# PROFINITE AND DISCRETE $G$-SPECTRA AND ITERATED HOMOTOPY FIXED POINTS

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Abstract. In chromatic homotopy theory, given $K \lhd G < G_n$, closed subgroups of the extended Morava stabilizer group $G_n$, and the Lubin-Tate spectrum $E_n$, which carries an action by $G_n$, the problem of understanding both the homotopy fixed point spectra of $E_n$ for the actions of $K$ and $G$ and the relationship between these two spectra is multifaceted. Let $E_n^{dhG}$ be the construction of Devinatz-Hopkins that behaves like continuous homotopy fixed points, and let $E_n^{hG}$ and $E_n^{h'G}$ be the continuous homotopy fixed points in the settings of profinite $G$-spectra and continuous $G$-spectra, respectively. Then there is (a) the strongly convergent Adams-type spectral sequence of Devinatz with abutment $\pi_*(E_n^{dhG})$ and $E_2$-term the continuous cohomology $H^s_c(G/K; \pi_t(E_n^{dhK}))$; (b) a descent spectral sequence abutting to $\pi_*(E_n^{h'G})$ with $E_2$-term $H^s_c(G/K; \pi_t(E_n^{h'K}))$; (c) the iterated continuous homotopy fixed points $(E_n^{h'K})^{h'G/K}$; and (d) equivalences $E_n^{dhG} \simeq E_n^{h'G} \simeq E_n^{hG}$. This rich situation motivates various natural questions about the existence of certain constructions and interrelations.

In this paper, we establish some connections between the aforementioned notions of homotopy fixed points and give various sets of sufficient conditions for the existence of well-behaved iterated continuous homotopy fixed points in the profinite setting. Our results yield the following answers to the questions referred to above. First, we prove that $E_n^{hK}$ is a profinite $G/K$-spectrum. This result shows that in the profinite setting $(E_n^{hK})^{hG/K}$ exists, and we show that it is just $E_n^{hG}$. Also, we complete the story begun by (a)–(d) and our just-described answers by proving that there is a continuous homotopy fixed point spectral sequence with abutment $\pi_*(E_n^{hG})$ and $E_2$-term $H^s_c(G/K; \pi_t(E_n^{hK}))$ that is isomorphic to the spectral sequences of (a) and (b). Our proof of well-behaved iteration for $E_n$ in the profinite setting possesses a technical simplicity in a certain aspect that is not enjoyed by the corresponding proofs in the settings of Devinatz-Hopkins (where $G/K$ must be finite) and continuous $G$-spectra.

## 1. Introduction

If $G$ is a (discrete) group acting on a spectrum $X$, one can form the homotopy fixed point spectrum $X^{hG}$. The spectrum $X^{hG}$ is defined as the $G$-fixed points of the function spectrum $F(EG_+, X)$, where $EG$ is a free contractible $G$-space. If $G$ carries a (non-discrete) topology with respect to which the action on $X$ is in some sense continuous, one would like to have constructions of (i) a continuous homotopy fixed point spectrum that respects the continuous action, and (ii) an associated homotopy fixed point spectral sequence whose $E_2$-term consists of

The first author was partially supported by a grant (LEQSF(2008-11)-RD-A-27) from the Louisiana Board of Regents during the first half of his work on this paper.

The second author was partially supported by Research Fellowship QU 317/1 of the German Research Foundation (DFG).

continuous cohomology groups. When $G$ is a profinite group, by building on earlier work – by Jardine [20, 21] and Thomason [32] (see the helpful paper [22] by Mitchell), and Goerss [15], in the case of discrete objects, and by the second author [24, 26] in the case of profinite objects, this construction problem has been studied (a) for discrete and continuous $G$-spectra in [5, 1], and (b) for profinite $G$-spectra in [27]. Motivated by the fact that a profinite $G$-set that is finite is also a discrete $G$-set, one of the purposes of this paper is to compare approaches (a) and (b) in certain cases and let the tools of each approach supplement the techniques of the other one.

It is a standard fact that if $H$ is any (discrete) group and $N$ is any normal subgroup, then for any $H$-space $\mathcal{X}$, the space $\mathcal{X}^{hN}$ can be identified with the $H/N$-space $\mathrm{Map}^N(EH, \mathcal{X})$, so that the iterated homotopy fixed point space $(\mathcal{X}^{hN})^{hH/N}$ is defined. Furthermore, it is well-known that $(\mathcal{X}^{hN})^{hH/N}$ is just $\mathcal{X}^{hH}$. Thus, another purpose of this paper is to take this familiar situation as a model and, for a profinite group $G$, consider the formation of iterated continuous homotopy fixed points in the setting of profinite $G$-spectra.

For the study of continuous actions by profinite groups in homotopy theory, a fundamental and motivating example is the action of the extended Morava stabilizer group $G_n$ on the Lubin-Tate spectrum $E_n$. We quickly review this example.

Let $p$ be a fixed prime, $n \geq 1$ an integer and $\mathbb{F}_{p^n}$ the field with $p^n$ elements. Let $S_n$ be the $n$th Morava stabilizer group, i.e. the automorphism group of the height $n$ Honda formal group law over $\mathbb{F}_{p^n}$. We denote by $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ the Galois group of $\mathbb{F}_{p^n}$ over $\mathbb{F}_p$ and let

$$G_n = S_n \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$$

be the semi-direct product. Let $K(n)$ be the $n$th Morava $K$-theory spectrum with $K(n)_* = \mathbb{F}_p[v_n^{\pm 1}]$, with $|v_n| = 2(p^n - 1)$. The Lubin-Tate spectrum $E_n$ is the $K(n)_*$-local Landweber exact spectrum whose coefficients are given by

$$E_{n*} = W(\mathbb{F}_{p^n})[[u_1, \ldots, u_{n-1}]][u, u^{-1}],$$

where $W(\mathbb{F}_{p^n})$ is the ring of Witt vectors of the field $\mathbb{F}_{p^n}$, $|u_i| = 0$ for all $i$ and $|u| = -2$. The group $G_n$ acts on the graded ring $E_{n*}$, and by Brown representability, this action is induced by an action of $G_n$ on $E_n$ by maps of ring spectra in the stable homotopy category. Furthermore, by work of Goerss, Hopkins and Miller (see [17, 29]), this homotopy action is induced by an action of $G_n$ on $E_n$ before passage to the stable homotopy category.

Now $G_n$ is a profinite group and each homotopy group $\pi_t E_n$ has the structure of a continuous profinite $G_n$-module. From Morava's change of rings theorem we know that the continuity of the action of $G_n$ on each $\pi_t E_n$ is an important property for stable homotopy theory (to view Morava's theorem in action, see, for example, [12, Section 2]). The most succinct way to convey the importance of this continuous action is to note that for any finite spectrum $Y$, there is a strongly convergent homotopy fixed point spectral sequence that has the form

$$H^s_c(G_n; \pi_t(E_n \wedge Y)) \Rightarrow \pi_{t-s}\big(L_{K(n)}(Y)\big),$$

where the $E_2$-term is continuous cohomology, $\pi_t(E_n \wedge Y)$ is a continuous profinite $G_n$-module (this structure is induced by $G_n$ acting diagonally on $E_n \wedge Y$, with $Y$ given the trivial $G_n$-action), and $L_{K(n)}(-)$ denotes Bousfield localization with respect to $K(n)$ (this result is [1, Corollary 8.2.4, Theorem 8.2.5], which depends on [12, Theorem 1]; see also [18, Proposition 7.4]). Therefore, to make sense of $E_n$ as a *continuous* $G_n$-spectrum is a fundamental problem.

For the closed subgroups $G$ of $G_n$, Devinatz and Hopkins [12] gave a construction of commutative $S$-algebras, here denoted by $E_n^{dhG}$, that behave like continuous homotopy fixed point spectra. However, the construction of the $E_n^{dhG}$ does not make use of a continuous action of $G$ on $E_n$. Using the construction of $E_n^{dhU}$ for open normal subgroups $U$ of $G_n$, a new and systematic definition of homotopy fixed points with respect to a continuous $G$-action, for arbitrary closed subgroups $G$, was given in [5]: we denote these continuous homotopy fixed points by $E_n^{h'G}$. The formation of the $E_n^{h'G}$ is based on the notion of discrete $G$-spectrum (a spectrum that is built out of simplicial discrete $G$-sets) and homotopy limits of towers of discrete $G$-spectra (such homotopy limits are the continuous $G$-spectra of [5]).

In [27], a different construction for a continuous homotopy fixed point spectrum $E_n^{hG}$ and its descent spectral sequence, independent of [12] and [5], has been obtained. The approach of [27] is to consider $E_n$ as an object in the category of profinite $G$-spectra, which, in contrast to the discrete $G$-spectra mentioned above, are $G$-spectra that are built out of simplicial profinite $G$-sets. In this framework, the profinite $G$-spectrum $E_n$ is a (homotopy) limit of certain spectra that are simultaneously discrete $G$-spectra and profinite $G$-spectra.

Each one of the above approaches has its own advantages (and drawbacks). But, as one might expect, there are equivalences

$$E_n^{dhG} \simeq E_n^{h'G} \simeq E_n^{hG}$$

for every closed subgroup $G$, by [1, Theorem 8.2.1] and [27, top of page 220], respectively.

Let us now go a step further and consider iterated homotopy fixed points. If $K$ is a closed normal subgroup of an arbitrary profinite group $G$ and $X$ is any profinite $G$-spectrum, the fixed points satisfy the equality $X^G = (X^K)^{G/K}$. Given this identity and the fact – mentioned earlier – that $(\mathcal{X}^{hN})^{hH/N}$ is always just $\mathcal{X}^{hH}$, a natural question to ask is if continuous homotopy fixed points satisfy a similar relationship: is there an equivalence

$$X^{hG} \simeq (X^{hK})^{hG/K} \tag{1}$$

between these two spectra? In the setting of profinite groups and for any object in some category of $G$-spectra, the question represented by (1) was first asked in [9, page 130] and, for the category of discrete $G$-spectra, the question was studied in detail in [7, 8]. The equivalence in (1) would simplify the analysis of the homotopy fixed points under $G$ by reducing it to the study of those under proper closed normal subgroups $K$ and the quotients $G/K$.

Unfortunately, it is in general not known that the homotopy fixed point spectrum $X^{hK}$ has the same topological characterization as the profinite $G$-spectrum $X$. For example, when the profinite group $G/K$ is not finite, $X^{hK}$ is in general not known to be a profinite $G/K$-spectrum. Thus, additional assumptions on $X$ and $G$ are necessary to hope for equivalence (1). These basic issues in the problem of iteration are considered in more detail in Sections 4.1 and 4.2.

In the case of the Lubin-Tate spectrum $E_n$, with $G$ now any closed subgroup of $G_n$, there are issues related to those mentioned above. For example, in [12] Devinatz and Hopkins did not obtain a construction of a continuous homotopy fixed point spectrum $(E_n^{dhK})^{hG/K}$ when $G/K$ is not finite. Nevertheless, by a sophisticated study of the structure of $E_n^{dhK}$ as a $E_n^{dhG}$-module, Devinatz [9] was able to construct a strongly convergent (Adams-type) Lyndon-Hochschild-Serre spectral sequence

$$H_c^*(G/K; \pi_*(E_n^{dhK})) \Rightarrow \pi_*(E_n^{dhG}), \tag{2}$$

with $E_2$-term given by continuous cohomology.

In [7], the first author was able to make sense of $E_n^{h'K}$ (as defined in [5]) as a continuous $G/K$-spectrum for an arbitrary closed normal subgroup $K$. Moreover, it was shown in [7] that there is an equivalence

$$E_n^{h'G} \simeq (E_n^{h'K})^{h'G/K}$$

and a descent spectral sequence

$$H_c^*(G/K; \pi_*(E_n^{h'K})) \Rightarrow \pi_*(E_n^{h'G}) \tag{3}$$

that is isomorphic to spectral sequence (2), by [7, Theorem 7.6].

Though it is somewhat of an oversimplification, let us describe the results of the preceding paragraph as taking place in the "world of continuous $G$-spectra" (this terminology is an adaptation of the "$G$-world" terminology of [21] (for example, see [21, page 211])). Then the main purpose of this paper is to show that analogous results hold in the setting of profinite $G$-spectra, by using the independent construction of continuous homotopy fixed points in [27]. To be precise, our main results are the following.

