$, $k \in K$, $k' \in K'$ are such that $g^{p^i} g', k^{p^i} k' \in H$, we have

$$[g^{p^i} g', k] = [g, k^{p^i} k'] \in H.$$

We can make some simplifying observations:

Remark 4. 1. In Theorem 4.1, if H is k-divisible, then we can restrict q_i to be different from k. This because if H is q_i divisible, we can replace $g^{q_i}g'$ and $k^{q_i}k'$ with some q_i-th power of an element of H, and then that commutator will certainly lie in H, and we can omit it from the given condition. In particular, if H is (or if G and K are) of exponent n, we can restrict q_i to numbers not relatively prime to n. If G and K both lie in $\mathcal{B}_{p^n}$, we can restrict the q_i to be powers of p. In that case, we can further restrict them to being powers p^i with $i < n$, since otherwise the corresponding commutator is trivial.

Remark 4. 2. Likewise, in Theorem 4.2, if G and K are both of finite exponent n, we can restrict the primes p to prime factors of n. If G and K are of exponent p^n, we can also restrict the prime powers to p^i, with $1 \leq i \leq n-1$.

Remark 4. 3. The conditions and simplifications given reduce, in the case $n = 1$, to the existence of a "coupled central series" of G and K over H, as described in Definition 1.1 and Theorem 1.2 in [11].

The characterization of weak and strong bases in $\mathcal{N}_2$ was given by Saracino:

THEOREM 4.3 (Saracino, Theorem 3.3 in [14]). *Let $G \in \mathcal{N}_2$. The following are equivalent:*

(a) G is a weak amalgamation base for $\mathcal{N}_2$.

(b) G satisfies $G' = Z(G)$, and $\forall g \in G, \forall n > 0$ ($g \in G^nG'$ or $\exists y \in G$ and $\exists k \in \mathbf{Z}$ such that ($y^n \equiv g^k \pmod{G'}$ and $[y, g] \neq e$)).

(c) G satisfies $G' = Z(G)$, and for all $g \in G$ and $n > 0$, either g has an n-th root modulo G' or else g has no n-th root in any overgroup $K \in \mathcal{N}_2$ of G.

(d) G is a strong amalgamation base for $\mathcal{N}_2$.

Remark 4. 4. Again it is not hard to verify that in Theorem 4.3 we can restrict n to prime powers. Moreover, the conditions for adjunction of roots show that if G is of exponent n, then an element of G has an n-th root in some $\mathcal{N}_2$-overgroup if and only if it is central. An easy way to verify it is to note that an element x which is a k-th power in some extension must commute with every element of exponent k in G; for if r is such a root in some overgroup, and y is of exponent k, then in that overgroup we have

$$[x,y] = [r^k, y] = [r, y^k] = [r, e] = e.$$

In fact, an x which is a k-th power in some $\mathcal{N}_2$-extension of G must commute with any element whose image in the abelianization of G has exponent k. So if G is of exponent n, any element to which we can adjoin an n-th root must be central, and of course we can always adjoin n-th roots to central elements. But if $G' = Z(G)$, then in such a group an element has an n-th root in some extension if and only if it is a commutator, and therefore will always lie in $G^nG' = G'$. Thus, for G of exponent p^k, we can restrict the prime powers in Theorem 4.3 to p^i, with $0 < i < k$.

Here we want a description of the amalgamation bases in $\mathcal{N}_2 \cap \mathcal{B}_{p^n}$, and we might also ask if there are any which are strong or weak bases there but not in $\mathcal{N}_2$. However, in contrast to Theorem 3.4, we have:

THEOREM 4.4 (cf. Theorem 3.4). *Let p be an odd prime, $n > 0$, and let $G \in \mathcal{N}_2 \cap \mathcal{B}_{p^n}$. The following are equivalent:*

(i) G is a weak amalgamation base for $\mathcal{N}_2$.
(ii) G is a weak amalgamation base for $\mathcal{N}_2 \cap \mathcal{B}_{p^n}$.
(iii) G satisfies $Z(G) = G'$ and for each i, $1 \leq i \leq n-1$, and each $g \in G$, either $g \in G^{p^i}G'$ or else there exists $c > 0$ and $y \in G$ with $y^{p^i} \equiv g^c \pmod{G'}$ and $[y,g] \neq e$.
(iv) G is a strong amalgamation base for $\mathcal{N}_2 \cap \mathcal{B}_{p^n}$.
(v) G is a strong amalgamation base for $\mathcal{N}_2$.

