# For the Ausoni-Rognes conjecture at $n = 1, p > 3$: a strongly convergent descent spectral sequence

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- $n \geq 1$
- $p$, a prime
- $E_n$ is the Lubin-Tate spectrum, with $\pi_*(E_n) = W(\mathbb{F}_{p^n})[\![u_1, \ldots, u_{n-1}]\!][u^{\pm 1}]$. Here:
  - $W(\mathbb{F}_{p^n})$ is the ring of Witt vectors of the field $\mathbb{F}_{p^n}$
  - the complete power series ring is in degree zero
  - $|u| = 2$ (this is the "choice" Ausoni makes)
- $\mathbb{G}_n$, the extended Morava stabilizer group
- $E_n$, a commutative $S$–algebra
- $\mathbb{G}_n$ acts on $E_n$ by maps of commutative $S$–algebras

- $(A$, a commutative $S$–algebra$) \mapsto (K(A)$, the algebraic $K$–theory spectrum of $A$, a commutative $S$–algebra$)$
  - $\Longrightarrow K(E_n)$, a commutative $S$–algebra
- By the functoriality of $K(-)$, $\mathbb{G}_n$ acts on $K(E_n)$ by maps of commutative $S$–algebras.
- $L_{K(n)}(S^0)$, the Bousfield localization of the sphere spectrum with respect to $K(n)$, the $n$th Morava $K$–theory spectrum.
- $\mathbb{G}_n$, a profinite group
- The $K(n)$–local unit map

$$L_{K(n)}(S^0) \to E_n$$

is a consistent profaithful $K(n)$–local profinite $\mathbb{G}_n$–Galois extension.

- $V_n$, a finite $p$–local complex of type $n+1$
- $v\colon \Sigma^d V_n \to V_n$, a $v_{n+1}$–self-map ($d$, some positive integer)
- $v$ induces a sequence

$$V_n \to \Sigma^{-d} V_n \to \Sigma^{-2d} V_n \to \cdots$$

  of maps of spectra
- We set

$$v_{n+1}^{-1} V_n = \operatorname*{colim}_{j \geq 0} \Sigma^{-jd} V_n,$$

  the colimit of the above sequence, the mapping telescope associated to the $v_{n+1}$–self-map $v$.

## Conjecture (Ausoni, Rognes)

The $\mathbb{G}_n$–Galois extension $L_{K(n)}(S^0) \to E_n$ induces a map

$$K(L_{K(n)}(S^0)) \wedge v_{n+1}^{-1} V_n \to (K(E_n))^{h\mathbb{G}_n} \wedge v_{n+1}^{-1} V_n$$

that is a weak equivalence, and associated with the target of this weak equivalence is a homotopy fixed point spectral sequence that has the form

$$E_2^{s,t} \Longrightarrow (V_n)_{t-s}((K(E_n))^{h\mathbb{G}_n})[v_{n+1}^{-1}],$$

with

$$E_2^{s,t} = H_c^s(\mathbb{G}_n; (V_n)_t(K(E_n))[v_{n+1}^{-1}]).$$

## Remark

We supplement the statement of the conjecture with the following comments:

- the $E_2$-term

$$E_2^{s,t} = H_c^s(\mathbb{G}_n; (V_n)_t(K(E_n))[v_{n+1}^{-1}])$$

  of its spectral sequence is given by continuous cohomology;
- its object $(K(E_n))^{h\mathbb{G}_n}$ is a continuous homotopy fixed point spectrum; and
- the conjecture is actually just a piece of the family of conjectures made by Ausoni and Rognes – we only stated the part that we have been focusing on.

- For every integer $t$, there is an isomorphism

$$(V_n)_t(K(E_n))[v_{n+1}^{-1}] \cong \pi_t(K(E_n) \wedge v_{n+1}^{-1} V_n).$$

- $\Longrightarrow$ When the conjectured spectral sequence

$$E_2^{s,t} \Longrightarrow (V_n)_{t-s}((K(E_n))^{h\mathbb{G}_n})[v_{n+1}^{-1}],$$

exists, since it is *for* homotopy fixed points, there should also be an equivalence

$$(K(E_n))^{h\mathbb{G}_n} \wedge v_{n+1}^{-1} V_n \simeq (K(E_n) \wedge v_{n+1}^{-1} V_n)^{h\mathbb{G}_n}.$$

- Obtaining this equivalence and a homotopy fixed point spectral sequence

$$E_2^{s,t} = H_c^s(\mathbb{G}_n; \pi_t(K(E_n) \wedge v_{n+1}^{-1} V_n)) \Longrightarrow \pi_{t-s}((K(E_n) \wedge v_{n+1}^{-1} V_n)^{h\mathbb{G}_n})$$

immediately implies the existence of the conjectured spectral sequence.

