THE ADAMS-NOVIKOV SPECTRAL SEQUENCE FOR $L_{K(n)}(X)$, WHEN X IS FINITE

DANIEL G. DAVIS

1. Summary

If X is a finite spectrum, there is an Adams-Novikov spectral sequence

$$H_c^s(S_n; \pi_t(E_n \wedge X)) \Rightarrow W(\mathbb{F}_{p^n}) \otimes_{\mathbb{Z}_p} \pi_*(L_{K(n)}X).$$

It is well-known that taking $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$-fixed points of this spectral sequence yields the spectral sequence

$$H_c^s(S_n \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p); \pi_t(E_n \wedge X)) \Rightarrow \pi_*(L_{K(n)}X).$$

Part of this last assertion is the statement that

$$H_c^s(S_n; \pi_t(E_n \wedge X))^{\mathrm{Gal}} \cong H_c^s(S_n \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p); \pi_t(E_n \wedge X)).$$

In this note, we give a proof of this isomorphism.

2. The details

Let E_n be the Lubin-Tate spectrum with $E_{n*} = W(\mathbb{F}_{p^n})[[u_1, ..., u_{n-1}]][u^{\pm 1}]$, where the degree of u is -2 and the complete power series ring over the Witt vectors is in degree zero. Let $G_n = S_n \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$, where S_n is the nth Morava stabilizer group.

Definition 2.1. Let $\mathcal{K}_{n,*}(X) = \varprojlim_I \pi_*(E_n \wedge M_I \wedge X)$, for an arbitrary spectrum X.

We recall Proposition 7.4 of [4]: If $\mathcal{K}_{n,*}(X)$ is finitely generated over E_{n*}, then there is a spectral sequence

$$E_2^{*,*} = H_c^{**}(S_n; \mathcal{K}_{n,*}(X)) \Rightarrow W(\mathbb{F}_{p^n}) \otimes_{\mathbb{Z}_p} \pi_*(L_{K(n)}X). \tag{2.1}$$

Remark 2.2. The second "$*$" in the superscript of the continuous cohomology above has the following meaning: if N_* is a graded G_n-module,

$$H_c^{*t}(S_n; N_*) = H_c^*(S_n; N_t).$$

Spectral sequence (2.1), constructed by Hopkins and Ravenel (see [3, §2.2], [5, pg. 241], [7, pg. 46], and the proof in [4, pg. 116]), is constructed by taking an inverse limit over $\{I\}$ of Adams-Novikov spectral sequences of the form

$$E_2^{*,*} = W(\mathbb{F}_{p^n}) \otimes_{\mathbb{Z}_p} \mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, BP_*(L_nX \wedge M_I)) \Rightarrow W(\mathbb{F}_{p^n}) \otimes_{\mathbb{Z}_p} \pi_*(L_nX \wedge M_I),$$

and applying the Morava change of rings isomorphism

$$W(\mathbb{F}_{p^n}) \otimes_{\mathbb{Z}_p} \mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, BP_*(L_nX \wedge M_I)) \cong H_c^{**}(S_n; \pi_t(E_n \wedge M_I \wedge X)).$$

Date: September 16, 2005.

Now assume that X is a finite spectrum, so that $\mathcal{K}_{n,*}(X) = \pi_*(E_n \wedge X)$ is a finitely generated E_{n*}-module.

Notice that Gal acts on the abutment of (2.1) by acting only on the Witt vectors. There is also a canonical action of Gal on the E_2 term: given $\sigma \in \mathrm{Gal}$, the automorphism

$$\sigma\colon \mathrm{Map}_c(S_n^k, \pi_*(E_n \wedge M_I \wedge X)) \to \mathrm{Map}_c(S_n^k, \pi_*(E_n \wedge M_I \wedge X))$$

is defined by

$$(\sigma \cdot h)(s_1, ..., s_k) = \sigma \cdot h(\sigma^{-1} s_1, ..., \sigma^{-1} s_k).$$

This induces the action of Gal on

$$H_c^s(S_n; \mathcal{K}_{n,*}(X)) \cong \varprojlim{}_I\, H_c^*(S_n; \pi_*(E_n \wedge M_I \wedge X)).$$

We recall the following definition from [1, Lem. 5.4] and [2, proof of Lem. 3.22].