Proof. Clearly $(v) \Rightarrow (i) \Rightarrow (ii)$ and $(v) \Rightarrow (iv) \Rightarrow (ii)$. We show that $(ii) \Rightarrow (iii) \Rightarrow (v)$.

Suppose that G does not satisfy *(iii)*. If $Z(G) \neq G'$, then we take an element $x \in Z(G) \setminus G'$. We let K_1 be an $\mathcal{N}_2 \cap \mathcal{B}_{p^n}$-overgroup in which x is a commutator, using Lemma 1.5, and we let K_2 be a $\mathcal{N}_2 \cap \mathcal{B}_{p^n}$-overgroup in which x is not central, say $G \coprod^{\mathcal{N}_2} Z/p^nZ$. Then the amalgam $(K_1, K_2; G)$ cannot be embedded in an $\mathcal{N}_2$-group, since x would have to be central by virtue of being a commutator in K_1, but would have to be noncentral since it is not central in K_2. So G cannot satisfy *(ii)*.

Suppose instead that $Z(G) = G'$, and that $g \in G$ is such that $g \notin G^{p^i}G'$, but there is some $\mathcal{N}_2$-extension with a p^i-th root for g. Note that since there is some extension with a p^i-th root for g, g must commute with every element of exponent p^i-th in G. Since G is of exponent p^n, this means that $x^{p^{n-i}}$ must be central in G, hence lies in the commutator of G.

Let K_1 be an overgroup in which g does not commute with an element of exponent p^i, for example, consider $G\coprod^{\mathcal{N}_2} Z/p^iZ$. Since $g\notin G^{p^i}G'$, g does not commute with the generator of the cyclic group. For K_2 we want a group of exponent p^n where x has a p^i-th root modulo K_2', since then we will have a problem: in any group into which we embedd the amalgam $(K_1,K_2;G)$, x would be a p^i-th power modulo the commutator, so it would have to commute with everything of exponent p^i, yet it does not do so in K_1.

To construct K_2 we again have to be a bit careful. First consider a nil-2 group in which x has a p^i-th root, and call such a root r. We may assume that the group is generated by G and r. Since $x^{p^{n-i}}$ lies in G', it is central, and we can adjoin a central p^n-th root t to that element. Since G is of exponent p^n, t^{p^i} is of exponent p^n; using Lemma 1.5, we can make t^{p^i} into a commutator, $[q_1,q_2]$, with q_i of exponent p^n. Now we consider the group generated by G, q_1, q_2, and rt^{-1}. Then

$$(rt^{-1})^{p^n} = r^{p^n}t^{-p^n} = x^{p^{n-i}}x^{-p^{n-i}} = e.$$

Therefore, this group is of exponent p^n, since it is generated by elements of exponent p^n; moreover, $x=(rt^{-1})^{p^i}[q_1,q_2]$, so x is indeed a p^i-th power modulo the commutator subgroup. Make this group K_2, and we are done.

Thus, if G does not satisfy *(iii)*, it cannot be a weak amalgamation base.

Now suppose that G satisfies *(iii)*. We use Theorem 4.2 to show that G is a strong amalgamation base in $\mathcal{N}_2$. Let K_1 and K_2 be overgroups of G. Then $K_1'\cap G\subseteq Z(G)=G'$, hence lies in the center of K_2, and likewise for $K_2'\cap G$. Suppose that $r\in K_1$, $r'\in K_1'$, $s\in K_2$, $s'\in K_2'$ are such that $r^qr'=x\in G$ and $s^qs'=y\in G$ for some prime power q. If $\gcd(p,q)=1$, then since G is q-divisible, we can write $x=g_1^q$ and $y=g_2^q$. Then:

$$[r^qr',s]=[g_1^q,s]=[g_1,s^qs']=[g_1,g_2^q]=[g_1,g_2]^q\in G$$

and likewise with $[r,s^qs']$. So we may assume that $q=p^k$. If $k\geq n$, then x and y must be central in G, and therefore they lie in G'. So $[r^qr',s]=[r,s^qs']=e$. We may therefore assume that $q=p^i$ with $1\leq i\leq n-1$.