- To make progress on the conjecture, one obstacle that must be overcome is that there are no known constructions of the (continuous) homotopy fixed point spectra

$$(K(E_n))^{h\mathbb{G}_n}, \ (K(E_n) \wedge v_{n+1}^{-1} V_n)^{h\mathbb{G}_n}$$

for any $n$ and $p$.

- Also, there are no known constructions of the two homotopy fixed point spectral sequences that we have referred to.

In this talk, we are reporting on progress on this conjecture for $n = 1$, $p \geq 5$, with

$$V_1 = V(1), \text{ the type 2 Smith-Toda complex } S^0/(p, v_1).$$

Thus, we have

$$E_1 = KU_p, \; p\text{–completed complex } K\text{–theory},$$

$$\mathbb{G}_1 = \mathbb{Z}_p^\times \cong \mathbb{Z}_p \times \mathbb{Z}/(p-1).$$

**N.B.**

Henceforth, we will use the term "descent spectral sequence" in place of "homotopy fixed point spectral sequence."

## Theorem 1 (D.)

Let $p \geq 5$. Given any closed subgroup $K$ of $\mathbb{Z}_p^\times$, there is a strongly convergent descent spectral sequence

$$E_2^{s,t} \Longrightarrow \pi_{t-s}\Big(\big(K(KU_p) \wedge v_2^{-1} V(1)\big)^{hK}\Big),$$

where

$$E_2^{s,t} = H_c^s(K; \pi_t(K(KU_p) \wedge V(1))[v_2^{-1}]),$$

with $E_2^{s,t} = 0$, for all $s \geq 2$ and any $t \in \mathbb{Z}$. Also, there is an equivalence of spectra

$$\big(K(KU_p) \wedge v_2^{-1} V(1)\big)^{hK} \simeq \operatorname*{colim}_{j \geq 0} (K(KU_p) \wedge \Sigma^{-jd} V(1))^{hK}.$$

## Remarks about the theorem

- Each subgroup $K$ is a profinite group.
- Both occurrences of $(-)^{hK}$ in the theorem are for the application of the (continuous) $K$–homotopy fixed points functor for the category of discrete $K$-spectra.
- The term "spectrum" means symmetric spectrum: the theorem, its proof, and the underlying theory are worked out in the setting of symmetric spectra of simplicial sets.
- Letting $K = \mathbb{Z}_p^\times$ gives some progress on the Ausoni-Rognes conjecture: for $n = 1$, $p \geq 5$, and $V_1 = V(1)$, we have obtained the desired descent spectral sequence

$$H_c^s(\mathbb{G}_n; \pi_t(K(E_n) \wedge v_{n+1}^{-1} V_n)) \Longrightarrow \pi_{t-s}\big((K(E_n) \wedge v_{n+1}^{-1} V_n)^{h\mathbb{G}_n}\big).$$

- The construction of $K(KU_p)^{h\mathbb{Z}_p^\times}$ remains open.

## Theorem 2 (D.)

For $p \geq 5$, there is a canonical map of symmetric spectra

$$\eta\colon K(L_{K(1)}(S^0)) \wedge v_2^{-1}V(1) \to (K(KU_p) \wedge v_2^{-1}V(1))^{h\mathbb{Z}_p^\times}.$$

## Remark

It is easy to see that if $\eta$ is a weak equivalence and if

$$(K(KU_p) \wedge v_2^{-1}V(1))^{h\mathbb{Z}_p^\times} \simeq K(KU_p)^{h\mathbb{Z}_p^\times} \wedge v_2^{-1}V(1),$$

then there would be an equivalence

$$K(L_{K(1)}(S^0)) \wedge v_2^{-1}V(1) \simeq K(KU_p)^{h\mathbb{Z}_p^\times} \wedge v_2^{-1}V(1),$$

which would be close to proving part of the Ausoni-Rognes conjecture in the $n = 1$, $p \geq 5$ case.

For the next result ...