**Theorem 1.1.** *Let $G$ be an arbitrary closed subgroup of $G_n$ and let $K$ be a closed normal subgroup of $G$. The continuous homotopy fixed point spectrum $E_n^{hK}$ has a model in the category of profinite $G/K$-spectra, there is an iterated continuous homotopy fixed point spectrum $(E_n^{hK})^{hG/K}$ and there is an equivalence*

$$E_n^{hG} \simeq (E_n^{hK})^{hG/K}.$$

The helpful notion of "model" that is used in the above result is explained in a precise way in Definition 4.1.

To be more explicit about the first two conclusions of Theorem 1.1 and to quickly illustrate that profinite iteration problems are not easy to solve, the proof of the theorem shows that the spectrum $E_n^{hK}$ can be identified with the profinite $G/K$-spectrum

$$\operatorname*{holim}_{k\geq 0} \operatorname*{holim}_{q\in\mathbb{Z}} F^s_{G/K}\Big(\operatorname*{colim}_{U\lhd_o G} \mathrm{Map}(EG, P^q E'_{n,I_k})^{KU}\Big), \tag{4}$$

so that

$$(E_n^{hK})^{hG/K} = \Big(\operatorname*{holim}_{k\geq 0} \operatorname*{holim}_{q\in\mathbb{Z}} F^s_{G/K}\Big(\operatorname*{colim}_{U\lhd_o G} \mathrm{Map}(EG, P^q E'_{n,I_k})^{KU}\Big)\Big)^{hG/K}.$$

In expression (4), all the undefined notation is carefully explained in later sections, but to gain a fairly complete understanding of what spectrum (4) is describing, it suffices to say that (a) in each of its applications above, $F^s_{G/K}(-)$ returns a profinite $G/K$-spectrum that is weakly equivalent to the $G/K$-spectrum that is its input and (b) morally, $P^q E'_{n,I_k}$ is "the $q$th Postnikov section of the $G_n$-spectrum $E_n/I_k$," where

$$\pi_*(E_n) \cong \lim_{k\geq 0} \pi_*(E_n)/I_k.$$

**Theorem 1.2.** *Let $G$ and $K$ be as in Theorem 1.1. There is a strongly convergent spectral sequence for iterated continuous homotopy fixed points*

$$H^s_c(G/K; \pi_t(E_n^{hK})) \Rightarrow \pi_{t-s}(E_n^{hG}),$$

*with $E_2$-term equal to the continuous cohomology of $G/K$ with coefficients the profinite $G/K$-module $\pi_t(E_n^{hK})$. This spectral sequence is isomorphic to the spectral sequences of* (2) *and* (3)*, from the $E_2$-term onward.*

The proofs of Theorems 1.1 and 1.2 are given in Section 5.

In order to prove Theorem 1.1 we provide various sets of sufficient conditions on a profinite group $G$ and a profinite $G$-spectrum $X$ that allow for the formation of iterated continuous homotopy fixed points and the obtaining of equivalence (1). The main tool is a certain comparison result between profinite and discrete homotopy fixed points. In more detail, under the assumption that $G$ has finite virtual cohomological dimension, Theorem 3.2 shows that for $G$-spectra that are built out of simplicial finite discrete $G$-sets, the two notions of continuous homotopy fixed points agree.

Another useful tool in our work is the notion of hyperfibrant discrete $G$-spectrum from [7]; we recall the definition and main properties of this concept in Remark 4.4. A third key tool is a certain type of profinite $G$-spectrum that has well-behaved Postnikov sections with respect to a closed normal subgroup $K$ (see Definition 4.7 for the details). These three tools are the primary input for the proof of Proposition 4.9. This proposition is the iteration result that serves as the springboard for the iteration results of Theorems 4.17 and 4.20, and Corollary 4.23.

Though the statement of Theorem 1.2 is clearly the result that one desires to more fully tie together $E_n^{dhG}$, $E_n^{hG}$ and $E_n^{h'G}$, its proof is quite intricate and a brief road map might be useful: the proof can be described as consisting of a chain of isomorphisms between spectral sequences.

It is worth noting that continuous iterated homotopy fixed points for $E_n$ are not just of purely theoretical interest. For example, [7, page 2883] (building on [10, page 133]) shows that certain instances of $(E_n^{h'K})^{h'G/K}$ play a useful role in the work of Devinatz [10, 11] on the major conjecture in chromatic homotopy theory that $\pi_*\big(L_{K(n)}(\mathbb{S}^0)\big)$ is a module of finite type over $\mathbb{Z}_p$. Also, given a continuous epimorphism $G_n \to \mathbb{Z}_p$ of groups with kernel $K$ and the topological generator 1 of $\mathbb{Z}_p$, [12, Proposition 8.1] shows that a model for $(E_n^{dhK})^{h\mathbb{Z}_p}$, the continuous $\mathbb{Z}_p$-homotopy fixed points of $E_n^{dhK}$, is given by taking the homotopy fiber of the map $E_n^{dhK} \xrightarrow{\mathrm{id}-1} E_n^{dhK}$ (this construction of the continuous $\mathbb{Z}_p$-homotopy fixed points is a special case of a well-known technique (for example, see [16, §2.2])), and this homotopy fiber sequence plays a role in constructing an interesting element in $\pi_{-1}\big(L_{K(n)}(\mathbb{S}^0)\big)$, for all $n$ and $p$ [12, Theorem 6].

Other examples of the importance of $(E_n^{h'K})^{h'G/K}$ occur in [33, end of §1.1] and [33, §5.5]. In the last reference, a doubly iterated homotopy fixed point spectrum

$$\Big((E_n^{h'K})^{h'G/K}\Big)^{h'G_n/G}$$

makes an appearance (we refer the reader to [33, §2.2, Corollary 3.24, §5.5] for the definitions of $K$ and $G$). Given these examples, we expect there to be situations where $(E_n^{hK})^{hG/K}$ will be a useful tool in chromatic theory.

We close our Introduction by pointing out a subtle feature of our proof that $(E_n^{hK})^{hG/K} \simeq E_n^{hG}$ is always valid (Theorem 1.1), a feature that is not enjoyed (i) by the proof in [7] that $(E_n^{h'K})^{h'G/K} \simeq E_n^{h'G}$ always holds, or (ii) by the proof in [12] that when $G/K$ is finite, $(E_n^{dhK})^{hG/K} \simeq E_n^{dhG}$ (see [12, Theorem 4, proof of Proposition 7.1]). To see this subtlety, we begin by noting that [12, Corollary 5.5] implies that if $U$ is an open subgroup of $G_n$, then there is an equivalence

$$L_{K(n)}(E_n^{dhU} \wedge E_n) \simeq \prod\nolimits_{G_n/U} E_n, \tag{5}$$

where the right-hand side is a finite product of $|G_n/U|$ copies of $E_n$. For our purposes here, we want to point out that the proof of (5) (see [12, pages 24–30]) is highly nontrivial and, in particular, it uses (see [12, page 29: the proof of Theorem 5.3]) the deep result due to Hopkins and Ravenel that there exists a finite spectrum $W$ with torsion-free $\mathbb{Z}_{(p)}$-homology, such that the continuous cohomology groups $H_c^{s,*}(U; \pi_*(E_n \wedge W)/I_n\pi_*(E_n \wedge W))$ vanish for all $s$ greater than some $s_0$ (see [28, Lemmas 8.3.5–8.3.7]). It is worth noting that this result of Hopkins and Ravenel played a key role in the proof of the very important smashing conjecture (which states that $L_{E(n)}(-)$ is a smashing localization, where $E(n)$ is the Johnson-Wilson spectrum; see [28, Theorem 7.5.6, Chapter 8]).

(To keep our explanation of the above point from being too long, the rest of our discussion is written in a style that assumes the reader has certain portions of [7, 12] readily available.)

The proof in [12] of the iteration result in (ii) above depends on (5). (The details for this assertion are as follows: [12, proof of Proposition 7.1] uses the isomorphism in [12, (6.5)], and the proof of this isomorphism uses (5) (see the second equality in [12, proof of Proposition 6.3]).) Similarly, the proof of the iteration result in (i) applies (5). (In detail: the result in (i) is [7, Theorem 7.3]; its proof depends on [7, Lemma 7.1]; the proof of this lemma uses [12, Proposition 6.3]; and as noted above, the proof of this proposition uses (5).)

However, the proof of Theorem 1.1 avoids using (5): this is not hard to see by going over the proof and by noting that in its use of the proof of Proposition 4.9 (see the proof of Theorem 4.20), it is able to utilize [7, proof of Lemma 4.9], thanks to the use of Postnikov towers (the properties of Postnikov towers that are relevant here are discussed in the first paragraph of Section 4.3). By contrast, the proof of the result in (i) (that is, [7, proof of Theorem 7.3]) also used [7, proof of Lemma 4.9], but to do so, it needed to apply (5), as described above. Thus, the proof of well-behaved iteration for $E_n$ in the profinite setting has the interesting technical advantage that it is simpler than the proofs referred to in (i) and (ii), in that the proof in the profinite setting does not depend on the deep result of Hopkins and Ravenel.

**Acknowledgements.** Both authors thank Ethan Devinatz for helpful remarks about the proof of Corollary 4.12.

## 2. Spectra with a continuous $G$-action

Everywhere in this paper, unless stated otherwise, $G$ denotes a profinite group. In this section we quickly review the categories of spectra with continuous $G$-action that we need for our work.

2.1. **Discrete $G$-spectra.** We summarize the most important properties of simplicial discrete $G$-sets and discrete $G$-spectra. More details can be found in [15], [5] and [1].

A $G$-set $S$ is called discrete if the action is continuous when $S$ is given the discrete topology. This is equivalent to requiring that the stabilizer of any element in $S$ be an open subgroup in $G$ and to asking that $S$ be equal to the colimit of fixed points

$$S = \operatorname*{colim}_U S^U$$

over the open subgroups $U$ of $G$. A simplicial discrete $G$-set is a simplicial object in the category of discrete $G$-sets. By defining morphisms as levelwise $G$-equivariant maps we obtain the category of simplicial discrete $G$-sets which we denote by $\mathcal{S}_G$.

In [15, Theorem 1.12], Goerss showed that there is a model structure on $\mathcal{S}_G$ for which the cofibrations are the monomorphisms and the weak equivalences are the morphisms whose underlying maps of simplicial sets are weak equivalences in the standard model structure on the category $\mathcal{S}$ of simplicial sets. The category $\mathcal{S}_{*G}$ of pointed simplicial discrete $G$-sets inherits a model structure from $\mathcal{S}_G$ in the usual way: a map is a weak equivalence (cofibration, fibration) if its underlying map in $\mathcal{S}_G$ is a weak equivalence (cofibration, fibration, respectively).

In order to stabilize the category $\mathcal{S}_{*G}$ we consider the category $\mathrm{Sp}(\mathcal{S}_{*G})$ of discrete $G$-spectra. An object $X$ of $\mathrm{Sp}(\mathcal{S}_{*G})$ consists of a sequence $\{X_n\}_{n\geq 0}$ of pointed simplicial discrete $G$-sets $X_n$ together with structure maps

$$\sigma_n : S^1 \wedge X_n \to X_{n+1}$$

in $\mathcal{S}_{*G}$, where we consider $S^1$ as a pointed simplicial discrete $G$-set with trivial $G$-action. A map $f : X \to Y$ of discrete $G$-spectra is a sequence of maps $f_n : X_n \to Y_n$ in $\mathcal{S}_{*G}$ which are compatible with the structure maps.

The model structure on $\mathcal{S}_{*G}$ is left proper and cellular: see [15] or [1, Lemma 2.1.3]. This allows one to use Hovey's stabilization methods [19] in order to prove the following theorem, as in [1, Theorem 2.2.1]; the initial proof using presheaves of spectra was given in [5].

**Theorem 2.1.** *The category $\mathrm{Sp}(\mathcal{S}_{*G})$ admits a model structure in which a map is a weak equivalence (cofibration) if its underlying map of Bousfield-Friedlander spectra is a weak equivalence (cofibration).*

**Remark 2.2.** For a discrete $G$-spectrum $X$, there is an induced action of $G$ on $\pi_*X$, such that each stable homotopy group $\pi_k X$ is a discrete $G$-module (see [5, Corollary 3.12]).