Clearly there are overgroups of K_1 in which x has a p^i-th root, so we must have $x\in G^{p^i}G'$. Let $g\in G$ such that $g^{p^i}\equiv x \pmod{G'}$. Likewise, there is an $h\in G$ such that $h^{p^i}\equiv y \pmod{G'}$. Then

$$[r^{p^i}r',s]=[x,s]=[g^{p^i},s]=[g,s^{p^i}]=[g,s^{p^i}s']=[g,h^p]=[g,h]^p\in G$$

and

$$[r,s^{p^i}s']=[r,y]=[r,h^{p^i}]=[r^{p^i},h]=[r^{p^i}r',h]=[g^p,h]=[g,h]^p\in G.$$

Thus the amalgam $(K_1, K_2; G)$ is strongly embeddable into an $\mathcal{N}_2$-group. This proves that G is a strong amalgamation base for $\mathcal{N}_2$, and proves the theorem. ▌

Remark 4. 5. It is worth noting that for the case $n = 1$, the conditions given reduce to $G' = Z(G)$. This is consistent with Maier's Theorem 2.1 in [11] and with Saracino's Proposition 3.4 in [14].

COROLLARY 4.5. *A group $G \in \mathcal{N}_2 \cap \mathcal{B}_{p^n}$ is a strong amalgamation base in $\mathcal{N}_2$ if and only if every amalgam of $\mathcal{N}_2 \cap \mathcal{B}_{p^n}$ groups with core G can be (strongly) embedded into an $\mathcal{N}_2$-group.*

5. COMMENTS AND QUESTIONS

Using Proposition 1.2, we see that the results in the previous two sections characterize amalgamation bases (weak, strong, and special) in the variety $\mathcal{N}_2 \cap \mathcal{B}_n$ for any odd number n. Since a group of exponent 2 is necessarily abelian, and in abelian groups of exponent two every amalgam can be strongly embedded, the results trivially extend to the case where n is not divisible by 4.

In order to extend it to all n, it would suffice to address the case of $n = 2^{i+1}$ with $i > 0$. Unfortunately, that case turns out to be much more complicated than the odd exponent case.

To make the arguments easier to state, we will write "no $\mathcal{N}_2$-extension of G has a 2^i-th root for x" rather than writing the equivalent condition given by Theorem 1.3.

One problem is clear when we note that in a group of exponent 2^{n+1}, the commutators are necessarily of exponent 2^n, and that it is no longer true that a group generated by elements of order 2^{n+1} is necessarily of order 2^{n+1}. If we attempt to proceed naively as we did above, we will get that the conditions under which an amalgam of exponent 2^{n+1} groups can be embedded *into a group of exponent* 2^{n+2} are the same as for $\mathcal{N}_2$, but it is unclear how to lower that last exponent by half.

Thus, for example, I do not know if the description of dominions is still true for groups of exponent 4 (in which case, dominions would be trivial, since squares are central in such a group).

There are other difficulties. For example, Corollary 4.5 no longer holds for groups of exponent 2^{n+1} (we will show an example below). One difficulty lies in the condition $Z(G) = G'$. In the case of odd exponent, if $Z(G) \neq G'$, we took a central element which was not a commutator and extended G in one direction to make that element a commutator, and in the other direction to make it non-central. However, if G is of exponent 2^{n+1}, there could be a central element of order 2^{n+1}; but all commutators are of exponent 2^n, so we would not be able to extend G to a group of exponent 2^{n+1} in which that element is a commutator. If we allow the K_i to be of exponent 2^{n+2}, so that commutators are of exponent 2^{n+1}, the difficulty disappears, although we can now only guarantee that the resulting amalgam is embeddable in a group of exponent 2^{n+3}. We will also be able to use this to handle a failure of the second part of the condition. Thus, by proceeding as we did above, we obtain the following result:

THEOREM 5.1. *Let G be an $\mathcal{N}_2$-group of exponent 2^{n+1}, with $n > 0$. The following are equivalent:*

(i) G is a weak amalgamation base for $\mathcal{N}_2$.

(ii) Every amalgam of $\mathcal{N}_2 \cap \mathcal{B}_{2^{n+2}}$ groups with core G is weakly embeddable in $\mathcal{N}_2$.

