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A CORRECTION TO A RESULT OF B. MAIER

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Abstract. In a 1985 paper, Berthold J. Maier gave necessary and sufficient conditions for the weak embeddability of amalgams of two nilpotent groups of class two over a common subgroup. Then he derived simpler conditions for some special cases. One of his subsequent results is incorrect, and we provide a counterexample. Finally, we provide a fix for the result.

In this note, $\mathcal{N}_2$ denotes the variety of nilpotent groups of class two, that is, groups G such that $[G,G] \subseteq Z(G)$. Recall that an *amalgam of A and B over the common subgroup D* consists of two groups, A and B, and a group D which is a subgroup of both A and B. The amalgam is *weakly embeddable* in $\mathcal{N}_2$ if and only if there is a $\mathcal{N}_2$-group G, such that A and B are subgroups of G and $D \subseteq A \cap B$ (inside G). We then say that G is a (weak) amalgam for A and B over D. If G satisfies the further condition that $D = A \cap B$, then G is said to be a *strong* amalgam.

In [1], Berthold J. Maier studied the question of weak embeddability of amalgams in $\mathcal{N}_2$. Note that when he says that an amalgam exists, he means that a *weak* amalgam exists. The Hauptsatz in [1] is a characterization of weak embeddability for amalgams in $\mathcal{N}_2$. We quote it here for reference:

Theorem 1 (Berthold J. Maier, Haupsatz in [1]). *Let $A, B \in \mathcal{N}_2$, with a common subgroup $D \leq A, B$. There exists a weak amalgam of A and B over D in $\mathcal{N}_2$ if and only if the following two conditions hold:*

1. *$A_2 \cap D \leq Z(B)$ and $B_2 \cap D \leq Z(A)$.*
2. *For all $k > 0$, $q_i > 0$, $x_i \in A$ and $x_i' \in A_2$ with $x_i^{q_i} x_i' \in D$, $y_i \in B$ and $y_i' \in B_2$ with $y_i^{q_i} y_i' \in D$, we have that for every $d \in D$,*

$$\prod_{i=1}^{k} [x_i, y_i^{q_i} y_i'] = d \Longleftrightarrow \prod_{i=1}^{k} [x_i^{q_i} x_i', y_i] = d.$$

In the above, A_2 is the commutator subgroup of A, $Z(A)$ is the center of A, and similarly for B; the commutator brackets are given by $[x, y] = x^{-1} y^{-1} x y$.

After the proof, Maier investigates whether the conditions may be relaxed given more information about A and B. For example, he proves that if A and B are torsion-free, then condition (2) above is superfluous. He also shows that if D is normal in either A or B, then it suffices to consider condition (2) with $k = 1$.

Then Maier looks at the case where D is either central or *co-central* in B. Maier defines a subgroup X of B to be co-central if there exists a central subgroup Z

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such that $\langle X, Z\rangle = B$. The case where D is central or co-central in A is covered by symmetry.

Maier's result reads:

Statement 2 (Satz 2 in [1]). *Let A and B in $\mathcal{N}_2$ with a common subgroup D.*

(1) *If D is co-central in B, then there exists an amalgam of A and B over D in $\mathcal{N}_2$ if and only if*
 (*) *For $q>0$, $a\in A$ with $a^q\in A_2D$, and $b\in B\setminus D$ with $b^q\in D$, we have $[a,b^q]=e$.*

(2) *If D is central in B, then there exists an amalgam of A and B over D in $\mathcal{N}_2$ if and only if $B_2\cap D\subseteq Z(A)$, and for $q>0$, $a\in A$ with $a^q\in A_2D$, and $b\in B$, $b'\in B_2$ with b^qb', we have $[a,b^qb']=e$.*

In fact, Part 1 of the statement above is incorrect. Although the condition given is sufficient, as Maier proves, it is not necessary. Part 2, on the other hand, is correct in its entirety.

First, we provide a counterexample to the statement in Part 1:

Example 3. Two groups, A and B in $\mathcal{N}_2$, with a common subgroup D, D co-central in B, which fails (*), and such that the amalgam of A and B over D is embeddable in $\mathcal{N}_2$.

Let D be the relatively free $\mathcal{N}_2$-group in two generators, x and y. The elements of the group can be uniquely written in the form

$$x^ay^b[y,x]^c; \qquad a,b,c\in\mathbb{Z}.$$

Let Z denote the infinite cyclic group, and Z/qZ the cyclic group of q elements. We denote the generator of the group by z.

Let $A=B=Z/qZ\times D$. Clearly, $Z/qZ\times D\times Z/qZ$ is a (strong) amalgam of A and B over D. Also, since $B=\langle D, Z/qZ\rangle$, and the Z/qZ factor of B is central, D is co-central in B.