**Remark 2.3.** There is also a version of discrete $G$-spectra based on symmetric spectra: see [1, §2.3]. But for the purposes of this paper it suffices to consider the model structure of the previous theorem on $\mathrm{Sp}(\mathcal{S}_{*G})$.

2.2. **Mapping spectra I.** In order to study homotopy fixed points we will need the following notion of mapping spectrum. Let $T$ be any set. Then the set $\mathrm{Map}_c(G,T)$ of continuous functions $G \to T$, where $T$ is regarded as a space with the discrete topology, is a discrete $G$-set with $G$-action given by $(gf)(h) = f(hg)$. If $Y$ is a simplicial set, the mapping space $\mathrm{Map}_c(G,Y)$ is defined to be the simplicial discrete $G$-set given in degree $m$ by

$$\mathrm{Map}_c(G,Y)_m = \mathrm{Map}_c(G,Y_m).$$

Now let $X$ be any spectrum. The continuous mapping spectrum $\mathrm{Map}_c(G, X)$ is defined to be the discrete $G$-spectrum whose $n$th space is

$$\mathrm{Map}_c(G, X_n).$$

It is not hard to see that there is an isomorphism of spectra

$$\mathrm{Map}_c(G, X) \cong \operatorname*{colim}_{N \lhd_o G} \prod_{G/N} X,$$

where the colimit is over the open normal subgroups of $G$. Also, if $X$ is a discrete $G$-spectrum, we again write $\mathrm{Map}_c(G, X)$ for the continuous mapping spectrum that is obtained as above, by just regarding $X$ as a spectrum.

2.3. **Profinite $G$-spectra.** A profinite space is a simplicial object in the category $\hat{\mathcal{E}}$ of profinite sets. Together with levelwise continuous maps, profinite spaces form a category that is denoted by $\hat{\mathcal{S}}$. If $\mathcal{S}$ denotes the category of simplicial sets, then the forgetful functor $|\cdot| : \hat{\mathcal{S}} \to \mathcal{S}$ has a left adjoint $\hat{(\cdot)} : \mathcal{S} \to \hat{\mathcal{S}}$ which we call profinite completion. There is a model structure on $\hat{\mathcal{S}}$ for which the cofibrations are the monomorphisms and a weak equivalence is a map $f$ which induces isomorphisms on $\pi_0$, the profinite fundamental group and on continuous cohomology with finite local coefficients. We refer to [24] and [26] for the details.

Let $S$ be a profinite set with a continuous map $\mu : G \times S \to S$ that satisfies the axioms of a group action. We call such an $S$ a profinite $G$-set. If $X$ is a profinite space and $G$ acts continuously on each $X_n$ such that the action is compatible with the structure maps, then we call $X$ a profinite $G$-space. We use $\hat{\mathcal{S}}_G$ to denote the category of profinite $G$-spaces with $G$-equivariant maps of profinite spaces as morphisms. If $X$ is a pointed profinite space with a continuous $G$-action that fixes the basepoint, then we call $X$ a pointed profinite $G$-space. We denote the corresponding category by $\hat{\mathcal{S}}_{*G}$.

The category $\hat{\mathcal{S}}_{*G}$ carries a fibrantly generated left proper simplicial model structure for which a map $f$ is a weak equivalence if and only if its underlying map is a weak equivalence in $\hat{\mathcal{S}}_*$ and is a cofibration if and only if $f$ is a levelwise injection and the action of $G$ on $Y_n - f(X_n)$ is free for each $n \geq 0$. The corresponding homotopy category is denoted by $\hat{\mathcal{H}}_{*G}$.

We would like to stabilize the category of profinite spaces. Since the simplicial circle $S^1 = \Delta^1/\partial\Delta^1$ is a simplicial finite set and hence an object in $\hat{\mathcal{S}}_*$, we may stabilize $\hat{\mathcal{S}}_*$ by considering sequences of pointed profinite spaces together with bonding maps for the suspension. In more detail, a profinite spectrum $X$ consists of a sequence of pointed profinite spaces $X_n \in \hat{\mathcal{S}}_*$ and maps $\sigma_n : S^1 \wedge X_n \to X_{n+1}$ in $\hat{\mathcal{S}}_*$ for $n \geq 0$. A morphism $f : X \to Y$ of spectra consists of maps $f_n : X_n \to Y_n$ in $\hat{\mathcal{S}}_*$ for $n \geq 0$ such that $\sigma_n(1 \wedge f_n) = f_{n+1}\sigma_n$. We denote by $\mathrm{Sp}(\hat{\mathcal{S}}_*)$ the corresponding category of profinite spectra.

There is a stable simplicial model structure on the category $\mathrm{Sp}(\hat{\mathcal{S}}_*)$. Also, the levelwise profinite completion functor is a left Quillen functor from Bousfield-Friedlander spectra $\mathrm{Sp}(\mathcal{S}_*)$ to $\mathrm{Sp}(\hat{\mathcal{S}}_*)$.

A profinite $G$-spectrum $X$ is a sequence of pointed profinite $G$-spaces $\{X_n\}$ together with pointed $G$-equivariant maps $S^1 \wedge X_n \to X_{n+1}$ for each $n \geq 0$, where $S^1$ is equipped with a trivial $G$-action. A map of profinite $G$-spectra $X \to Y$ is a collection of maps $X_n \to Y_n$ in $\hat{\mathcal{S}}_{*G}$ compatible with the structure maps of $X$ and $Y$. The following theorem was proved in [26].

**Theorem 2.4.** *There is a stable left proper simplicial model structure on* $\mathrm{Sp}(\hat{\mathcal{S}}_{*G})$ *in which a map between fibrant profinite $G$-spectra is an equivalence if and only if it is an equivalence in* $\mathrm{Sp}(\hat{\mathcal{S}}_*)$.

Finally, if $X$ is a profinite $G$-spectrum, then there is an induced action of $G$ on each stable profinite homotopy group $\pi_k(R_G X)$, where $R_G$ denotes a fibrant replacement functor for profinite $G$-spectra: this $G$-action is compatible with the profinite structure and each stable profinite homotopy group $\pi_k(R_G X)$ is a continuous profinite $G$-module. Thus, the topological $G$-module structure of $\pi_k(R_G X)$ reflects the character of $X$ as a profinite $G$-spectrum and, to ease our notation, we will write just $\pi_k X$ for this $G$-module.

2.4. **Mapping spectra II.** For a detailed discussion of continuous mapping spectra we refer the reader to [27]. Here we summarize only the basic definitions.

For $X, Y \in \hat{\mathcal{S}}_*$, the mapping space $\mathrm{map}_{\hat{\mathcal{S}}_*}(X,Y)$ is defined to be the simplicial set whose set of $n$-simplices is given as the set of maps

$$\mathrm{map}_{\hat{\mathcal{S}}_*}(X,Y)_n = \mathrm{Hom}_{\hat{\mathcal{S}}_*}(X \wedge \Delta[n]_+, Y).$$

For $X, Y \in \hat{\mathcal{S}}_{*G}$, the mapping space $\mathrm{map}_{\hat{\mathcal{S}}_{*G}}(X,Y)$ is defined to be the simplicial set whose set of $n$-simplices is given as the set of maps

$$\mathrm{map}_{\hat{\mathcal{S}}_{*G}}(X,Y)_n = \mathrm{Hom}_{\hat{\mathcal{S}}_{*G}}(X \wedge \Delta[n]_+, Y)$$

where $\Delta[n]_+$ is considered as a pointed profinite $G$-space with trivial $G$-action.

Let $Y$ be a profinite space and $W$ be a pointed profinite space. The functor $\hat{\mathcal{S}} \to \hat{\mathcal{S}}_*$ that sends $Y$ to $Y_+$ (defined by adding a disjoint basepoint) is the left adjoint of the functor that forgets the basepoint. As in [27], we will use the notation $\mathrm{Map}(Y,W)$ for the pointed simplicial set $\mathrm{map}_{\hat{\mathcal{S}}_*}(Y_+, W)$ whose basepoint is the map $Y_+ \to * \to W$. This defines a functor

$$\mathrm{Map}(-,-) : \hat{\mathcal{S}}^{\mathrm{op}} \times \hat{\mathcal{S}}_* \to \mathcal{S}_*.$$

For a profinite space $Y$ and a profinite spectrum $X$, we denote by $\mathrm{Map}(Y,X)$ the spectrum whose $n$th space is given by the pointed simplicial set $\mathrm{Map}(Y, X_n)$. This defines a functor

$$\mathrm{Map}(-,-) : \hat{\mathcal{S}}^{\mathrm{op}} \times \mathrm{Sp}(\hat{\mathcal{S}}_*) \to \mathrm{Sp}(\mathcal{S}_*).$$

Now let $Y$ be a profinite $G$-space and let $W$ be a pointed profinite $G$-space. The pointed simplicial set $\mathrm{Map}_G(Y,W)$ is defined to be the pointed simplicial set $\mathrm{map}_{\hat{\mathcal{S}}_{*G}}(Y_+, W)$ with basepoint equal to the map $Y_+ \to * \to W$. This defines a functor

$$\mathrm{Map}_G(-,-) : \hat{\mathcal{S}}_G^{\mathrm{op}} \times \hat{\mathcal{S}}_{*G} \to \mathcal{S}_*.$$

When $Y$ is a profinite $G$-space and $W$ is a pointed profinite $G$-space, we equip the pointed simplicial set $\mathrm{Map}(Y,W)$ with a $G$-action by defining $(gf)(y) := gf(g^{-1}y)$. With this $G$-action, $\mathrm{Map}_G(Y,W)$ is the pointed space of $G$-fixed points of the pointed space $\mathrm{Map}(Y,W)$.

If $Y$ is a profinite $G$-space and $X$ is a profinite $G$-spectrum, then $\mathrm{Map}_G(Y,X)$ is the spectrum whose $n$th space is given by the pointed simplicial set $\mathrm{Map}_G(Y, X_n)$. This construction yields a functor

$$\mathrm{Map}_G(-,-) : \hat{\mathcal{S}}_G^{\mathrm{op}} \times \mathrm{Sp}(\hat{\mathcal{S}}_{*G}) \to \mathrm{Sp}(\mathcal{S}_*).$$

## 3. Comparing continuous homotopy fixed points

In this section we recall the definition of homotopy fixed points for each of discrete and profinite $G$-spectra and show that they agree in certain cases where they are both defined. Our recollections start with the profinite case, for which details can be found in [27]. As usual $G$ denotes a profinite group.

3.1. **Homotopy fixed points of profinite $G$-spectra.** A very convenient feature of the profinite approach is that the universal classifying space $EG$ of our profinite group (given, as usual, in degree $n$ by $G^{n+1}$) is naturally a profinite $G$-space. Thus, for a profinite $G$-spectrum $X$ it is possible to form the continuous mapping spectrum $\mathrm{Map}_G(EG, X)$. Moreover, $EG$ is a cofibrant profinite $G$-space, since $G$ acts freely in each degree, and hence, we can consider $EG$ as a cofibrant resolution of a point in $\hat{\mathcal{S}}_G$. If $X$ is a fibrant profinite $G$-spectrum, then $\mathrm{Map}_G(EG, X)$ is a fibrant spectrum, giving a homotopically well-behaved version of the fixed points $\mathrm{Map}_G(\{*\}, X)$. Thus, we let $R_G$ denote a fibrant replacement functor in $\mathrm{Sp}(\hat{\mathcal{S}}_{*G})$. In [27], for any profinite $G$-spectrum $X$, the continuous homotopy fixed points of $X$ under $G$ were defined to be

$$X^{hG} := \mathrm{Map}_G(EG, R_G X). \tag{6}$$

One advantage of the construction in (6) is that the associated descent spectral sequence arises naturally from the filtration of $EG$ just as in the classical case for finite groups. This descent spectral sequence has the form

$$E_2^{s,t} = H_c^s(G; \pi_t X) \Rightarrow \pi_{t-s} X^{hG} \tag{7}$$

where the $E_2$-term is the continuous cohomology of $G$ with coefficients in the profinite $G$-module $\pi_t X$.

