GALOIS THEORY, COMMUTATIVE RINGS, AND CHROMATIC HOMOTOPY THEORY

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1. Introduction

I'd like to describe some algebra. Given two abelian groups A and B, one can form their tensor product $A \otimes B$. This tensor product has nice properties: in particular, $\otimes$ gives a symmetric monoidal product on the category of abelian groups, so that, for example,

$$A \otimes B \cong B \otimes A, \quad (A \otimes B) \otimes C \cong A \otimes (B \otimes C), \quad A \otimes \mathbb{Z} \cong A.$$

I will be talking about certain objects in homotopy theory that also have a symmetric monoidal product: they behave like the tensor product that I've described. I will denote the tensor product in homotopy theory by using the symbol $\wedge$, for smash product.

Roughly speaking, a *spectrum* X is a collection

$$\{X_0, X_1, X_2, ...\}$$

of pointed topological spaces, such that there are continuous maps

$$S^1 \wedge X_n \to X_{n+1},$$

for each $n \geq 0$. For example, S^0, the *sphere spectrum*, is the collection

$$\{S^0, S^1, S^2, ...\},$$

where

$$S^1 \wedge S^n \to S^{n+1}$$

is a homeomorphism for each n.

In algebra, a commutative ring is identical to a commutative $\mathbb{Z}$-algebra. We can express this by saying that a commutative ring is a commutative algebra object in the category of abelian groups. One can define commutative algebra object here by writing down various diagrams involving the tensor product and saying that all of these diagrams commute. If all of the tensor products in these diagrams are changed to smash products and if the diagrams are regarded as being in the category of spectra, then any object for which all of these diagrams commute is a commutative ring object in the category of spectra. We call it a commutative ring.

Now I want to introduce the notion of a Galois extension.

Definition 1. *Let $R \to T$ be a homomorphism of commutative rings, so that T is a commutative R-algebra, and let G be a finite group acting on T from the left through R-algebra homomorphisms. If the canonical maps $R \to T^G$ and $T \otimes_R T \to \prod_G T$ are isomorphisms, then $R \to T$ is a G-Galois extension.*

Example 2. *Let G be a finite group acting on a field E by automorphisms, then $E^G \hookrightarrow E$ is a G-Galois extension.*

Definition 3. *In the category of spectra, a G-Galois extension of commutative rings is basically the same thing as before: one just replaces homomorphism with map, fixed points with homotopy fixed points, tensor product with smash product, and isomorphism with equivalence.*

Definition 4 (Rognes, simplified some)**.** *Let $G = \lim_\alpha G/N_\alpha$ be a profinite group. Then the commutative ring spectrum E is a profinite G-Galois extension of the commutative ring spectrum A if there is a collection $\{E_\alpha\}_\alpha$, where each E_α is a G/N_α-Galois extension of A and $E = \operatorname{colim}_\alpha E_\alpha$. This implies that $E \wedge_A E \cong \operatorname{colim}_\alpha \prod_{G/N_\alpha} E$.*

One of the deepest and most-studied problems in algebraic topology is the computation of the stable homotopy groups $\pi_*(S^0)$ of the sphere spectrum. These groups are quite important: for example, the enumeration of the distinct smooth structures on S^q (type of manifold structure, up to orientation-preserving diffeomorphism), for $q \geq 5$, has been reduced to the computation of these stable homotopy groups.

In chromatic theory, progress in computing these groups is primarily made by computing $\pi_*(\widehat{L}S^0)$. Here, $\widehat{L}S^0$ is a complete commutative ring:

$$\widehat{L}S^0 = \lim_n (L_n S^0)/I^n,$$

where the right-hand side is the I-adic completion of the commutative ring $L_n S^0$. In chromatic homotopy theory, the main object that we want to understand is $\widehat{L}S^0$.

2. Resolutions

Let

$$F\colon J \to \mathrm{Sp}, \quad j \mapsto F_j$$

be a functor from the indexing category J to the category of spectra. There is a machine

$$E_2^{s,t} = \lim_j^s(\pi_t(F_j)) \Rightarrow \pi_*(\lim_j F_j),$$

called the Bousfield-Kan spectral sequence, that helps with the computation of the stable homotopy groups of $\lim_j F_j$.

There are certain types of diagrams of spectra, known as *resolutions*, that give rise to functors F and associated B-K spectral sequences. For about 25 years, the most fruitful resolution for understanding $\widehat{L}S^0$ has the form stated in the following theorem.

Theorem 5 (Morava, Annals, 85)**.** *If E_n is the Lubin-Tate spectrum, then $\widehat{L}S^0$ has resolution*

$$\mathrm{Map}_c(G_n, E_n)^{G_n} \to \mathrm{Map}_c(G_n^2, E_n)^{G_n} \to \mathrm{Map}_c(G_n^3, E_n)^{G_n} \to \cdots.$$

Here, G_n is ... and E_n is a continuous G_n-spectrum.