Definition 2.3. Let N be a $W(\mathbb{F}_{p^n})$-module and a Gal-module. If Gal acts by $\mathbb{Z}_p$-module automorphisms and if $\sigma(cm) = (\sigma c)(\sigma m)$, whenever $\sigma \in \mathrm{Gal}$, $c \in W(\mathbb{F}_{p^n})$, and $m \in N$, then N is called a twisted Gal-$W(\mathbb{F}_{p^n})$ module.

We will make use of the following, a version of [1, Lem. 5.4].

Lemma 2.4. *If N is a twisted* Gal-$W(\mathbb{F}_{p^n})$ *module, then the inclusion $N^{\mathrm{Gal}} \hookrightarrow N$ extends to a* Gal-*equivariant isomorphism*

$$W(\mathbb{F}_{p^n}) \otimes_{\mathbb{Z}_p} N^{\mathrm{Gal}} \to N$$

of $W(\mathbb{F}_{p^n})$-modules.

Lemma 2.5. *If N is a twisted* Gal-$W(\mathbb{F}_{p^n})$ *module with the discrete topology, then, for any $j \geq 0$,* $\mathrm{Map}_c(S_n^j, N)$ *is a twisted* Gal-$W(\mathbb{F}_{p^n})$ *module.*

Proof. It is clear that $\mathrm{Map}_c(S_n^j, N)$ is a $W(\mathbb{F}_{p^n})$-module, and the Gal-action is as above.

We verify that the module structure is twisted:

$$\begin{aligned}(\sigma \cdot ch)(s_1, ..., s_j) &= \sigma \cdot ((ch)(\sigma^{-1} s_1, ..., \sigma^{-1} s_j)) = (\sigma c)(\sigma \cdot h(\sigma^{-1} s_1, ..., \sigma^{-1} s_j)) \\ &= (\sigma c)((\sigma \cdot h)(s_1, ..., s_j)) = ((\sigma c)(\sigma \cdot h))(s_1, ..., s_j).\end{aligned}$$

Finally, the following shows that Gal acts by $\mathbb{Z}_p$-homomorphisms, using the fact that Gal acts trivially on $\mathbb{Z}_p$: for any p-adic integer c,

$$(\sigma \cdot (ch))(s_1, ..., s_j) = c(\sigma \cdot h(\sigma^{-1} s_1, ..., \sigma^{-1} s_j)) = (c(\sigma \cdot h))(s_1, ..., s_j).$$

□

Lemma 2.6. *If N is a finite twisted* Gal-$W(\mathbb{F}_{p^n})$ *module with the discrete topology, then*

$$H^s(\mathrm{Gal}; N) = 0, \quad s > 0.$$

Proof. Since $S_n \lhd_o G_n$, the continuous projection $G_n \to G_n/S_n \cong \mathrm{Gal}$ makes N a discrete G_n-module. Thus, the hypotheses imply that N is a discrete twisted G_n-$W(\mathbb{F}_{p^n})$ module [2, Def. 3.21].

Recall [op. cit., pg. 23] that given a discrete twisted G_n-$W(\mathbb{F}_{p^n})$ module N and a closed subgroup H of G_n, there is a cochain complex $D_H^*(N)$ with $H^*(D_H^*(-))$ equivalent to $H_c^*(H; -)$ on the category of discrete twisted G_n-$W(\mathbb{F}_{p^n})$ modules. The cochains are given by

$$D_H^j(N) = \mathrm{Map}_c^{\ell}(S_n^{j+1}, N)^H,$$

where the action on $h \in \mathrm{Map}_c^\ell(S_n^k, N)$, by $(g,\sigma) \in S_n \rtimes \mathrm{Gal} = G_n$, is given by