One way to describe the above spectral sequence is as follows. Let $X$ be a fibrant profinite $G$-spectrum. We can consider $\mathrm{Map}_G(EG, R_G X)$ as coming from a cosimplicial spectrum $\mathrm{Map}_G(G^{\bullet+1}, X)$ whose $n$th spectrum is $\mathrm{Map}_G(G^{n+1}, X)$. Then there is an equivalence

$$X^{hG} \simeq \operatorname*{holim}_{\Delta} \mathrm{Map}_G(G^{\bullet+1}, X)$$

and spectral sequence (7) is isomorphic to the spectral sequence associated to the tower of spectra

$$\{\mathrm{Tot}_k\left(\mathrm{Map}_G(G^{\bullet+1}, X)\right)\}_k.$$

We refer the reader to [27, §3] for the proofs and more details.

3.2. **Homotopy fixed points of discrete $G$-spectra.** For discrete $G$-spectra, the bad news is that, in general, $EG$ is not a simplicial discrete $G$-set. But the good news (for example, see [15, Lemma 2.3, Corollary 2.4]) is that a one-point space is a cofibrant object in $\mathcal{S}_G$. Thus, instead of using $EG$, the homotopy fixed points of a discrete $G$-spectrum $X$ are defined as the fixed points of a fibrant replacement $X_{f,G}$ in $\mathrm{Sp}(\mathcal{S}_{*G})$. We denote these homotopy fixed points by $X^{h_d G}$ in order to distinguish them from the previous construction. Thus, as in [5], we set

$$X^{h_d G} := (X_{f,G})^G.$$

It is a nice feature of this definition that it is clear that homotopy fixed points are the right derived functor of the right Quillen functor $(-)^G : \mathrm{Sp}(\mathcal{S}_{*G}) \to \mathrm{Sp}(\mathcal{S}_*)$.

However, in order to obtain a descent spectral sequence it is more convenient to consider a different description of homotopy fixed points. Let $\Gamma_G(-)$ be the endofunctor

$$X \mapsto \mathrm{Map}_c(G, X) =: \Gamma_G(X)$$

on discrete $G$-spectra (the object $\mathrm{Map}_c(G, X)$ is defined in Section 2.2). The iterated application of $\Gamma_G(-)$ defines a cosimplicial object

$$(\Gamma_G X)^\bullet = \mathrm{Map}_c(G^{\bullet+1}, X)$$

in discrete $G$-spectra, with

$$(\Gamma_G X)^j \cong \mathrm{Map}_c(G^{j+1}, X),$$

for each $j \geq 0$.

We say that $G$ has finite cohomological dimension if there exists a positive integer $r$ such that $H_c^s(G; M) = 0$, for all $s \geq r$ and every discrete $G$-module $M$. We say that $G$ has finite virtual cohomological dimension if $G$ contains an open subgroup $U$ that has finite cohomological dimension. Provided $G$ has finite virtual cohomological dimension and $X \xrightarrow{\simeq} Y$ is a weak equivalence in $\mathrm{Sp}(\mathcal{S}_{*G})$ with $Y$ fibrant as a spectrum, it follows from [5] that there is an equivalence

$$X^{h_d G} \simeq \operatorname*{holim}_{\Delta} \mathrm{Map}_c(G^{\bullet+1}, Y)^G \tag{8}$$

and a descent spectral sequence

$$E_2^{s,t} = H_c^s(G; \pi_t X) \Rightarrow \pi_{t-s} X^{h_d G} \tag{9}$$

whose $E_2$-term is the continuous cohomology of $G$ with coefficients in the discrete $G$-module $\pi_t X$.

3.3. **Comparison of homotopy fixed points.** In this section, we show under mild assumptions on the profinite group $G$ that the two notions of continuous homotopy fixed points coincide in several situations in which they are both defined.

**Definition 3.1.** *A fibrant profinite $G$-spectrum $X$ is called an $f$-$G$-spectrum if each space $X_n$ is a simplicial finite discrete $G$-set.*

Let $X$ be an $f$-$G$-spectrum. Since $X$ is fibrant as a profinite $G$-spectrum, the homotopy groups of $X$ are all finite discrete $G$-modules, by [26, proof of Proposition 3.9]. Thus, an $f$-$G$-spectrum is an $f$-spectrum in the sense of [4, page 5] (that is, each homotopy group of $X$ is finite), which explains part of the motivation for the terminology of Definition 3.1. Since $X$ is both a profinite and a discrete $G$-spectrum, we have our two different notions of continuous homotopy fixed points at hand.

**Theorem 3.2.** *Let $G$ be a profinite group with finite virtual cohomological dimension and let $X$ be an $f$-$G$-spectrum. There is an equivalence of spectra*

$$X^{hG} \simeq X^{h_d G}. \tag{10}$$

*Proof.* By [5], since $X$ is fibrant as a spectrum and $G$ has finite virtual cohomological dimension, we can use the homotopy limit $\mathrm{holim}_\Delta \mathrm{Map}_c(G^{\bullet+1}, X)^G$ as a model for $X^{h_d G}$. There is an isomorphism of cosimplicial spectra

$$\mathrm{Map}_G(G^{\bullet+1}, X) \cong \mathrm{Map}_c(G^{\bullet+1}, X)^G. \tag{11}$$

By [27, Proposition 3.23], this shows that we have equivalences

$$X^{hG} \simeq \operatorname{holim}_{\Delta} \operatorname{Map}_G(G^{\bullet+1}, X) \simeq \operatorname{holim}_{\Delta} \operatorname{Map}_c(G^{\bullet+1}, X)^G.$$

Hence there is an equivalence of spectra $X^{hG} \simeq X^{h_dG}$, as desired. □

**Remark 3.3.** Let $X$ be any $f$-$G$-spectrum. It is worth noting that even when $G$ does not have finite virtual cohomological dimension, [6, Theorem 3.5] shows that the spectrum $\operatorname{colim}_{N \lhd_o G} \bigl(\operatorname{holim}_{\Delta} \operatorname{Map}_c(G^{\bullet+1}, X)\bigr)^N$, a colimit over the open normal subgroups of $G$, is a fibrant discrete $G$-spectrum, and hence,

$$\begin{aligned} X^{hG} &\simeq \operatorname{holim}_{\Delta} \operatorname{Map}_c(G^{\bullet+1}, X)^G \\ &\cong \Bigl( \operatorname{colim}_{N \lhd_o G} \bigl(\operatorname{holim}_{\Delta} \operatorname{Map}_c(G^{\bullet+1}, X)\bigr)^N \Bigr)^G \\ &\simeq \Bigl( \operatorname{colim}_{N \lhd_o G} \bigl(\operatorname{holim}_{\Delta} \operatorname{Map}_c(G^{\bullet+1}, X)\bigr)^N \Bigr)^{h_dG}, \end{aligned}$$

so that $X^{hG}$ can always be regarded as being the $G$-homotopy fixed points of some discrete $G$-spectrum.

After a few preparatory comments, we recall a theorem that gives an example of a way that $f$-$G$-spectra arise. We call a spectrum $X \in \mathrm{Sp}(\mathcal{S}_*)$ a $G$-spectrum (without taking any topology into account) if each space $X_n$ is a pointed $G$-space and the $G$-actions are compatible with the bonding maps $S^1 \wedge X_n \to X_{n+1}$.

**Definition 3.4.** *A $G$-spectrum $Z$ is called $\pi$-finite if all its homotopy groups are finite.*

Let $X$ be an arbitrary $f$-$G$-spectrum: since the homotopy groups of $X$ are all finite discrete $G$-modules, the underlying $G$-spectrum of $X$ is $\pi$-finite. With respect to this conclusion, the following converse was proved in [26], Theorem 5.15, for the case when $G$ is strongly complete (that is, if every subgroup of finite index is open in $G$).

**Theorem 3.5.** *Let $G$ be a strongly complete profinite group and let $X$ be a $\pi$-finite $G$-spectrum. Then there is a $G$-equivariant map*

$$\varphi^s : X \to F^s_G X$$

*of spectra from $X$ to an $f$-$G$-spectrum $F^s_G X$ such that $\varphi^s$ is a stable equivalence of underlying spectra.*

*The assignment $X \mapsto F^s_G X$ is functorial in the sense that given a $G$-equivariant map $h : X \to Y$ between $\pi$-finite $G$-spectra, there is a map $F^s_G(h)$ in $\mathrm{Sp}(\hat{\mathcal{S}}_{*G})$ such that the diagram*

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d] & Y \arrow[d] \\ F^s_G X \arrow[r, "F^s_G(h)"'] & F^s_G Y \end{tikzcd}$$

*of underlying spectra commutes.*

The above theorem motivates the following definition.

**Definition 3.6.** *Let $G$ be a strongly complete profinite group and let $X$ be a $\pi$-finite $G$-spectrum. Then both $(F^s_G X)^{hG}$ and $(F^s_G X)^{h_d G}$ can be formed and it is natural to define*
$$X^{hG} := (F^s_G X)^{hG}$$
*and*
$$X^{h_d G} := (F^s_G X)^{h_d G}.$$

The following result is immediate from Theorem 3.2.

**Corollary 3.7.** *Let $G$ be a strongly complete profinite group with finite virtual cohomological dimension and let $X$ be a $\pi$-finite $G$-spectrum. Then there is an equivalence*
$$X^{hG} \simeq X^{h_d G}.$$

## 4. Iterated homotopy fixed point spectra in the profinite setting

4.1. **Recollections of basic facts and the main problem.** We begin by recalling some material about iterated homotopy fixed points from [27]. Let $K$ be a closed subgroup of $G$ and let $N(K)$ be the normalizer of $K$ in $G$. Also, let $X$ be any fibrant profinite $G$-spectrum. The composition
$$\mathrm{Map}_K(EG, X) \xrightarrow{\simeq} \mathrm{Map}_K(EK, X) \xrightarrow{\simeq} \mathrm{Map}_K(EK, R_K X) = X^{hK}$$
of weak equivalences of spectra shows that it is natural to make the identification
$$X^{hK} = \mathrm{Map}_K(EG, X).$$
With this identification, it is clear that the profinite quotient group $N(K)/K$ acts on $X^{hK}$. Note that by setting $K = G$, our discussion shows that there is the useful identification
$$X^{hG} = \mathrm{Map}_G(EG, X).$$

To simplify our notation, we now assume that $K$ is a closed normal subgroup of $G$. There is a canonical map
$$X^{hG} \to X^{hK}$$
that is defined by
$$X^{hG} = \mathrm{Map}_G(EG, X) = \mathrm{Map}(EG, X)^G \to \mathrm{Map}(EG, X)^K = X^{hK},$$
and it is easy to see that this map factors into the identity map
$$X^{hG} = \mathrm{Map}(EG, X)^G \xrightarrow{=} \big(\mathrm{Map}(EG, X)^K\big)^{G/K} = (X^{hK})^{G/K}$$
followed by the natural inclusion $\big(\mathrm{Map}(EG, X)^K\big)^{G/K} \to X^{hK}$. When the $G/K$-spectrum $X^{hK}$ is a profinite $G/K$-spectrum, then composition of the above identity map with the canonical map
$$(X^{hK})^{G/K} \to (X^{hK})^{hG/K}$$
yields a map
$$X^{hG} \to (X^{hK})^{hG/K}. \tag{12}$$
Thus, a natural problem in the theory of profinite $G$-spectra is to show that whenever $X^{hK}$ is a profinite $G/K$-spectrum, map (12) to the iterated continuous homotopy fixed points $(X^{hK})^{hG/K}$ is an equivalence.

The first issue in studying map (12) is that $X^{hK}$ does not in general carry the structure of a *profinite* $G/K$-spectrum. This observation relies on the fact that the set of continuous maps between two profinite sets is in general not a profinite set itself. But there are interesting cases when it can be shown that (12) is an equivalence.

To understand the simplest case, we consider the situation when $K$ is open in $G$, so that the group $G/K$ is finite. Whether or not $X^{hK}$ is a profinite $G/K$-spectrum, since $G/K$ is a finite discrete space, the fibrant spectrum $X^{hK}$ is automatically a discrete $G/K$-spectrum, and hence, there is always the canonical map

$$X^{hG} \to F(E(G/K)_+, X^{hK})^{G/K} = (X^{hK})^{h_d G/K}$$

(as written above, $(X^{hK})^{h_d G/K}$ is equal to the "usual" homotopy fixed point spectrum for the discrete group $G/K$; we remark that in forming this spectrum, no fibrant replacement of $X^{hK}$ is needed since $X^{hK}$ is already a fibrant spectrum) that is defined to be the composition of the aforementioned identity map with the canonical map $(X^{hK})^{G/K} \to (X^{hK})^{h_d G/K}$. Then an adjunction argument suffices to prove that the map $X^{hG} \to (X^{hK})^{h_d G/K}$ is an equivalence (see [27], §3.5).

