A LYNDON-HOCHSCHILD-SERRE-TYPE SPECTRAL SEQUENCE FOR DISCRETE G-SPECTRA

DANIEL G. DAVIS

ABSTRACT. These are some notes for my talk in the bell show at the conference Structured Ring Spectra – TNG, on August 4th, 2011, in Hamburg, Germany.

1. Notation for the talk

- $G$ is a profinite group with finite vcd (that is, finite virtual cohomological dimension): there exists some $U <_o G$ and some natural number $l$ such that $H^s_c(U; M) = 0$, for all $s > l$ and all discrete $U$–modules $M$.
- Note: in practice, the above is not too restrictive a hypothesis because it is satisfied by many of the profinite groups that one cares about, such as any compact $p$-adic analytic Lie group.
- $H$ and $K$ are closed subgroups of $G$, with $H \triangleleft K$. This implies that $K/H$ is a profinite group.
- $\text{Spt}_G$ is the model category of discrete $G$–spectra.

2. Motivation for our theorem

Let

$$G = G_n,$$

$$= S_n \rtimes \text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p),$$

= the extended Morava stabilizer group

and let $Z$ be any finite spectrum.

Work of Ethan Devinatz, Mike Hopkins, Mark Behrens, and myself shows that there is a strongly convergent descent spectral sequence that has the form

$$E_2^{s,t} = H^s_c(K/H; \pi_t((E_n \wedge Z)^{hH})) \Rightarrow \pi_{t-s}(X^{hK}).$$

This is referred to as a “Lyndon-Hochschild-Serre spectral sequence” because the abutment is the total right derived functor of $K$–fixed points and the $E_2$–term is the $K/H$–continuous cohomology of the total right derived functor of $H$–fixed points.

Now let $X \in \text{Spt}_G$. The above spectral sequence leads one to ask if there is a descent spectral sequence that has the form

$$H^s_c(K/H; \pi_t(X^{hH})) \Rightarrow \pi_{t-s}(X^{hK}).$$
3. Dog-gone-it, back to reality

When \( X \) is a totally hyperfibrant discrete \( G \)-spectrum and \( K/H \) has finite vcd, then it is known that the desired descent spectral sequence exists. But, in general, we are not able to say that this spectral sequence exists.

For example:
- there are cases where \( X^{hH} \) is not a discrete \( K/H \)-spectrum;
- in general, it is not known how to view \( \pi_t(X^{hH}) \) as a topological \( K/H \)-module; and
- in general, it is not known how to define
  \[ (X^{hH})^{hK/H}, \]
  which is the way one expects to build the above spectral sequence if it exists.

Nevertheless, it is still possible, in general, by using the speaker’s framework of delta-discrete \( K/H \)-spectra, to build a descent spectral sequence for

\[ (X^{hH})^{b_{K/H}} \xrightarrow{\simeq} X^{hK} \]

that is a Lyndon-Hochschild-Serre-type spectral sequence.

4. Two definitions to help state our result

**Definition 4.1.** Let \( P = \lim_{\alpha} P_{\alpha} \) be a profinite set, so that each \( P_{\alpha} \) is a finite set. Then let

\[ \text{Map}_c(P, X) := \colim_{\alpha} \prod_{\alpha} X, \]

where the colimit and product are formed in \( \text{Spt}_G \) (and hence, in this case, in the category of spectra). Thus, \( \text{Map}_c(P, X) \in \text{Spt}_G \).

**Definition 4.2.** Let \( \hat{X} = \colim_{N \to \hat{O}_G} (X^N)_f \), where \((-)_f \) denotes fibrant replacement in the category of spectra. Then \( \hat{X} \in \text{Spt}_G \) and there is a map \( X \to \hat{X} \) that is a weak equivalence in \( \text{Spt}_G \).

5. Our result

**Theorem 5.1.** There is a conditionally convergent descent spectral sequence

\[ E_{2}^{s,t} = H^s\left[ \pi_t(\text{Map}_c(K/H^s, \hat{X})^{hH}) \right] \implies \pi_{t-s}(X^{hK}), \]

with

\[ E_{2}^{s,t} = H^t\left[ \pi_t(X^{hH}) \to \pi_t(\text{Map}_c(K/H, \hat{X})^{hH}) \to \cdots \right]. \]

This spectral sequence has the desired abutment and the \( E_2 \)-term involves the total right derived functor of \( H \)-fixed points and an expression that is related to the continuous cochains for \( K/H \) of continuous cohomology, so this spectral sequence is of Lyndon-Hochschild-Serre-type.

This is the nicest way I know to write the general spectral sequence, but its form is not meant to imply that the cochain complex above comes from the usual simplicial object \( K/H^\bullet \), because it does not, and the definition of the cochain complex does (as it should) involve the \( K/H \)-action on the discrete \( K/H \)-spectra \( \text{Map}_c(G^\bullet, \hat{X})^H \).
Remark 5.2. One can ask: if $X^{hH}$ is a discrete $K/H$–spectrum and $K/H$ has finite vcd, which is what happens in the cases of interest in chromatic stable homotopy theory, then is the above spectral sequence isomorphic to the usual descent spectral sequence

$$H^s_c(K/H; \pi_t(X^{hH})) \Longrightarrow \pi_{t-s}(X^{hH})^{hK/H}.$$ 

I expect to be able to show that in the main case when this happens (that is, when $X$ is a totally hyperfibrant discrete $G$–spectrum), this is indeed correct, but this is work in progress.