$$((g,\sigma)h)(s_1, ..., s_k) = \sigma \cdot h(\sigma^{-1}(g^{-1}s_1), \sigma^{-1}s_2, ..., \sigma^{-1}s_k),$$

and the differential $\delta\colon \mathrm{Map}_c^\ell(S_n^{j+1}, N)^H \to \mathrm{Map}_c^\ell(S_n^{j+2}, N)^H$ is defined by

$$\begin{aligned}\delta h(s_0, s_1, ..., s_{j+1}) =& \big(\textstyle\sum_{i=0}^{j}(-1)^i h(s_0, ..., s_i s_{i+1}, ..., s_{j+1})\big) \\ &+ (-1)^{j+1} s_{j+1}^{-1} h(s_0, ..., s_j).\end{aligned}$$

Again, we point out that in the above, $\sigma^{-1}s_i$ refers to the action of Gal on S_n, instead of being the product of two elements in G_n.

When $H = \mathrm{Gal}$, the Gal-action specified above is identical to the action defined previously. Also, we need to verify that the Gal-action on the cochain complex $\mathrm{Map}_c(S_n^{*+1}, N)$ is by maps of cochain complexes. We compute:

$$\begin{aligned}(\sigma \cdot \delta(h))(s_0, ..., s_{j+1}) &= \sigma \cdot (\delta(h)(\sigma^{-1}s_0, ..., \sigma^{-1}s_{j+1})) \\ &= \big(\textstyle\sum_{i=0}^{j}(-1)^i \sigma \cdot h(\sigma^{-1}s_0, ..., (\sigma^{-1}s_i)(\sigma^{-1}s_{i+1}), ..., \sigma^{-1}s_{j+1})\big) \\ &\quad + (-1)^{j+1}\sigma \cdot ((\sigma^{-1}s_{j+1})^{-1} h(\sigma^{-1}s_0, ..., \sigma^{-1}s_j)).\end{aligned}$$

Recall that for $s \in S_n$, $\sigma s = \sigma s \sigma^{-1}$, where the right-hand side is a product of elements in G_n, and for $(s,\sigma), (s',\sigma') \in S_n \rtimes \mathrm{Gal}$, $(s,\sigma)(s',\sigma') = (s(\sigma \cdot s'), \sigma\sigma')$. Therefore,

$$(\sigma^{-1}s_i)(\sigma^{-1}s_{i+1}) = \sigma^{-1}s_i\sigma\sigma^{-1}s_{i+1}\sigma = \sigma^{-1} \cdot (s_i s_{i+1}).$$

Also,

$$\sigma \cdot ((\sigma^{-1}s_{j+1})^{-1} h(\sigma^{-1}s_0, ..., \sigma^{-1}s_j)) = (\sigma(\sigma^{-1}s_{j+1})^{-1}) \cdot h(\sigma^{-1}s_0, ..., \sigma^{-1}s_j),$$

where $\sigma(\sigma^{-1}s_{j+1})^{-1}$ is a product of elements in G_n. Thus,

$$\begin{aligned}\sigma(\sigma^{-1}s_{j+1})^{-1} &= \sigma(\sigma^{-1}s_{j+1}\sigma)^{-1} = (1,\sigma)(\sigma^{-1}s_{j+1}^{-1}\sigma, 1) \\ &= (\sigma \cdot (\sigma^{-1}s_{j+1}^{-1}\sigma), \sigma) = (s_{j+1}^{-1}, \sigma).\end{aligned}$$

Also,

$$\begin{aligned}(\delta(\sigma h))(s_0, ..., s_{j+1}) &= \big(\textstyle\sum_{i=0}^{j}(-1)^i(\sigma h)(s_0, ..., s_i s_{i+1}, ..., s_{j+1})\big) + \\ &\quad (-1)^{j+1}s_{j+1}^{-1}(\sigma h)(s_0, ..., s_j) \\ &= \big(\textstyle\sum_{i=0}^{j}(-1)^i \sigma \cdot h(\sigma^{-1}s_0, ..., \sigma^{-1} \cdot (s_i s_{i+1}), ..., \sigma^{-1}s_{j+1})\big) + \\ &\quad (-1)^{j+1}s_{j+1}^{-1}(\sigma \cdot h(\sigma^{-1}s_0, ..., \sigma^{-1}s_j)).\end{aligned}$$