Furthermore, if $X^{hK}$ is a fibrant profinite $G/K$-spectrum, then the fibrant replacement map $X^{hK} \to R_{G/K} X^{hK}$ of profinite $G/K$-spectra, when regarded as a map of spectra, is a weak equivalence between fibrant objects that is $G/K$-equivariant, and hence, the map $(X^{hK})^{G/K} \to (X^{hK})^{h_d G/K}$ can be identified with the map

$$(X^{hK})^{G/K} \to \mathrm{Map}_{G/K}(E(G/K), R_{G/K} X^{hK}) = (X^{hK})^{hG/K}.$$

Therefore, whenever $X^{hK}$ is a fibrant profinite $G/K$-spectrum, the weak equivalence $X^{hG} \xrightarrow{\simeq} (X^{hK})^{h_d G/K}$ can be identified with map (12) to the iterated continuous homotopy fixed point spectrum $(X^{hK})^{hG/K}$, giving our first case of when (12) is a weak equivalence.

4.2. **Some definitions and observations that help with understanding the problem of iteration.** Now we give some preliminary considerations that lead to our second case of when map (12) is an equivalence.

**Definition 4.1.** *Let $G$ be a profinite group and $X$ a $G$-spectrum. We say that $X$ has a model in the category of profinite $G$-spectra (or that $X$ has a profinite $G$-model $X'$) if there exists a fibrant profinite $G$-spectrum $X'$ and a zigzag of $G$-equivariant morphisms of $G$-spectra between $X$ and $X'$, such that each morphism is a weak equivalence of spectra.*

**Remark 4.2.** Let $G$ and $X$ be as in Definition 4.1. Since the homotopy groups of a fibrant profinite $G$-spectrum are profinite $G$-modules, when the homotopy groups $\pi_t X$ are not profinite $G$-modules, then the model described above cannot exist for $X$. Also, we point out that if $G$ is strongly complete and $X$ is $\pi$-finite, then the discrete $G$-spectrum $F_G^s X$ is a model for $X$ as a profinite $G$-spectrum.

**Definition 4.3.** *Let $G$ be a profinite group and $K$ a closed normal subgroup of $G$. Let $X$ be a fibrant profinite $G$-spectrum such that $X^{hK}$ has a profinite $G/K$-model $X'(K)$. Then we define the iterated continuous homotopy fixed points of $X$ to be*

$$(X^{hK})^{hG/K} := (X'(K))^{hG/K}.$$

As at the beginning of §4.1, we let $X$ be an arbitrary fibrant profinite $G$-spectrum. By letting $\{U_j\}_j$ be the collection of open normal subgroups of $G$, we can write

$G = \lim_j G/U_j$. For each $j$, we let

$$K_j := KU_j,$$

an open normal subgroup of $G$. For indices $i, j$ such that $U_j \subset U_i$, there is the canonical surjection $G/K_j \to G/K_i$, the natural $G/K_j$-equivariant map

$$X^{hK_i} = \mathrm{Map}(EG, X)^{K_i} \to \mathrm{Map}(EG, X)^{K_j} = X^{hK_j}$$

between fibrant spectra, and the isomorphism $G/K \cong \lim_j G/K_j$. Also, there is the induced map

$$(X^{hK_i})^{h_d G/K_i} \xrightarrow{\simeq} (X^{hK_j})^{h_d G/K_j}$$

that is equal to the canonical composition

$$(X^{hK_i})^{h_d G/K_i} \to F(E(G/K_i)_+, X^{hK_i})^{G/K_j} \to F(E(G/K_j)_+, X^{hK_j})^{G/K_j}$$

and is a weak equivalence (for each $j$, since $G/K_j$ is a finite group, $(X^{hK_j})^{h_d G/K_j}$ is equivalent to $X^{hG}$, by [27], as discussed earlier). Hence by taking the colimit over all $j$, we obtain an equivalence

$$X^{hG} \simeq \operatorname*{colim}_j (X^{hK_j})^{h_d G/K_j}.$$

Hence, if $X^{hK}$ is a profinite $G/K$-spectrum and the colimit on the right-hand side above is equivalent to $(X^{hK})^{hG/K}$ (this is plausible if there is an equivalence of the right-hand side with $F(\lim_j E(G/K_j)_+, \mathrm{colim}_j X^{hK_j})^{G/K}$), then it would follow that (12) is an equivalence for the closed normal subgroup $K$. Unfortunately, this is not always the case.

**Remark 4.4.** For the duration of this remark (and the next), we let $K$ be an arbitrary closed subgroup of $G$. A problem similar to what was just described above occurs for discrete $G$-spectra: in [7] a discrete $G$-spectrum $Y$ is called hyperfibrant if the canonical map $\mathrm{colim}_j Y^{h_d K_j} \to Y^{h_d K}$ is an equivalence for all $K$. In [loc. cit.], it was shown that if $Y$ is a hyperfibrant discrete $G$-spectrum and $H$ is any closed subgroup of $G$ that is normal in $K$, then the identification $Y^{h_d H} = \mathrm{colim}_j Y^{h_d HU_j}$ makes $Y^{h_d H}$ a discrete $K/H$-spectrum, and hence, $(Y^{h_d H})^{h_d K/H}$ is defined, and if $K$ is normal in $G$, then

$$(Y^{h_d K})^{h_d G/K} \simeq Y^{h_d G}.$$

**Remark 4.5.** As in Remark 4.4, let $K$ be any closed subgroup of $G$. We recall how a version of the issue discussed above is handled in the corresponding "pro-setting" in [14, Section 11.1]. If $Y$ is a pro-$G$-spectrum, in addition to considering the $K$-homotopy fixed point pro-spectrum $Y^{hK}$, Fausk uses a technique that was reviewed in Remark 4.4: he defines the $K$-$G$-homotopy fixed point pro-spectrum $Y^{h_G K}$ to be $\mathrm{hocolim}_j (Y_f)^{hK_j}$, where $Y_f$ is a fibrant replacement of $Y$. Then Fausk shows that when $K$ is normal in $G$, there is an equivalence

$$(Y^{h_G K})^{hG/K} \simeq Y^{hG}$$

in the Postnikov model structure on pro-spectra. In [14, Lemma 11.4], Fausk describes a situation when $Y^{h_G K}$ can be identified with $Y^{hK}$.

Given the difficulties described above, our next step in studying the problem of iteration is to consider a special case. To help with this, it is useful to observe that because $X$ is a fibrant profinite $H$-spectrum for each closed subgroup $H$ of $G$, there is a canonical map

$$\operatorname*{colim}_j X^{hK_j} \xrightarrow{\cong} \operatorname*{colim}_j \mathrm{Tot}(\mathrm{Map}(K_j^\bullet, X)) \to \mathrm{Tot}(\mathrm{Map}(K^\bullet, X)) \xrightarrow{\cong} X^{hK},$$

where here $K$ is any closed subgroup of $G$ (for the isomorphisms, see [27, Proposition 3.23]). A version of the following result in the setting of discrete $G$-spectra was obtained in [7, end of §3].

**Proposition 4.6.** *Let $G$ be a profinite group and $K$ a closed subgroup. Let $X$ be an $f$-$G$-spectrum and $q$ an integer such that $\pi_t X = 0$ for all $t \geq q$. Then the canonical map*

$$\operatorname*{colim}_j X^{hK_j} \xrightarrow{\simeq} X^{hK}$$

*is an equivalence of spectra.*

*Proof.* For each closed subgroup $H$ of $G$, by [27, Proposition 3.23] the homotopy spectral sequence for $\mathrm{Tot}(\mathrm{Map}(H^\bullet, X))$ is a descent spectral sequence $E_r^{*,*}(H)$ that has the form

$$E_2^{s,t}(H) = H_c^s(H; \pi_t X) \Rightarrow \pi_{t-s}(X^{hH}),$$

where the continuous cohomology group has coefficients in the finite discrete $H$-module $\pi_t X$. For the subgroups $K$ and all $K_j$, the associated spectral sequences assemble to yield a map of conditionally convergent spectral sequences

$$\operatorname*{colim}_j E_r^{*,*}(K_j) \to E_r^{*,*}(K).$$

Since $K = \bigcap_j K_j$, the $E_2$-terms satisfy

$$\operatorname*{colim}_j E_2^{s,t}(K_j) = \operatorname*{colim}_j H_c^s(K_j; \pi_t X) \cong H_c^s(K; \pi_t X) = E_2^{s,t}(K).$$

By [22, Proposition 3.3], the spectral sequence $\mathrm{colim}_j\, E_r^{*,*}(K_j)$ has abutment equal to the colimit of the abutments $\pi_*(X^{hK_j})$ if there exists a fixed $m$ such that $H_c^s(K_j; \pi_t X) = 0$ for all $t \geq m$, all $s \geq 0$ and all $j$, and by hypothesis, this condition is satisfied. Hence, since the map of spectral sequences is an isomorphism from the $E_2$-terms onward, the map of abutments $\mathrm{colim}_j\, \pi_*(X^{hK_j}) \to \pi_*(X^{hK})$ is an isomorphism. □

4.3. **The notion of a $K$-Postnikov $G$-spectrum and its use with iterated continuous homotopy fixed points.** Let $X$ be an $f$-$G$-spectrum and let $K$ be a closed normal subgroup of $G$. As in [27, Section 3.2], for each integer $q$, let

$$P^q X := \mathrm{cosk}_q X$$

be the $q$th Postnikov section of $X$ in $\mathrm{Sp}(\hat{\mathcal{S}}_{*G})$: since $X$ is an $f$-$G$-spectrum, each of $X$ and the Postnikov sections $P^q X$, for every $q$, are $f$-$H$-spectra, where $H$ is any closed subgroup of $G$. As explained in [7, page 2888, end of §3], by applying the fact that each discrete $G$-spectrum $P^q X$ satisfies a coconnectivity condition, we see that

- for any closed subgroup $H$ in $G$, by [8, Theorem 7.2], there is an equivalence
$$(P^q X)^{h_d H} \simeq \operatorname*{holim}_{\Delta} \mathrm{Map}_c(H^{\bullet+1}, P^q X)^H,$$
and hence, Remark 3.3 shows that
$$(P^q X)^{h_d H} \simeq (P^q X)^{hH},$$

for every integer $q$;

- each $P^q X$ is a hyperfibrant discrete $G$-spectrum; and thus,
- there is the natural identification

$$(P^q X)^{h_d K} = \operatorname*{colim}_j (P^q X)^{h_d K_j},$$

showing that each $(P^q X)^{h_d K}$ is a discrete $G/K$-spectrum.

The above discussion shows that we can regard each $G/K$-spectrum $(P^q X)^{hK}$ as a discrete $G/K$-spectrum by the identification

$$(P^q X)^{hK} = (P^q X)^{h_d K}$$

(this identification can also be obtained by using Proposition 4.6).

Before giving our next result, we introduce some helpful terminology. We note that though the following definition is not short, it is also not complicated: the crux of the notion it defines ("$K$-Postnikov $G$-spectrum") is almost completely captured by its property (a) below, and each of properties (b), (c) and (d) is just a basic compatibility condition that one would expect to be satisfied when the zigzags in the defining data satisfy a minimal level of naturality.

