ARBITRARY MORAVA MODULES, THEIR ADAMS SPECTRAL SEQUENCE, AND CONTINUOUS GROUP COHOMOLOGY

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Abstract. These are the notes that I wrote up for myself, for my Colloquium on February 10th, 2011, in the Department of Mathematics at the University of Louisiana at Lafayette.

§1. Spaces and spectra

I'd like to start by explaining what a *spectrum* is.

Def. Let X be a topological space and let x_0 be any point in X. Then the pair (X, x_0) is a *pointed space* and x_0 is the *basepoint* of the pointed space. For simplicity, we write just X for the pointed space (X, x_0). A *map of pointed spaces* is a continuous function $X \to Y$ between pointed spaces such that $f(x_0) = y_0$, where the point y_0 is the basepoint of Y.

Now we give a fundamental example of a pointed space.

Ex. For each integer $n \geq 0$, let

$$S^n = \text{the } \textit{unit } n\textit{-sphere},$$

a subspace of $\mathbb{R}^{n+1}$, with basepoint the $(n+1)$-tuple $(1, 0, 0, ..., 0)$. Thus, S^0 is the space ((draw the two points -1 and 1 on the real line and circle the basepoint 1)); S^1 is the unit circle ((draw the unit circle in the xy–plane and draw the basepoint $(1, 0)$)); and S^2 is the unit sphere ((draw the unit sphere and draw the basepoint $(1, 0, 0)$)).

Def. Let I be the unit interval $[0, 1]$. Let X be a pointed space. Then the (*reduced*) *suspension* of X, ΣX, is the pointed space

$$\frac{X \times I}{(X \times \{0\}) \cup (X \times \{1\}) \cup (\{x_0\} \times I)},$$

with basepoint the point that the "denominator" gets collapsed to.

Def. $\Sigma S^0 \approx S^1$ ((illustrate this)). $\Sigma S^1 \approx S^2$ ((illustrate this)). In general, for all $n \geq 0$, $\Sigma S^n \approx S^{n+1}$. For each $n < 0$, define $S^n = *$, a point. Then it is natural to consider together all the spaces $\{S^n\}_{n \in \mathbb{Z}}$ and note that each space S^n "at level n" is related to "level $n+1$" by pointed homeomorphisms $\Sigma S^n \to S^{n+1}$, when $n \geq 0$, or, when $n < 0$, by pointed maps $\Sigma S^n = \Sigma * = * \to S^{n+1}$. All of this data taken together is known as S^0, the *sphere spectrum.*

More generally, there is the following definition.

Def. Let $\{X_n\}_{n \in \mathbb{Z}}$ be a collection of pointed spaces X_n and, for each integer n, suppose that there is a pointed map $\Sigma X_n \to X_{n+1}$. All of this data taken together defines the *spectrum* X.

I would like to point out that spectra are almost always quite exotic objects. We have a good feel for what all the spaces that constitute S^0 are like and we all know what some of them look like. But this situation is rare. Usually, for most of the spectra that we study in chromatic homotopy theory, we have no idea what the spaces actually look like.

Related to this comment, I'd like to mention an example. There is the spectrum tmf, an inverse limit of spectra associated to generalized elliptic curves. Its construction takes at least 200 pages of work using the latest tools in the field. There is a major conjecture that says that its nth space is equivalent to the space of 2–dimensional supersymmetric quantum field theories of degree $-n$.

Now we want to define what a *map* between spectra is. This notion is the analogue, for spectra, of the concept of a continuous function between topological spaces. First of all, let's note the following.

Remark: If $f\colon W \to Z$ is a pointed map between pointed spaces, one can see from the definition of suspension that f induces a map $\Sigma f\colon \Sigma W \to \Sigma Z$ that sends (w, t) to $(f(w), t)$. Notice that Σf is a pointed map.

Def. A *map of spectra* $f\colon X \to Y$ *of degree* k is a sequence of pointed maps $f_n\colon X_n \to Y_{n-k}$, such that for each integer n, ((now draw the commutative diagram stating that $\Sigma X_n \to \Sigma Y_{n-k} \to Y_{n-k+1}$ equals $\Sigma X_n \to X_{n+1} \to Y_{n-k+1}$)).

§2. Certain homotopy groups: the main object we want to understand

If $q, r\colon W \to Z$ are maps of topological spaces, then we say that $q \simeq r$ ((say "q is homotopic to r")) if there is a continuous function $H\colon W \times I \to Z$, such that $H\big|_{W\times\{0\}} = q$ and $H\big|_{W\times\{1\}} = r$. This is just a rigorous way of saying that the map q can be continuously transformed into the map r. For example, ((draw a picture of a path $\phi\colon W = I \to \mathbb{R}^2$ and another path $\psi\colon I \to \mathbb{R}^2$, say that $\phi \simeq \psi$, and justify this by drawing how ϕ is continuously transformed into ψ)). The relation $\simeq$ is an equivalence relation and one can form the set $\{W, Z\}$ of equivalence classes of maps $W \to Z$ under the equivalence relation $\simeq$.

Def. Given spectra X and Y, there is an analogous construction for maps of spectra $X \to Y$ of degree k:

$$[X, Y]_k = \text{the set of equivalence classes of maps } X \to Y \text{ of degree } k,$$

under the relation of homotopy, modified for spectra. It turns out that $[X, Y]_k$, for every integer k, is an abelian group.

Now we consider a special case of these abelian groups.

Def. Given a spectrum X,

$$\pi_k(X) = [S^0, X]_k$$

is the *kth stable homotopy group of* X. When $l > k + 1$, $\pi_k(S^0) = \pi_{k+l}(S^l)$ is just the $(k + l)$th homotopy group of the l–sphere. We write $\pi_*(X)$ for the graded group $\{\pi_k(X)\}_{k\in\mathbb{Z}}$.