Since $s_{j+1}^{-1}\sigma = (s_{j+1}^{-1}, 1)(1, \sigma) = (s_{j+1}^{-1}, \sigma)$, we see that $\sigma \cdot \delta(h) = \delta(\sigma h)$, so that σ is indeed a map of cochain complexes.

Since $\mathrm{Map}_c(S_n^j, N)$ is a twisted Gal-$W(\mathbb{F}_{p^n})$ module, there is a Gal-equivariant isomorphism

$$W(\mathbb{F}_{p^n}) \otimes_{\mathbb{Z}_p} \mathrm{Map}_c(S_n^j, N)^{\mathrm{Gal}} \to \mathrm{Map}_c(S_n^j, N)$$

of $W(\mathbb{F}_{p^n})$-modules. As in [1, proof of Lem. 5.15], since Gal acts by maps of cochain complexes and the above isomorphism is induced by the inclusion, there is actually an isomorphism

$$W(\mathbb{F}_{p^n}) \otimes_{\mathbb{Z}_p} \mathrm{Map}_c(S_n^{*+1}, N)^{\mathrm{Gal}} \to \mathrm{Map}_c(S_n^{*+1}, N)$$

of cochain complexes. Since $W(\mathbb{F}_{p^n}) \cong \bigoplus_{i=1}^n \mathbb{Z}_p$ is a free $\mathbb{Z}_p$-module, there is an isomorphism

$$\begin{aligned}\bigoplus_{i=1}^n H^*(\mathrm{Gal}; N) &\cong \bigoplus_{i=1}^n H^*(\mathrm{Map}_c(S_n^{*+1}, N)^{\mathrm{Gal}}) \\ &\cong H^*(\mathrm{Map}_c(S_n^{*+1}, N)) \cong H^*(\{e\}; N).\end{aligned}$$

Because $H^s(\{e\}; N) = 0$, whenever $s > 0$, the conclusion of the theorem follows. □

Theorem 2.7. *If X is a finite spectrum, then for all s and t,*

$$H_c^s(S_n; \pi_t(E_n \wedge M_I \wedge X))^{\mathrm{Gal}} \cong H_c^s(G_n; \pi_t(E_n \wedge M_I \wedge X)).$$

Proof. Consider the canonical restriction map

$$H_c^s(G_n; \pi_t(E_n \wedge M_I \wedge X)) \to H_c^s(S_n; \pi_t(E_n \wedge M_I \wedge X))^{\mathrm{Gal}},$$

and the Lyndon-Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p(\mathrm{Gal}; H_c^q(S_n; \pi_t(E_n \wedge M_I \wedge X))) \Rightarrow H_c^{p+q}(G_n; \pi_t(E_n \wedge M_I \wedge X)).$$

It is enough to prove that $E_2^{p,q} = 0$ for all $p \geq 1, q \geq 0$, since this implies that

$$\begin{aligned}H_c^q(G_n; \pi_t(E_n \wedge M_I \wedge X)) &\cong H^0(\mathrm{Gal}; H_c^q(S_n; \pi_t(E_n \wedge M_I \wedge X))) \\ &\cong H_c^q(S_n; \pi_t(E_n \wedge M_I \wedge X))^{\mathrm{Gal}}.\end{aligned}$$

As a finite G_n-module, $\pi_t(E_n \wedge M_I \wedge X)$ is also an $E_{n,0}$-module with the $E_{n,0}$-module structure map G_n-equivariant (since G_n acts on E_{n*} by ring automorphisms). Then $W(\mathbb{F}_{p^n}) \subset E_{n,0}$ implies that $\pi_t(E_n \wedge M_I \wedge X)$ is a twisted Gal-$W(\mathbb{F}_{p^n})$ module.