**Definition 4.7.** *Let $X$ be an $f$-$G$-spectrum and let $K$ be a closed normal subgroup of $G$. Then we say that $X$ is a $K$-Postnikov $G$-spectrum if there is an inverse system $\{X^q(K)\}_{q\in\mathbb{Z}}$ of $G/K$-spectra that has the following properties*:

(a) *for each $q$, the spectrum $X^q(K)$ is an $f$-$G/K$-spectrum and a model for the $G/K$-spectrum $(P^q X)^{hK}$ in the category of profinite $G/K$-spectra*;

(b) *for each $q$, there is an equivalence $\big((P^q X)^{h_d K}\big)^{h_d G/K} \simeq (X^q(K))^{h_d G/K}$*;

(c) *there is an equivalence*

$$\operatorname*{holim}_q \big((P^q X)^{h_d K}\big)^{h_d G/K} \simeq \operatorname*{holim}_q (X^q(K))^{h_d G/K};$$

*and*

(d) *the fibrant profinite $G/K$-spectrum $\operatorname{holim}_q X^q(K)$ is a model for the $G/K$-spectrum $\operatorname{holim}_q (P^q X)^{hK}$ in the category of profinite $G/K$-spectra.*

*Before closing out this definition, we remark that it is easy to see that one would expect properties* (b), (c) *and* (d) *to hold* (*though they are not entirely trivial*) *on the basis of knowing just that* (a) *alone holds.*

**Remark 4.8.** If $X$ is a $K$-Postnikov $G$-spectrum, then because each $X^q(K)$ is a discrete $G/K$-spectrum, it follows automatically that each map $X^q(K) \to X^{q-1}(K)$ has the requisite continuity properties for being a morphism in the category of profinite $G/K$-spectra, and hence, the inverse system $\{X^q(K)\}_{q\in\mathbb{Z}}$ is a diagram in that category.

Our next result illustrates the utility of Definition 4.7. After proving this result, we give a discussion of when an $f$-$G$-spectrum possesses the properties required by this definition.

Several key ingredients in the next result and its proof are from an unpublished manuscript by the first author (however, the proof below does not depend in any way on this manuscript) that uses Postnikov towers to study the problem of iterated homotopy fixed points in the setting of discrete $G$-spectra. The aforementioned ingredients and manuscript build on the idea that Postnikov towers are a helpful tool for building homotopy fixed point spectra for profinite group actions, and as far as we know, this idea is primarily due to [14] (and [15, 21]).

**Proposition 4.9.** *Let $G$ be a profinite group and let $K$ be a closed normal subgroup of $G$ such that $G/K$ has finite virtual cohomological dimension. If $X$ is a $K$-Postnikov $G$-spectrum, then there is an equivalence*

$$X^{hG} \simeq (X^{hK})^{hG/K}.$$

*Proof.* There is the following chain of equivalences:

$$\begin{aligned} X^{hG} &= \mathrm{Map}_G(EG, \lim_q P^q X) \simeq \operatorname*{holim}_q (P^q X)^{hG} \simeq \operatorname*{holim}_q (P^q X)^{h_d G} \\ &\simeq \operatorname*{holim}_q \big(\operatorname*{colim}_j (P^q X)^{h_d K_j}\big)^{h_d G/K} \simeq \operatorname*{holim}_q (X^q(K))^{h_d G/K} \\ &\simeq \operatorname*{holim}_q (X^q(K))^{hG/K} \cong \mathrm{Map}_{G/K}(E(G/K), \operatorname*{holim}_q X^q(K)) \\ &= \big(\operatorname*{holim}_q X^q(K)\big)^{hG/K} = (X^{hK})^{hG/K}. \end{aligned}$$

Above, the second equivalence is from an application of the fact that each $P^q X$ is a fibrant profinite $G$-spectrum and, as in [27, page 205, (9)], $\{\mathrm{Map}_G(EG, P^q X)\}_{q\in\mathbb{Z}}$ is a tower of fibrations of fibrant objects in spectra; the fourth equivalence follows from [7, proof of Lemma 4.9]; the sixth equivalence is due to Theorem 3.2; the fibrancy of $\operatorname{holim}_q X^q(K)$ as a profinite $G/K$-spectrum implies the next-to-last equivalence (an equality); and the last equivalence, an identity that is an example of Definition 4.3, is because the equivalences

$$\operatorname*{holim}_q X^q(K) \simeq \operatorname*{holim}_q (P^q X)^{hK} \simeq \lim_q (P^q X)^{hK} \cong \mathrm{Map}_K(EG, \lim_q P^q X) = X^{hK}$$

show that $\operatorname{holim}_q X^q(K)$ is a model for $X^{hK}$ in the category of profinite $G/K$-spectra. □

**Remark 4.10.** Line two of the above proof shows that if $G$ is an arbitrary profinite group and $X$ is an $f$-$G$-spectrum, then there is an equivalence

$$X^{hG} \simeq \operatorname*{holim}_q (P^q X)^{h_d G} \simeq \operatorname*{holim}_{q\geq 0} (P^q X)^{h_d G} = \Big(\operatorname*{holim}_{q\geq 0} P^q X\Big)^{hG},$$

where the rightmost expression is the $G$-homotopy fixed point spectrum of the continuous $G$-spectrum $\operatorname{holim}_{q\geq 0} P^q X$ (in the sense of [5]; $\{P^q X\}_{q\geq 0}$ is a tower of discrete $G$-spectra that are fibrant as spectra). This observation gives a version of Theorem 3.2 with the hypothesis on cohomological dimension removed.

As in Proposition 4.9, we continue to let $K$ be a closed normal subgroup of $G$ and we let $X$ be an $f$-$G$-spectrum. Proposition 4.9 shows that it is quite useful when $X$ is a $K$-Postnikov $G$-spectrum and so we now give a discussion about when this occurs.

Suppose that $(P^q X)^{hK}$ is a $\pi$-finite $G/K$-spectrum, for each integer $q$, with $G/K$ strongly complete. (It is worth noting that if $G$ is strongly complete, then so is $G/K$.) By Proposition 4.6, the $G/K$-equivariant map

$$(P^q X)^{hK} = \mathrm{Map}(EG, P^q X)^K \xleftarrow{\simeq} \operatorname*{colim}_j \mathrm{Map}(EG, P^q X)^{K_j} = \operatorname*{colim}_j (P^q X)^{hK_j}$$

is a weak equivalence of spectra. The continuous surjection $G/K \to G/K_j$ makes the discrete $G/K_j$-spectrum $\mathrm{Map}(EG, P^q X)^{K_j}$ a discrete $G/K$-spectrum, so that the $\pi$-finite $G/K$-spectrum $\operatorname{colim}_j \operatorname{Map}(EG, P^q X)^{K_j}$ is a discrete $G/K$-spectrum. Notice that the inverse system

$$\{F^s_{G/K}\big(\operatorname*{colim}_j \operatorname{Map}(EG, P^q X)^{K_j}\big)\}_{q\in\mathbb{Z}} \tag{13}$$

is a diagram of profinite $G/K$-spectra, each of which is an $f$-$G/K$-spectrum and a profinite $G/K$-model for $(P^q X)^{hK}$. Since $(P^q X)^{hH} \simeq (P^q X)^{h_d H}$ for all closed subgroups $H$, the discrete $G/K$-spectrum $\operatorname{colim}_j \operatorname{Map}(EG, P^q X)^{K_j}$ can be identified with the discrete $G/K$-spectrum $(P^q X)^{h_d K} = \operatorname{colim}_j (P^q X)^{h_d K_j}$. Because

$$\operatorname*{colim}_j \operatorname{Map}(EG, P^q X)^{K_j} \xrightarrow{\simeq} F^s_{G/K}\big(\operatorname*{colim}_j \operatorname{Map}(EG, P^q X)^{K_j}\big)$$

is a weak equivalence of discrete $G/K$-spectra, there is a weak equivalence

$$\big(\operatorname*{colim}_j \operatorname{Map}(EG, P^q X)^{K_j}\big)^{h_d G/K} \xrightarrow{\simeq} \big(F^s_{G/K}\big(\operatorname*{colim}_j \operatorname{Map}(EG, P^q X)^{K_j}\big)\big)^{h_d G/K}.$$

The preceding two sentences show that the system in (13) satisfies property (b) of Definition 4.7, and additionally, it is easy to see that properties (c) and (d) are satisfied, completing a proof of the following result.

**Theorem 4.11.** *Let $K$ be a closed normal subgroup of the profinite group $G$ and let $X$ be an $f$-$G$-spectrum. If $G/K$ is strongly complete and, for every $q \in \mathbb{Z}$, $(P^q X)^{hK}$ has finite homotopy groups, then $X$ is a $K$-Postnikov $G$-spectrum.*

Notice that in Theorem 4.11, the hypotheses of the first sentence imply that for each $q \in \mathbb{Z}$, since $P^q X$ is an $f$-$K$-spectrum, $\pi_t(P^q X)$ is a finite discrete $K$-module for every integer $t$.

**Corollary 4.12.** *Let $G$, $K$, and $X$ be as in the first sentence of Theorem* 4.11 *and suppose that $G/K$ is strongly complete. If $H^s_c(K; \pi_t(X))$ is a finite group for all $s \geq 0$ and every integer $t$, then $X$ is a $K$-Postnikov $G$-spectrum.*

*Proof.* By Theorem 4.11, it suffices to show that for each integer $q$, the spectrum $(P^q X)^{hK}$ has finite homotopy groups. We consider the conditionally convergent descent spectral sequence

$$E_2^{s,t} = H^s_c(K; \pi_t(P^q X)) \Rightarrow \pi_{t-s}((P^q X)^{hK}).$$

For all $s \geq 0$ and any $t > q$, the vanishing of $\pi_t(P^q X)$ implies that $E_2^{s,t} = 0$. It follows that the filtration on the abutment is finite (for example, see [32, Lemma 5.48]). Since any nonzero terms on the $E_2$-page of the above spectral sequence are finite, we obtain the desired conclusion. □

**Remark 4.13.** In Corollary 4.12, the hypotheses imply that $\pi_t(X)$ is a finite discrete $K$-module for every integer $t$. Then it is worth noting that in Corollary 4.12, the finiteness condition on the continuous cohomology groups is plausible: for example, by [31, Proposition 4.2.2], if $K$ is of type $p$-$\mathbf{FP}_\infty$ and $M$ is a finite discrete ($p$-torsion) $\mathbb{Z}_p[[K]]$-module, then $H^s_c(K; M)$ is finite, for all $s \geq 0$ (we refer the reader to [31] for more details about this result).

Let $q$ be any fixed integer. We consider in more detail the condition that $(P^q X)^{hK}$ has finite homotopy groups. As above, we assume that $K$ is closed and normal in $G$ and $X$ is an $f$-$G$-spectrum.

Suppose that $K$ contains an open normal subgroup $U_K$ such that the abelian groups $H^s_c(U_K; \pi_t(P^qX))$ are finite for all $s \geq 0$ and all $t \leq q$. As recalled earlier (from [27]), since $P^qX$ is a fibrant profinite $K$-spectrum, there is a weak equivalence

$$(P^qX)^{hK} \xrightarrow{\simeq} \big((P^qX)^{hU_K}\big)^{h_d K/U_K}.$$

By arguing as in the proof of Corollary 4.12, the descent spectral sequence

$$H^s_c(U_K; \pi_t(P^qX)) \Rightarrow \pi_{t-s}((P^qX)^{hU_K})$$

yields that $(P^qX)^{hU_K}$ is a $\pi$-finite $K/U_K$-spectrum.

**Remark 4.14.** Without making any additional assumptions, we pause to consider the $\pi$-finite $K/U_K$-spectrum $(P^qX)^{hU_K}$. Since the finite group $K/U_K$ is naturally discrete, it is strongly complete, and hence, by Theorem 3.5, there is a $K/U_K$-equivariant map

$$(P^qX)^{hU_K} \xrightarrow{\simeq} F^s_{K/U_K}((P^qX)^{hU_K})$$

that is a weak equivalence, with target equal to an $f$-$K/U_K$-spectrum, yielding a particularly nice model for $(P^qX)^{hU_K}$. It follows that there are equivalences of spectra

$$(P^qX)^{hK} \simeq \big((P^qX)^{hU_K}\big)^{h_d K/U_K} \simeq \operatorname*{holim}_{K/U_K} F^s_{K/U_K}((P^qX)^{hU_K}).$$

The last spectrum above can be regarded as a homotopy limit of a fibrant profinite spectrum in the category of profinite spectra, showing that $(P^qX)^{hK}$ can be regarded as a profinite spectrum.

Now we go a little further with the above conclusion that $(P^qX)^{hU_K}$ is a $\pi$-finite $K/U_K$-spectrum. If for all integers $t$, $H^s(K/U_K; \pi_t((P^qX)^{hU_K}))$ is zero for all $s \geq r$, for some positive integer $r$, and finite for $0 \leq s < r$, then once again, the descent spectral sequence

$$H^s(K/U_K; \pi_t((P^qX)^{hU_K})) \Rightarrow \pi_{t-s}((P^qX)^{hK})$$

yields that $(P^qX)^{hK}$ has finite homotopy groups, as desired.