Though this resolution is very useful, it is quite complicated due to its infinite length, so that experts hunt for a different type of resolution that has finite length. Computing $\pi_*(\widehat{L}S^0)$ is very difficult and currently a lot of effort is being spent on understanding the $n = 2$, $p = 3$ case, which is already very complicated. Fortunately, in this case there is a short resolution. We use Σ^n to denote taking the smash product with the n-sphere.

Theorem 6 (GHMR, Annals, 2005)**.** *There is a short resolution with special properties that has the form*

$$\widehat{L}S^0 \to E_2^{hG_{24}} \to (\Sigma^8 E_2^{hSD_{16}}) \vee E_2^{hG_{24}} \to (\Sigma^8 E_2^{hSD_{16}}) \vee (\Sigma^{40} E_2^{hSD_{16}})$$
$$\to (\Sigma^{40} E_2^{hSD_{16}}) \vee (\Sigma^{48} E_2^{hG_{24}}) \to \Sigma^{48} E_2^{hG_{24}}.$$

Here, G_{24} and SD_{16} are subgroups of G_2; G_{24} has order 24 and $SD_{16} = \mathbb{F}_9^\times \rtimes \mathbb{Z}/2$ is the semidihedral group of order 16.

This resolution is very useful because, by looking at the B-K spectral sequence associated to this resolution, a tremendous amount of data associated to the $n = 2, p = 3$ computation is organized and better understood. Notice that this resolution has self-duality: ignoring suspensions, the second and sixth terms are the same and the third and fifth terms are the same, and the suspensions of terms that are paired together sum to 48. Now the natural problem left open here is how to explain the self-duality and the appearance of the number 48.

3. Understanding the GHMR resolution

To produce this resolution the authors used many tools and one of them is a lemma. Before stating this lemma, we need some algebra. Let A, B, and C be abelian groups and let $\mathrm{Hom}(A, B)$ be the abelian group of group homomorphisms $A \to B$. Then there is an isomorphism

$$\mathrm{Hom}(A \otimes B, C) \cong \mathrm{Hom}(A, \mathrm{Hom}(B, C)).$$

With spectra, there are analogous constructions: if X, Y, and Z are spectra, then there is a function spectrum $F(X, Y)$ of maps of spectra $X \to Y$. Similarly, there is an isomorphism

$$F(X \wedge Y, Z) \cong F(X, F(Y, Z)).$$

Lemma 7 (GHMR)**.** *Let H and K be closed subgroups of G_n, with K finite. Then there is an isomorphism*

$$F(E_n^{hH}, E_n^{hK}) \cong (E_n[[G_n/H]])^{hK},$$

where K acts diagonally on $E_n[[G_n/H]]$.

The proof of this result uses the fact that the objects involved are spectra and E_n-module spectra; no further algebraic structure on the spectra was needed. Given this result, it is natural to ask if the result is still true when K is any closed subgroup and not just finite. This turns out to be true and it is a special case of the following result.

Theorem 8 (M. Behrens-D)**.** *Let E be a k-local consistent faithful profinite G-Galois extension of finite vcd and let H and K be closed subgroups of G. Then there is an equivalence*

$$F_A((E^{hH})_k, (E^{hK})_k) \cong ((E[[G/H]])^{hK})_k.$$

In proving this, we needed to use the existence of additional algebraic structure: in particular, we needed to use

$$\mu\colon (E^{hN})_k \wedge_A (E^{hN})_k \to (E^{hN})_k,$$

the multiplication map of the A-algebra $(E^{hN})_k$, where N is an open normal subgroup of G.

Mark used this result to help with proving the following theorem (a 59-page paper).

I could mention the model category structures that we develop.

Theorem 9. *(B, Topology, 2006) When $n = 2$ and $p = 3$, there is a cofiber sequence*

$$F(Q(2), \widehat{L}S^0) \to \widehat{L}S^0 \to Q(2),$$

where $Q(2)$ is defined by using elliptic curves and a simplicial stack.

This sequence has duality in that the first term is the Spanier-Whitehead dual of the third term. Mark used this sequence to explain the self-duality that I described earlier. Along the way, Mark also explained the appearance of the 48.

Let G be a totally disconnected locally compact group G. Such groups include profinite groups.

Theorem 10 (B-Lawson, to appear, AMS Memoir)**.** *The category of smooth G-spectra is a model category where a morphism f is a cofibration (weak equivalence) if it is a cofibration (weak equivalence) of spectra.*

Behrens and Lawson use stacks, the theory of buildings, homotopy fixed points, the above model category, and other tools to make it possible to use the arithmetic of Shimura varieties to help with understanding the stable homotopy groups of spheres.