By the preceding result, we only have to prove that

$$\mathcal{H} = H_c^q(S_n; \pi_t(E_n \wedge M_I \wedge X))$$

is a finite twisted Gal-$W(\mathbb{F}_{p^n})$ module with the discrete topology. Since S_n is a compact p-adic analytic group, by [8, Prop. 4.2.2], $\mathcal{H}$ is a finite discrete abelian group. By [6, Remark 7.2.2], Gal acts on $\mathcal{H}$ by automorphisms, and it is easy to check that these automorphisms are $\mathbb{Z}_p$-homomorphisms. Since each $\mathrm{Map}_c(S_n^j, \pi_t(E_n \wedge M_I \wedge X))$ is a $W(\mathbb{F}_{p^n})$-module, the maps in the cochain complex of continuous cochains are $W(\mathbb{F}_{p^n})$-homomorphisms, so that $\mathcal{H}$ is a $W(\mathbb{F}_{p^n})$-module. Also, it is easy to check that $\mathcal{H}$ is twisted, since the continuous cochains are. Thus, $\mathcal{H}$ is a finite twisted Gal-$W(\mathbb{F}_{p^n})$ module and a discrete abelian group. □

Corollary 2.8. *If X is a finite spectrum, then, for all s and t,*

$$H_c^s(S_n; \mathcal{K}_{n,t}(X))^{\mathrm{Gal}} \cong H_c^s(G_n; \pi_t(E_n \wedge X)).$$

Proof. We have:

$$\begin{aligned}H_c^s(S_n; \mathcal{K}_{n,t}(X))^{\mathrm{Gal}} &\cong \varprojlim{}_{\mathrm{Gal}} \varprojlim{}_I H_c^s(S_n; \pi_t(E_n \wedge M_I \wedge X)) \\ &\cong \varprojlim{}_I H_c^s(G_n; \pi_t(E_n \wedge M_I \wedge X)).\end{aligned}$$

□

Therefore, as stated in [7, pg. 46], after taking Gal-fixed points, the spectral sequence in (2.1) becomes

$$H_c^*(G_n; \pi_*(E_n \wedge X)) \Rightarrow \pi_*(L_{K(n)}X). \tag{2.2}$$

References

[1] Ethan S. Devinatz. Morava's change of rings theorem. In *The Čech centennial (Boston, MA, 1993)*, pages 83–118. Amer. Math. Soc., Providence, RI, 1995.

[2] Ethan S. Devinatz and Michael J. Hopkins. Homotopy fixed point spectra for closed subgroups of the Morava stabilizer groups. *Manuscript, 1999*.

[3] M. J. Hopkins and B. H. Gross. The rigid analytic period mapping, Lubin-Tate space, and stable homotopy theory. *Bull. Amer. Math. Soc. (N.S.)*, 30(1):76–86, 1994.

[4] Michael J. Hopkins, Mark Mahowald, and Hal Sadofsky. Constructions of elements in Picard groups. In *Topology and representation theory (Evanston, IL, 1992)*, pages 89–126. Amer. Math. Soc., Providence, RI, 1994.

[5] Mark Hovey. Bousfield localization functors and Hopkins' chromatic splitting conjecture. In *The Čech centennial (Boston, MA, 1993)*, pages 225–250. Amer. Math. Soc., Providence, RI, 1995.

[6] Luis Ribes and Pavel Zalesskii. *Profinite groups*. Springer-Verlag, Berlin, 2000.

[7] N. P. Strickland. On the p-adic interpolation of stable homotopy groups. In *Adams Memorial Symposium on Algebraic Topology, 2 (Manchester, 1990)*, pages 45–54. Cambridge Univ. Press, Cambridge, 1992.

[8] Peter Symonds and Thomas Weigel. Cohomology of p-adic analytic groups. In *New horizons in pro-p groups*, pages 349–410. Birkhäuser Boston, Boston, MA, 2000.