(iii) $G' = Z(G)$ and for every $g \in G$ and every i, $1 \leq i \leq n$, either $g \in G^{2^i}G'$ or else no $\mathcal{N}_2$-extension of G has a 2^i-th root for g.

(iv) Every amalgam of $\mathcal{N}_2 \cap \mathcal{B}_{2^{n+2}}$ groups with core G is strongly embeddable in $\mathcal{N}_2$.

(v) G is a strong amalgamation base for $\mathcal{N}_2$.

Another difficulty lies in trying to characterize the elements $x \in G$ which can be a 2^i-th power modulo the commutator in an extension of G of exponent 2^{n+1}. Of course, it must be true that there is some $\mathcal{N}_2$-extension where x is a 2^i-th power; but we also get two extra conditions that are not implied by that statement: x^{2^n} must be trivial (if $i > 0$), and $x^{2^{n-i}}$ must be central in G (in fact, in the extension).

Although Corollary 4.5 no longer holds for power of two exponent, we can still characterize the groups $G \in \mathcal{N}_2 \cap \mathcal{B}_{2^{n+1}}$ for which every amalgam of $\mathcal{N}_2 \cap \mathcal{B}_{2^{n+1}}$-groups with core G are embeddable in $\mathcal{N}_2$. We define

$$\Omega^i(G) = \{g \in G \mid g^{2^i} = e\}.$$

THEOREM 5.2. *Let $G \in \mathcal{N}_2 \cap \mathcal{B}_{2^{n+1}}$, with $n > 0$. Then the following are equivalent:*

(i) Every amalgam $(K_1, K_2; G)$ of $\mathcal{N}_2 \cap \mathcal{B}_{2^{n+1}}$ groups is weakly embeddable in $\mathcal{N}_2$.

(ii) Every amalgam $(K_1, K_2; G)$ of $\mathcal{N}_2 \cap \mathcal{B}_{2^{n+1}}$ groups is strongly embeddable in $\mathcal{N}_2$.

(iii) $\Omega^n(Z(G)) = G^{2^n}G'$ and for each $x \in G$ and each i, $1 \leq i \leq n-1$, one of the following holds:

(a) $x \notin G^{2^i}G'$; *or*

(b) $x^{2^{n-i}} \notin Z(G)$; *or*

(c) $x^{2^n} \neq e$; *or*

(d) No $\mathcal{N}_2$ *extension of* G *has a* 2^i*-th root for* x.

Proof. Note that for any $G \in \mathcal{N}_2 \cap \mathcal{B}_{2^{n+1}}$, $G^{2^n}G' \subseteq \Omega^n(Z(G))$, as every 2^n-th power is central and of exponent 2, and every commutator is central and of exponent 2^n.

Clearly $(ii) \Rightarrow (i)$. Suppose G fails to satisfy (iii). If $\Omega^n(Z(G)) \neq G^{2^n}G'$, let $g \in G$ be central of exponent 2^n, but not in $G^{2^n}G'$. Since g is of exponent 2^n and central, we can extend G to a group where g is a commutator; moreover, we can make $g = [q_1, q_2]$ with q_i of exponent 2^n, so the resulting group is still of exponent 2^{n+1}. Let that group be K_1. Then let $K_2 = G \coprod^{\mathcal{N}_2} Z/2^nZ$; K_2 is also of exponent 2^{n+1}, and g is not central in K_2. Thus $(K_1, K_2; G)$ cannot be weakly embedded.

If $\Omega^n(Z(G)) = G^{2^n}G'$, let $x \in G$ be such that $x \notin G^{2^i}G'$, but such that $x^{2^n} = e$, $x^{2^{n-i}}$ is central in G, and there is some extension of G where x has a 2^i-th root. Let K_1 be obtained from G by first adjoining a 2^i-th root r to x; then adjoining central 2^n-th root t to $x^{2^{n-i}}$. Note that t^{2^i} is central of exponent 2^n, since $t^{2^{i+n}} = x^{2^n} = e$, so we can make t^{2^i} into a commutator $[q_1, q_2]$, with q_i of exponent 2^n. Then the group generated by G, rt^{-1}, q_1 and q_2 is of exponent 2^{n+1} (since rt^{-1} is also of exponent 2^n), and here $x = r^{2^i}[q_1, q_2]$. Let $K_2 = G \coprod^{\mathcal{N}_2} Z/2^iZ$; then, in K_2, x does not commute with an element of exponent 2^i, which gives that $(K_1, K_2; G)$ is not embeddable in $\mathcal{N}_2$. This proves that $(i) \Rightarrow (iii)$.