These stable homotopy groups of the sphere are quite important in various parts of math, including finding critical points of smooth maps from $\mathbb{R}^{k+l+1} \to \mathbb{R}^{l+1}$ and understanding the group of smooth structures on k–spheres and certain other manifolds.

Furthermore, these stable homotopy groups of the sphere are incredibly complicated to compute. For example, for $k > 63$, usually not much is known.

Def. Let $n \geq 0$ and let p be any prime. Then the spectrum $L_{K(n)}(X)$ is the $K(n)$–*localization of* X. Here, the spectrum $L_{K(n)}(X)$ is the inverse limit

$$\lim_{j\geq 0}\left[L_n(X)/I_0 \leftarrow L_n(X)/I_1 \leftarrow \cdots \leftarrow L_n(X)/I_j \leftarrow \cdots\right],$$

where each spectrum $L_n(X)/I_j$ is a "spectrum version" of the quotient ring of a commutative ring by an ideal. (More precisely, each $L_n(X)/I_j$ is the cofiber of a map, a notion analogous to the way that R/I is the cokernel of the ring monomorphism $I \hookrightarrow R$, where I is an ideal of the commutative ring R.)

It turns out that one of the best ways to make progress in computing $\pi_*(S^0)$ is to be able to compute

$$\pi_*(L_{K(n)}(X)),$$

for various X, for all n and all primes p. (I should note that part of the algorithm for obtaining $\pi_*(S^0)$ from various $\pi_*(L_{K(n)}(X))$ is conjectural, but the previous statement is widely believed by experts in homotopy theory.)

More generally, to help with computing $\pi_*(X)$ for all spectra X, we would like to understand $\pi_*(L_{K(n)}(X))$ for all spectra X, for all n and p.

§3. The $K(n)$–local E_n–Adams spectral sequence

This spectral sequence is a machine for computing $\pi_*(L_{K(n)}(X))$, where X is any spectrum. This spectral sequence has the form

$$\{E_2^{s_2,t_2}, E_3^{s_3,t_3}, E_4^{s_4,t_4}, \ldots \mid \text{each } s_i, t_i \in \mathbb{Z}\} \approx E_\infty^{s,t} \Longrightarrow \pi_*(L_{K(n)}(X)),$$

where $E_2^{s_2,t_2} = 0$ whenever $s_2 < 0$. For each $r \geq 2$, there are differentials

$$d_r\colon E_r^{s_r,t_r} \to E_r^{s_r+r,t_r+r-1}.$$

This spectral sequence is basically a sacred object in the field of chromatic homotopy theory. For various X, many people have done a tremendous number of computations with it.

For each X, its E_∞–term has a horizontal vanishing line: there exists some s_0 such that for all $s > s_0$, $E_\infty^{s,t} = 0$.

Here is how the E_∞–term is defined. Set

$$Z_\infty^s = \bigcap_{r\geq 2} Z_r^s, \quad B_\infty^s = \bigcup_{r\geq 2} B_r^s, \quad \text{and} \quad E_\infty^s = Z_\infty^s / B_\infty^s.$$

There is a nested sequence

$$\cdots \subset F^{s+1} \subset F^s \subset F^{s-1} \subset \cdots$$

of subgroups of $\pi_k(L_{K(n)}(X))$ such that

$$\pi_k(L_{K(n)}(X)) = \bigcup_s F^s \text{ and } \bigcap_s F^s = 0.$$

Since the spectral sequence is strongly convergent, we can obtain $\pi_k(L_{K(n)}(X))$ from certain subquotients as

$$\pi_k(L_{K(n)}(X)) = \lim_{s\in\mathbb{Z}} \operatorname{colim}_{s'\leq s} F^{s'}/F^s,$$

where, for all s,

$$E_\infty^s \cong F^s/F^{s+1}.$$

Given groups N and H, we say that a group G is an *extension of N by H* if there is a short exact sequence

$$0 \to N \to G \to H \to 0.$$

The above features of the Adams spectral sequence mean that $\pi_k(L_{K(n)}(X))$ can be obtained from the E_∞–page up to group extension.

§4. What does the E_2–term look like?

By a recent theorem of Takeshi Torii and myself, the $K(n)$–local E_n–Adams spectral sequence is isomorphic to a certain descent spectral sequence. The input for our result was work by Mark Behrens, Ethan Devinatz, Mike Hopkins, John Rognes, and myself.

A recent result of mine that depends on the above result of Takeshi and myself is the following.

Theorem. Let X be any spectrum and let $\mathcal{X}_i = E_n \wedge X \wedge M_i$, where $i \geq 0$. Then there is a long exact sequence

$$\begin{aligned}
0 \to H^0\big[\lim_i{}^1 \operatorname{Map}^c(G_n^*, \pi_{k+1}(\mathcal{X}_i))\big] &\to E_2^{0,k} \to \lim_i\big(\pi_k(\mathcal{X}_i)\big)^{G_n} - \cdots \\
\cdots \to H^1\big[\lim_i{}^1 \operatorname{Map}^c(G_n^*, \pi_{k+1}(\mathcal{X}_i))\big] &\to \cdots \\
\cdots \to H^s\big[\lim_i{}^1 \operatorname{Map}^c(G_n^*, \pi_{k+1}(\mathcal{X}_i))\big] &\to E_2^{s,k} \to H^s_{\mathrm{cts}}(G_n; \lim_i \pi_k(\mathcal{X}_i)) - \cdots \\
\cdots \to H^{s+1}\big[\lim_i{}^1 \operatorname{Map}^c(G_n^*, \pi_{k+1}(\mathcal{X}_i))\big] &\to \cdots .
\end{aligned}$$