Let $G$ be a profinite group, with $\mathcal{N} = \{N_\alpha\}_{\alpha \in \Lambda}$ an inverse system of open normal subgroups of $G$ that satisfies

(a) the maps in the diagram $\mathcal{N}$ (indexed by the directed poset $\Lambda$) are given by inclusions (that is, $\alpha_1 \leq \alpha_2$ in $\Lambda$ if and only if $N_{\alpha_2}$ is a subgroup of $N_{\alpha_1}$), and

(b) the intersection $\bigcap_{\alpha \in \Lambda} N_\alpha$ is the trivial group $\{e\}$.

Also, let $X$ be a $G$–spectrum such that the $G$–module $\pi_t(X)$ is a discrete $G$–module, for every $t \in \mathbb{Z}$.

## Theorem (D.)

*Let $G$ and $X$ be as on the previous slide. Suppose that the map*

$$\lambda^s_{\pi_t(X)} \colon H^s_c(N_\alpha; \pi_t(X)) \to H^s(N_\alpha; \pi_t(X))$$

*is an isomorphism for all $s \geq 0$, every integer $t$, and each $\alpha \in \Lambda$. If*

- *there exists a natural number $r$, such that for all integers $t$ and every $\alpha \in \Lambda$, $H^s_c(N_\alpha; \pi_t(X)) = 0$, for all $s > r$; or*
- *there exists some fixed integer $l$, such that $\pi_t(X) = 0$, for all $t > l$,*

*then there is a zigzag of $G$–equivariant maps*

$$X \xrightarrow{\simeq} \operatorname{holim}_{\Delta} \mathrm{Sets}(G^{\bullet+1}, X_f) \xleftarrow{\simeq} X^{\mathrm{dis}}_{\mathcal{N}}$$

*that are weak equivalences in $\mathrm{Sp}^\Sigma$, with $X^{\mathrm{dis}}_{\mathcal{N}} \in \Sigma\mathrm{Sp}_G$.*

For Theorem 1, we didn't need the full power of the preceding theorem; we only needed a simpler version ...

## Definition

A spectrum $X$ is an *$f$–spectrum* if $\pi_*(X)$ is degreewise finite.

## Theorem (D.)

*Let $p$ be any prime and let $H$ be any finite discrete group. If $X$ is a $(\mathbb{Z}_p \times H)$–spectrum and an $f$–spectrum, then there is a zigzag*

$$X \xrightarrow{\simeq} X' \xleftarrow{\simeq} X_{\mathcal{N}}^{\mathrm{dis}}$$

*of $(\mathbb{Z}_p \times H)$–spectra and $(\mathbb{Z}_p \times H)$–equivariant maps that are weak equivalences of symmetric spectra, and $X_{\mathcal{N}}^{\mathrm{dis}}$ is a discrete $(\mathbb{Z}_p \times H)$–spectrum.*

$$X = X_{\mathcal{N}}^{\mathrm{dis}} \implies X^{h(\mathbb{Z}_p \times H)} := (X_{\mathcal{N}}^{\mathrm{dis}})^{h(\mathbb{Z}_p \times H)}$$

The construction of $X_{\mathcal{N}}^{\mathrm{dis}}$ is elementary!:

$$X_{\mathcal{N}}^{\mathrm{dis}}$$

is equal to

$$\operatorname*{colim}_{m\geq 0} \operatorname*{holim}_{[n]\in\Delta} \Big(\underbrace{\mathrm{Sets}(\mathbb{Z}_p \times H, \cdots, \mathrm{Sets}(\mathbb{Z}_p \times H,}_{(n+1)\text{ times}} X_f \underbrace{)\cdots)}_{\substack{(n+1)\\ \text{times}}}\Big)^{(p^m\mathbb{Z}_p)\times\{e\}},$$

where each $(p^m\mathbb{Z}_p) \times \{e\}$ is an (open normal) subgroup of $\mathbb{Z}_p \times H$ and $p^m\mathbb{Z}_p$ has its usual meaning.

Now we build on these tools for the situation of filtered colimits.

Let $G$ be any profinite group and $X$ any $G$–spectrum.

**Definition**

If $G$, $X$, and $\mathcal{N}$ (an inverse system of open normal subgroups of $G$) satisfy the hypotheses of either of the last two theorems, then we say that the triple $(G, X, \mathcal{N})$ is *suitably finite*.