4.4. **More cases of well-behaved iteration via applications of Proposition 4.9.** To obtain these additional cases, we begin with the following definition, a natural extension of Definition 3.6.

**Definition 4.15.** *If $G$ is a strongly complete profinite group and $X$ is a $\pi$-finite $G$-spectrum, then $F^s_GX$ is both a fibrant profinite $H$-spectrum and a discrete $H$-spectrum for any closed subgroup $H$ in $G$, and thus, for such $H$, we define*

$$X^{hH} := (F^s_GX)^{hH}, \quad X^{h_dH} := (F^s_GX)^{h_dH}.$$

In the above definition, when the closed subgroup $H$ is a proper subgroup of $G$ that is strongly complete, then since $X$ is a $\pi$-finite $H$-spectrum, the definition also yields that $X^{hH} := (F^s_HX)^{hH}$ and $X^{h_dH} := (F^s_HX)^{h_dH}$. Thus, in the remark below, we show that our two definitions of $X^{hH}$ agree with each other and that our two definitions of $X^{h_dH}$ are equivalent.

**Remark 4.16.** Let $G$ and $X$ be as in Definition 4.15 and let $H$ be a strongly complete closed subgroup of $G$. Then there is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & F_H^s X \arrow[d, "\simeq"] \\ F_G^s X \arrow[r, "\simeq"] & F_H^s(F_G^s X) \end{tikzcd}$$

of weak equivalences in spectra that are $H$-equivariant. It is easy to see that

$$F_G^s X \xrightarrow{\simeq} F_H^s(F_G^s X) \xleftarrow{\simeq} F_H^s X$$

is a zigzag of weak equivalences in the category of discrete $H$-spectra, and hence,

$$(F_G^s X)^{h_d H} \simeq (F_H^s X)^{h_d H},$$

as desired. Now, as in [27, Section 3.2], for each integer $q$, we let $P^q$ denote the functor

$$\mathrm{cosk}_q \colon \mathrm{Sp}(\hat{\mathcal{S}}_{*H}) \to \mathrm{Sp}(\hat{\mathcal{S}}_{*H}), \quad Z \mapsto P^q Z := \mathrm{cosk}_q Z.$$

Notice that

$$(F_G^s X)^{hH} \simeq \operatorname*{holim}_q (P^q(F_G^s X))^{hH} \simeq \operatorname*{holim}_q (P^q(F_G^s X))^{h_d H},$$

as in the proof of Proposition 4.9. Similarly, we have

$$(F_H^s X)^{hH} \simeq \operatorname*{holim}_q (P^q(F_H^s X))^{h_d H}.$$

Also, by considering homotopy groups, it is easy to see that for each integer $q$,

$$P^q(F_G^s X) \xrightarrow{\simeq} P^q(F_H^s(F_G^s X)) \xleftarrow{\simeq} P^q(F_H^s X)$$

is a zigzag of weak equivalences in the category of discrete $H$-spectra. It follows that

$$(P^q(F_G^s X))^{h_d H} \xrightarrow{\simeq} (P^q(F_H^s(F_G^s X)))^{h_d H} \xleftarrow{\simeq} (P^q(F_H^s X))^{h_d H}$$

is a zigzag of weak equivalences between fibrant spectra that is natural in $q$, and hence,

$$(F_G^s X)^{hH} \simeq (F_H^s X)^{hH}.$$

**Theorem 4.17.** *Let $G$ be a strongly complete profinite group with $K$ a closed normal subgroup of $G$ such that $G/K$ has finite virtual cohomological dimension. If $X$ is a $\pi$-finite $G$-spectrum such that $F_G^s X$ is a $K$-Postnikov $G$-spectrum, then there is an equivalence of spectra*

$$X^{hG} \simeq (X^{hK})^{hG/K}.$$

*Proof.* By Proposition 4.9, we have

$$X^{hG} = (F_G^s X)^{hG} \simeq \big((F_G^s X)^{hK}\big)^{hG/K} = (X^{hK})^{hG/K}. \qquad \square$$

Now we want to apply Proposition 4.9 to suitable homotopy limits of profinite $G$-spectra: to do this, we need the following definition, whose key components are properties (a) and (b) below.

**Definition 4.18.** *Let $K$ be a closed normal subgroup of $G$ and let $J$ be a small category. We say that $\{X_\beta\}_{\beta \in J}$ is a diagram of $K$-Postnikov $G$-spectra that is natural in $\beta$ whenever*

(a) *$\{X_\beta\}_{\beta \in J}$ is a $J$-shaped diagram of profinite $G$-spectra, such that*

(b) *for each $\beta$, $X_\beta$ is a $K$-Postnikov $G$-spectrum, with $\{X^q_\beta(K)\}_{q\in\mathbb{Z}}$ denoting the inverse system of profinite $G/K$-models associated to $X_\beta$,*
(c) *there is a $(J\times\{q\})$-shaped diagram $\{X^q_\beta(K)\}_{\beta,q}$ of profinite $G/K$-spectra,*
(d) *there is an equivalence*

$$\operatorname*{holim}_\beta \operatorname*{holim}_q \big((P^q X_\beta)^{h_d K}\big)^{h_d G/K} \simeq \operatorname*{holim}_\beta \operatorname*{holim}_q (X^q_\beta(K))^{h_d G/K},$$

*and*
(e) *the fibrant profinite $G/K$-spectrum $\operatorname{holim}_\beta \operatorname{holim}_q X^q_\beta(K)$ is a model for the $G/K$-spectrum $\operatorname{holim}_\beta \operatorname{holim}_q (P^q X_\beta)^{hK}$ in the category of profinite $G/K$-spectra.*

**Remark 4.19.** In the above definition, we point out that properties (c), (d) and (e) are criteria that one would expect to be satisfied on the basis of just (a) and (b) alone: they are just expressing the requirement that the input data have enough naturality to be practically useful (e.g., properties (d) and (e) are saying that properties (c) and (d), respectively, in Definition 4.7 are natural in $\beta$).

Notice that if $\{X_\beta\}_{\beta\in J}$ is a diagram of $K$-Postnikov $G$-spectra that is natural in $\beta$, then $\operatorname{holim}_\beta X_\beta$ is a fibrant profinite $G$-spectrum.

**Theorem 4.20.** *Let $G$ be a profinite group and $K$ a closed normal subgroup of $G$ such that $G/K$ has finite virtual cohomological dimension. If $\{X_\beta\}_{\beta\in J}$ is a diagram of $K$-Postnikov $G$-spectra that is natural in $\beta$, then there is an equivalence of spectra*

$$\big(\operatorname*{holim}_\beta X_\beta\big)^{hG} \simeq \Big(\big(\operatorname*{holim}_\beta X_\beta\big)^{hK}\Big)^{hG/K}$$

*and there is a conditionally convergent spectral sequence*

$$E_2^{s,t} = H^s_c\big(G/K; \pi_t\big((\operatorname*{holim}_\beta X_\beta)^{hK}\big)\big) \Rightarrow \pi_{t-s}\big((\operatorname*{holim}_\beta X_\beta)^{hG}\big),$$

*where the $E_2$-term is the continuous cohomology of $G/K$ with coefficients the profinite $G/K$-module $\pi_t\big((\operatorname{holim}_\beta X_\beta)^{hK}\big)$.*

*Proof.* The $f$-$G$-spectra $X_\beta$ satisfy the hypotheses of Proposition 4.9. Hence for each $\beta$, we have an equivalence

$$(X_\beta)^{hG} \simeq ((X_\beta)^{hK})^{hG/K} = \big(\operatorname*{holim}_q X^q_\beta(K)\big)^{hG/K}.$$

Since taking $H$-homotopy fixed points commutes with homotopy limits of fibrant profinite $H$-spectra, for any profinite group $H$, by [27, Proposition 3.12], and the $G/K$-spectrum $(\operatorname{holim}_\beta X_\beta)^{hK}$ is easily seen to have $\operatorname{holim}_\beta \operatorname{holim}_q X^q_\beta(K)$ as a profinite $G/K$-model, it follows that there is an equivalence

$$(\operatorname*{holim}_\beta X_\beta)^{hG} \simeq \big(\operatorname*{holim}_\beta \operatorname*{holim}_q X^q_\beta(K)\big)^{hG/K} = \Big((\operatorname*{holim}_\beta X_\beta)^{hK}\Big)^{hG/K}.$$

The second assertion now follows from the above equivalence: by applying (7), there is a homotopy fixed point spectral sequence

$$E_2^{s,t} = H^s_c\big(G/K; \pi_t\big(\operatorname*{holim}_\beta \operatorname*{holim}_q X^q_\beta(K)\big)\big) \Rightarrow \pi_{t-s}\big((\operatorname*{holim}_\beta X_\beta)^{hG}\big).$$

There is a $G/K$-equivariant isomorphism

$$\pi_t\big((\operatorname*{holim}_\beta X_\beta)^{hK}\big) \cong \pi_t\big(\operatorname*{holim}_\beta \operatorname*{holim}_q X^q_\beta(K)\big)$$

of abelian groups, and thus, $\pi_t\big((\operatorname{holim}_\beta X_\beta)^{hK}\big)$ can be identified with the profinite $G/K$-module $\pi_t\big(\operatorname{holim}_\beta \operatorname{holim}_q X_\beta^q(K)\big)$. $\square$

The following definition extends Definition 4.15 to homotopy limits of diagrams.

**Definition 4.21.** *Let $G$ be a strongly complete profinite group. Also, let $\{X_\beta\}_{\beta\in J}$, with $J$ a small category, be a diagram of $G$-spectra, such that for each $\beta$, $X_\beta$ is a $\pi$-finite $G$-spectrum that is fibrant as a spectrum. Then we define*

$$(\operatorname*{holim}_\beta X_\beta)^{hH} := (\operatorname*{holim}_\beta F_G^s(X_\beta))^{hH},$$

*where $H$ is any closed subgroup of $G$.*

**Remark 4.22.** We point out that the construction in Definition 4.21 has the following desired properties: the canonical map

$$\operatorname*{holim}_\beta X_\beta \xrightarrow{\simeq} \operatorname*{holim}_\beta F_G^s(X_\beta)$$

is a weak equivalence of spectra that is $G$-equivariant, so that $\operatorname{holim}_\beta X_\beta$ has the fibrant profinite $G$-spectrum $\operatorname{holim}_\beta F_G^s(X_\beta)$ as a model in the category of profinite $G$-spectra; if the closed subgroup $H$ is finite, there are equivalences

$$(\operatorname*{holim}_\beta X_\beta)^{h_dH} \xrightarrow{\simeq} (\operatorname*{holim}_\beta F_G^s(X_\beta))^{h_dH} \simeq (\operatorname*{holim}_\beta F_G^s(X_\beta))^{hH} = (\operatorname*{holim}_\beta X_\beta)^{hH},$$

yielding agreement between the "classical" and "profinite" homotopy fixed points (the first and last terms, respectively, above); and if $H$ is any closed subgroup that is strongly complete, then

$$\begin{aligned}(\operatorname*{holim}_\beta F_G^s(X_\beta))^{hH} &\simeq \operatorname*{holim}_\beta (F_G^s(X_\beta))^{hH} \simeq \operatorname*{holim}_\beta (F_H^s(X_\beta))^{hH}\\ &\simeq (\operatorname*{holim}_\beta F_H^s(X_\beta))^{hH},\end{aligned}$$

so that the two possible definitions of $(\operatorname{holim}_\beta X_\beta)^{hH}$ (the first and last terms in the preceding chain of equivalences) are equivalent to each other.

Given Definition 4.21, the following result is immediate from Theorem 4.20.