To prove that $(iii) \Rightarrow (ii)$, let $(K_1, K_2; G)$ be an amalgam of $\mathcal{N}_2 \cap \mathcal{B}_{2^{n+1}}$ groups, and assume G satisfies (iii). We verify Maier's conditions for strong embeddability.

The elements of $K_1' \cap G$ are central, and of exponent 2^n, so they lie in $\Omega^n(Z(G)) = G^{2^n}G'$. However, $G^{2^n}G'$ is central in K_2, so $K_1' \cap G \subseteq Z(K_2)$. A symmetric argument yields $K_2' \cap G \subseteq Z(K_1)$.

Now suppose that $k_1 \in K_1$, $k_1' \in K_1'$, $k_2 \in K_2$, $k_2' \in K_2'$ and $q > 0$ is a prime power such that $k_1^q k_1', k_2^q k_2' \in G$. Clearly we may assume that q is a power of 2, and moreover that $q = 2^i$ with $1 \leq i \leq n-1$ (2^n-th powers are central in G and of exponent at most 2, so they lie in $\Omega^n(Z(G))$, and so are central in each K_i). Let $x = k_1^q k_1'$, $y = k_2^q k_2'$. It is easy to verify that $x^{2^n} = y^{2^n} = e$ and that $x^{2^{n-i}}$ and $y^{2^{n-i}}$ are central in G, so we must have $x, y \in G^{2^i}G'$. Now we can proceed as before to get that $[k_1^q k_1', k_2] = [k_1, k_2^q k_2'] \in G$, and so the amalgam is strongly embeddable in $\mathcal{N}_2$. ∎

Remark 5. 1. For $n = 1$, condition *(ii)* reduces to the condition $\Omega^1(Z(G)) = G^2G'$.

Example 5.1. For example, the group

$$G = \langle a, b, c \mid a^4 = b^4 = c^4 = [a, b]^2 = [a, c] = [b, c] = e; c^2 = [a, b] \rangle$$

is such that every amalgam of groups of exponent 4 with core G is embeddable (into a group of exponent at most 8). This because $\Omega^1(Z(G)) = \langle a^2, b^2, [a, b] \rangle$. However, it is clearly not a strong amalgamation base for $\mathcal{N}_2$, since c is central but not a commutator. This example shows that Corollary 4.5 does not hold for power-of-two exponent.

The results above show that we will encounter new difficulties in trying to extend the results in Sections 2 and 4 to $\mathcal{N}_2 \cap \mathcal{B}_{2^{n+1}}$. We therefore have the following questions:

Question 5.3. *What are the conditions for weak and strong embeddability of amalgams of $\mathcal{N}_2 \cap \mathcal{B}_{2^{n+1}}$ groups? What is a description of dominions in the variety $\mathcal{N}_2 \cap \mathcal{B}_{2^{n+1}}$? What is a characterization the weak, strong, and special amalgamation bases for the varieties $\mathcal{N}_2 \cap \mathcal{B}_{2^{n+1}}$?*

The varieties we have considered so far are not the only subvarieties of $\mathcal{N}_2$. The subvariety of $\mathcal{N}_2$ are in one-to-one correspondence with pairs of nonnegative integers (m, n), where $n | m/\gcd(2, m)$. To each such pair corresponds the variety defined by the identities:

$$x^m = [x_1, x_2]^n = [x_1, x_2, x_3] = e$$

(see for example [5] and [13]). So far we have only considered the cases of $m = n$ with m odd, and $m = n = 0$. So we also ask

Question 5.4. *For the subvarieties of $\mathcal{N}_2$ we have not considered, find conditions for weak and strong embeddabilty of amalgams. Describe dominions in those subvarieties, and characterize the weak, strong, and special amalgamation bases.*

We note that if n is not square free, we already know that there are nontrivial dominions in such a variety (Theorem 9.92 in [8]), so describing dominions and characterizing special amalgamation bases is a nontrivial problem there.

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