Let $a=(e,x)\in A$, and let $b=(z,y)\in B\setminus D$. Then $a^q=(e,x^q)\in D$, and $b^q=(e,y^q)\in D$. However,

$$[a,b^q]=[x,y^q]=[x,y]^q\neq e.$$

An easy way to fix the problem is to modify condition (*). Note first that a subgroup H of G is co-central if and only if $\langle H, Z(G)\rangle$ equals the whole group. To fix Maier's result, instead of asking that $b\in B\setminus D$, we request merely that it lie in the center of B.

To prove that this modification will work, we need to quote another result of Maier's:

Theorem 4 (Korollar 3 in [1]). *Let A and B in $\mathcal{N}_2$ with a common subgroup D, and assume that D is normal in either A or B. Then there exists an amalgam of A and B over D in $\mathcal{N}_2$ if and only if the following two conditions hold:*

(a) *$A_2\cap D\subseteq Z(B)$ and $B_2\cap D\subseteq Z(A)$.*

(b) *For $q>0$, $a\in A$, $a'\in A_2$ with $a^qa'\in D$, $b\in B$, $b'\in B_2$ with $b^qb'\in D$, we have*

$$[a^qa',b]=[a,b^qb']\in D.$$

We can now provide a corrected version of Statement 2, Part 1:

Theorem 5. *Let A and B in $\mathcal{N}_2$, and let D be a common subgroup of A and B. Assume further that D is co-central in B. There exists a (weak) amalgam of A and B over D in $\mathcal{N}_2$ if and only if the following condition holds:*

(**) *For $q > 0$, $a \in A$ with $a^q \in A_2D$, and $z \in Z(B)$ with $z^q \in D$, we have $[a, z^q] = e$.*

Proof. First we prove necessity. Let G be an amalgam of A and B over D, with D co-central in B. Let q, a, and z be as in (**). Since $a^q a' \in D \subseteq B$, and $z \in Z(B)$, $[a^q a', z] = e$. But in G the commutator bracket acts bilinearly on $G^{\rm ab} \times G^{\rm ab}$, so

$$\begin{aligned} [a, z^q] &= [a^q, z] \\ &= [a^q a', z] \\ &= e. \end{aligned}$$

This proves (**), giving necessity.

We will prove sufficiency by showing that the hypothesis that D is co-central in B together with (**) imply (a) and (b) from Theorem 4. Note that a co-central subgroup is necessarily normal, so Theorem 4 applies.

To prove (a), let $a \in A_2 \cap D$. Then a is necessarily in $Z(D)$, since A lies in $\mathcal{N}_2$; thus a commutes with every element of D, and when considered inside of B, it must commute with everything in the center of B. Since B is generated by D and $Z(B)$, it follows that $a \in Z(B)$. Thus, $A_2 \cap D \subseteq Z(B)$.

For the other inclusion, note that since D is co-central and B is nilpotent of class two, it follows that $B_2 = D_2$. In particular,

$$B_2 \cap D = D_2 \subseteq A_2 \subseteq Z(A).$$

To prove (b), let $a \in A$, $a' \in A_2$, $b \in B$, $b' \in B_2$, and $q > 0$ such that $a^q a'$ and $b^q b'$ both lie in D. We want to show that

$$[a^q a', b] = [a, b^q b'] \in D.$$

Since $B_2 = D_2$, note that $b^q b' \in D$ if and only if $b^q \in D$, and that $[a, b^q b'] = [a, b^q]$ in A. So we may assume without loss of generality that $b' = e$.

Write $b = dz$, with $d \in D$ and $z \in Z(B)$. Then $b^q = d^q z^q$, so $b^q \in D$ implies that $z^q \in D$. Then condition (**) applies to a and z, so we conclude that $[a, z^q] = e$.

Therefore, in A, we have:

$$\begin{aligned} [a, b^q] &= [a, (dz)^q] \\ &= [a, d^q z^q] \\ &= [a, d^q][a, z^q] \\ &= [a, d^q] \\ &= [a^q, d] \\ &= [a^q a', d]. \end{aligned}$$

The third step can be done because both d^q and z^q lie in D. The next to last and last steps follow because $A \in \mathcal{N}_2$.

On the other hand, in B we have:

$$\begin{aligned} [a^q a', b] &= [a^q a', dz] \\ &= [a^q a', d][a^q a', z] \\ &= [a^q a', d] \end{aligned}$$

since z is central in B.

Therefore, $[a, b^q] = [a^q a', b]$. Moreover, since $[a^q a', b] \in B_2 = D_2 \subseteq D$, the two commutators lie in D. This proves condition (b), and thus the theorem. $\square$

References

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