**Definition**

Let $G$ be a profinite group with $\mathcal{N}$ a fixed inverse system of open normal subgroups of $G$, and let $\{X_\mu\}_\mu$ be a filtered diagram of $G$–spectra such that for each $\mu$, $(G, X_\mu, \mathcal{N})$ is a suitably finite triple and $X_\mu$ is a fibrant spectrum. We refer to $(G, \{X_\mu\}_\mu, \mathcal{N})$ as a *suitably filtered triple*.

Let $(G, \{X_\mu\}_\mu, \mathcal{N})$ be a suitably filtered triple. There is a zigzag of $G$–equivariant maps

$$\operatorname*{colim}_{\mu} X_\mu \xrightarrow{\simeq} \operatorname*{colim}_{\mu} \operatorname*{holim}_{\Delta} \mathrm{Sets}(G^{\bullet+1}, (X_\mu)_f) \xleftarrow{\simeq} \operatorname*{colim}_{\mu} (X_\mu)^{\mathrm{dis}}_{\mathcal{N}}$$

that are weak equivalences in $\Sigma\mathrm{Sp}$. The composition

$$\operatorname*{colim}_{\mu} \pi_t(X_\mu) \xrightarrow{\cong} \pi_t(\operatorname*{colim}_{\mu} (X_\mu)^{\mathrm{dis}}_{\mathcal{N}}) \xrightarrow{\cong} \operatorname*{colim}_{\mu} \pi_t((X_\mu)^{\mathrm{dis}}_{\mathcal{N}})$$

consists of two isomorphisms in the category of discrete $G$–modules (in particular, each of the above abelian groups is a discrete $G$–module).

## Definition

Given a suitably filtered triple $(G, \{X_\mu\}_\mu, \mathcal{N})$, we have seen that the $G$–spectrum $\operatorname{colim}_\mu X_\mu$ can be identified with the discrete $G$–spectrum $\operatorname{colim}_\mu (X_\mu)^{\mathrm{dis}}_{\mathcal{N}}$. Thus, it is natural to define

$$(\operatorname*{colim}_\mu X_\mu)^{hG} = (\operatorname*{colim}_\mu (X_\mu)^{\mathrm{dis}}_{\mathcal{N}})^{hG}.$$

We can extend this definition to an arbitrary closed subgroup $K$ in $G$: since the $K$–spectrum $\operatorname{colim}_\mu X_\mu$ can be regarded as the discrete $K$–spectrum $\operatorname{colim}_\mu (X_\mu)^{\mathrm{dis}}_{\mathcal{N}}$, we define

$$(\operatorname*{colim}_\mu X_\mu)^{hK} = (\operatorname*{colim}_\mu (X_\mu)^{\mathrm{dis}}_{\mathcal{N}})^{hK}.$$

We say that a profinite group $G$ has *finite virtual cohomological dimension* ("finite v.c.d.") if $G$ contains an open subgroup that has finite c.d.

**Theorem (D.)**

*Let $G$ be a profinite group with finite v.c.d. If $(G, \{X_\mu\}_\mu, \mathcal{N})$ is a suitably filtered triple and $K$ is a closed subgroup of $G$, then there is a conditionally convergent descent spectral sequence $E_r^{*,*}(K)$ that has the form*

$$E_2^{s,t}(K) = H_c^s(K; \pi_t(\operatorname{colim}_\mu X_\mu)) \Longrightarrow \pi_{t-s}\big((\operatorname{colim}_\mu X_\mu)^{hK}\big).$$

Ausoni showed that $K(ku_p) \wedge V(1)$ is an $f$–spectrum.

$$\Longrightarrow$$

$K(KU_p) \wedge V(1)$ is an $f$–spectrum.

Then our tools give

$$K(KU_p) \wedge v_2^{-1} V(1) = \operatorname*{colim}_{j \geq 0} \left( \left( K(KU_p) \wedge \Sigma^{-jd} V(1) \right)_f \right)_{\mathcal{N}}^{\mathrm{dis}} \in \Sigma\mathrm{Sp}_{\mathbb{Z}_p^\times}$$

and

$$\left( K(KU_p) \wedge v_2^{-1} V(1) \right)^{hK} = \left( \operatorname*{colim}_{j \geq 0} \left( \left( K(KU_p) \wedge \Sigma^{-jd} V(1) \right)_f \right)_{\mathcal{N}}^{\mathrm{dis}} \right)^{hK}.$$