**Corollary 4.23.** *Let $G$ be a strongly complete profinite group and let $K$ be a closed normal subgroup of $G$ with $G/K$ having finite virtual cohomological dimension. Also, let $\{X_\beta\}_{\beta\in J}$, where $J$ is a small category, be a diagram of $G$-spectra, with each $X_\beta$ both a fibrant spectrum and a $\pi$-finite $G$-spectrum. If the $J$-shaped diagram $\{F_G^s(X_\beta)\}_\beta$ of profinite $G$-spectra is a diagram of $K$-Postnikov $G$-spectra that is natural in $\beta$, then there is an equivalence of spectra*

$$\big(\operatorname*{holim}_\beta X_\beta\big)^{hG} \simeq \Big(\big(\operatorname*{holim}_\beta X_\beta\big)^{hK}\Big)^{hG/K}$$

*and a conditionally convergent spectral sequence*

$$E_2^{s,t} = H_c^s\big(G/K; \pi_t\big((\operatorname*{holim}_\beta X_\beta)^{hK}\big)\big) \Rightarrow \pi_{t-s}\big((\operatorname*{holim}_\beta X_\beta)^{hG}\big),$$

*whose $E_2$-term is the continuous cohomology of $G/K$ with coefficients the profinite $G/K$-module $\pi_t\big((\operatorname{holim}_\beta X_\beta)^{hK}\big)$.*

## 5. Iterated continuous homotopy fixed points for $E_n$

We return to our main example of the extended Morava stabilizer group $G_n$ and its continuous action on the Lubin-Tate spectrum $E_n$. Let $BP$ be the Brown-Peterson spectrum for the fixed prime $p$. Its coefficient ring is $BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$, where $v_i$ has degree $2(p^i - 1)$. There is a map

$$r : BP_* \to E_{n*} = W(\mathbb{F}_{p^n})[[u_1, \ldots, u_{n-1}]][u, u^{-1}],$$

defined by $r(v_i) = u_i u^{1-p^i}$ for $i < n$, $r(v_n) = u^{1-p^n}$ and $r(v_i) = 0$ for $i > n$, that makes $E_{n*}$ a $BP_*$-module. Let $I$ be an ideal in $BP_*$ of the form $(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}})$. Such ideals form a cofiltered system and their images in $E_{n*}$ under $r$ provide each $\pi_t E_n$ with the structure of a continuous profinite $G_n$-module: explicitly, we have

$$\pi_t E_n \cong \lim_I \pi_t E_n / I \pi_t E_n.$$

In fact, for $t$ odd these groups vanish, for $t$ even each quotient $\pi_t E_n / I \pi_t E_n$ is a finite discrete $G_n$-module, and the above decomposition of $\pi_t E_n$ as a limit of these finite discrete $G_n$-modules is compatible with the $G_n$-action.

In the collection of ideals $I$, we can fix a descending chain of ideals

$$I_0 \supset I_1 \supset \cdots \supset I_k \supset \cdots$$

with an associated tower

$$M_{I_0} \leftarrow M_{I_1} \leftarrow \cdots \leftarrow M_{I_k} \leftarrow \cdots$$

of generalized Moore spectra with trivial $G_n$-action, such that

$$L_{K(n)}(\mathbb{S}^0) \simeq \operatorname{holim}_k \big(L_{E_n}(M_{I_k})\big)_f\,,$$

where $L_{K(n)}$ denotes Bousfield localization with respect to $K(n)$ and $(-)_f$ is a fibrant replacement functor for Bousfield-Friedlander spectra, and for each $k \geq 0$, $BP_*(M_{I_k}) = BP_*/I_k$ (see [18]). It is useful to note that the last condition (about Brown-Peterson homology) implies that for each $k$,

$$\pi_t(E_n \wedge M_{I_k}) \cong \pi_t E_n / I_k \pi_t E_n$$

for all $t$. Also, as in [18], there is an equivalence

$$E_n \simeq \operatorname{holim}_k (E_n \wedge M_{I_k})_f$$

which is induced by the isomorphism $E_n \cong E_n \wedge \mathbb{S}^0$ in the stable homotopy category and the $G_n$-equivariant map

$$E_n \wedge \mathbb{S}^0 \xrightarrow{\simeq} \operatorname{holim}_k (E_n \wedge M_{I_k})_f$$

whose underlying map of spectra is a stable equivalence.

Now the profinite group $G_n$ is strongly complete (see e.g. [5, page 330]) and each $E_n \wedge M_{I_k}$ is a $\pi$-finite $G_n$-spectrum. Thus, the functorial replacement of Theorem 3.5 yields a fibrant profinite $G_n$-spectrum $F^s_{G_n}((E_n \wedge M_{I_k})_f)$ built out of pointed simplicial finite discrete $G_n$-sets and a $G_n$-equivariant map

$$(E_n \wedge M_{I_k})_f \xrightarrow{\simeq} F^s_{G_n}((E_n \wedge M_{I_k})_f) =: E'_{n,I_k}$$

which is a stable equivalence of underlying spectra. It follows that there is an equivalence

$$\operatorname{holim}_k (E_n \wedge M_{I_k})_f \xrightarrow{\simeq} \operatorname{holim}_k E'_{n,I_k}$$

of spectra, and the target of this equivalence can be regarded as a homotopy limit in the category of profinite $G_n$-spectra. Thus, because of the equivalence

$$E_n \simeq \operatorname*{holim}_k E'_{n,I_k},$$

we denote by $E'_n$ the fibrant profinite $G_n$-spectrum

$$E'_n := \operatorname*{holim}_k E'_{n,I_k}.$$

In [27] the continuous homotopy fixed points $E_n^{hG}$ under a closed subgroup $G$ of $G_n$ are defined as

$$E_n^{hG} := (E'_n)^{hG}.$$

These homotopy fixed points satisfy

$$E_n^{hG} \simeq \operatorname*{holim}_k (E'_{n,I_k})^{hG}, \tag{14}$$

since each $E'_{n,I_k}$ is a fibrant profinite $G$-spectrum.

Given the above setup, we are now ready to give a proof of Theorem 1.1.

*Proof of Theorem 1.1.* Recall that $G$ is an arbitrary closed subgroup of $G_n$ and $K$ is any closed normal subgroup of $G$. Since $G_n$ is a $p$-adic analytic group, [13, Theorem 9.6] implies that $G$ and the quotient group $G/K$ are $p$-adic analytic groups. Since $G$ is $p$-adic analytic, it contains an open pro-$p$ subgroup $H$ that is finitely generated. It follows that $H$ is strongly complete. If $G$ is not strongly complete, then it contains no open subgroups that are strongly complete, and hence, $G$ is strongly complete. This conclusion implies that $G/K$ is strongly complete. Also, since $G/K$ is $p$-adic analytic, it has finite virtual cohomological dimension (for example, see [5, page 330]).

Since $E'_{n,I_k}$ is an $f$-$G_n$-spectrum, it is an $f$-$G$-spectrum, and it follows from [31, Proposition 4.2.2] (as recalled in Remark 4.13) that $H^s_c(K; \pi_t(E'_{n,I_k}))$ is finite for all $s$ and $t$. Then, by applying Corollary 4.12, we can conclude that $E'_{n,I_k}$ is a $K$-Postnikov $G$-spectrum for each $k$. For each $k$ and any integer $q$, let

$$L^q_k = F^s_{G/K}\big(\operatorname*{colim}_j \operatorname{Map}(EG, P^q E'_{n,I_k})^{K_j}\big):$$

the diagram $\{L^q_k\}_{q\in\mathbb{Z}}$ is the inverse system of profinite $G/K$-models associated to the $K$-Postnikov $G$-spectrum $E'_{n,I_k}$. (To avoid any confusion, we remind the reader that $\{U_j\}_j$ is the collection of open normal subgroups of $G$ (not $G_n$) and $K_j = KU_j$.) It is easy to see that $\{E'_{n,I_k}\}_k$ is a diagram of $K$-Postnikov $G$-spectra that is natural in $k$, and hence, Theorem 4.20 yields the equivalence

$$E_n^{hG} = \big(\operatorname*{holim}_k E'_{n,I_k}\big)^{hG} \simeq \Big(\big(\operatorname*{holim}_k E'_{n,I_k}\big)^{hK}\Big)^{hG/K} = \big(E_n^{hK}\big)^{hG/K}. \qquad \square$$

We now give the proof of Theorem 1.2. Our proof continues the setup and notation that was established above in the proof of Theorem 1.1.

*Proof of Theorem 1.2.* Since $\{E'_{n,I_k}\}_k$ is a diagram of $K$-Postnikov $G$-spectra that is natural in $k$, Theorem 4.20 shows that there exists a conditionally convergent spectral sequence that has the form

$$H^s_c(G/K; \pi_t(E_n^{hK})) \Rightarrow \pi_{t-s}(E_n^{hG}). \tag{15}$$

Now let $H$ denote any closed subgroup of $G_n$ and notice that as at the end of the first paragraph of the proof of Theorem 1.1, $H$ has finite virtual cohomological dimension. Thus, by applying Theorem 3.2 to the $f$-$H$-spectrum $E'_{n,I_k}$, we obtain for each $k$ the equivalence $(E'_{n,I_k})^{hH} \simeq (E'_{n,I_k})^{h_dH}$. We can build upon this equivalence to obtain, for each $k$, the equivalences

$$(E'_{n,I_k})^{hH} \simeq (E'_{n,I_k})^{h_dH} \simeq \big(\big(\operatorname*{colim}_{N \lhd_o G_n} E_n^{dhN}\big) \wedge M_{I_k}\big)^{h_dH} \simeq E_n^{dhH} \wedge M_{I_k}, \tag{16}$$

where the above colimit is over the open normal subgroups of $G_n$, the second equivalence follows from the fact that the composition

$$\big(\operatorname*{colim}_{N \lhd_o G_n} E_n^{dhN}\big) \wedge M_{I_k} \xrightarrow{\simeq} E_n \wedge M_{I_k} \xrightarrow{\simeq} (E_n \wedge M_{I_k})_f \xrightarrow{\simeq} E'_{n,I_k}$$

is a weak equivalence in the category of discrete $H$-spectra (the key ingredient here is that the first map above is a weak equivalence of spectra: this is due to [12] and is made explicit in [5, Theorem 6.3, Corollary 6.5]), and the last equivalence in (16) is by [5, Corollary 9.8] and [1, Theorem 8.2.1]. By [9, proof of Lemma 3.5], the spectrum $E_n^{dhH} \wedge M_{I_k}$ has finite homotopy groups, and hence, so does the spectrum $(E'_{n,I_k})^{hH}$.

Our last conclusion implies that $\pi_*\big((E'_{n,I_k})^{hK}\big)$ and $\pi_*\big((E'_{n,I_k})^{hG}\big)$ are degreewise finite. Therefore, it follows from (14) (and from a second application of (14) with $G$ set equal to $K$), as in [18, proof of Proposition 7.4] (see also the beginning of the proof of Theorem 7.6 in [7]), that the spectral sequence in (15) is the inverse limit over $\{k\}$ of conditionally convergent spectral sequences $E_r(k) \equiv E_r^{*,*}(k)$ that have the form

$$E_2^{s,t}(k) = H_c^s\big(G/K; \pi_t\big((E'_{n,I_k})^{hK}\big)\big) \Rightarrow \pi_{t-s}\big((E'_{n,I_k})^{hG}\big),$$

where for each $k$, $E_r(k)$ is constructed as an instance of the descent spectral sequence in (7), with abutment equal to $\pi_*\big((\operatorname{holim}_q L_k^q)^{hG/K}\big)$.

We pause to introduce some helpful terminology: if $Z^\bullet$ is a cosimplicial spectrum that is fibrant in each codegree, then we refer to the conditionally convergent homotopy spectral sequence

$$E_2^{s,t} = H^s\big[\pi_t(Z^*)\big] \Rightarrow \pi_{t-s}\big(\operatorname*{holim}_{\Delta} Z^\bullet\big)$$

as the homotopy spectral sequence for $\operatorname{holim}_\Delta Z^\bullet$. For clarity later, we go ahead and point out that as defined above, whenever we use this terminology, the relevant homotopy limit (that is, $\operatorname{holim}_\Delta Z^\bullet$) is always indexed by $\Delta$.

By [27, Proposition 3.20 and Lemma 3.21, (b)], spectral sequence $E_r(k)$ can be regarded as the homotopy spectral sequence for

$$\operatorname*{holim}_{\Delta} \operatorname{Map}_{G/K}\big((G/K)^{\bullet+1}, \operatorname*{holim}_q L_k^q\big).$$

Thus, [27, Lemma 3.5] implies that $E_r(k)$ is the homotopy spectral sequence for

$$\operatorname*{holim}_{\Delta} \operatorname*{holim}_q \operatorname{Map}_{G/K}\big((G/K)^{\bullet+1}, L_k^q\big).$$

Since $L_k^q$ is an $f$-$G/K$-spectrum, the isomorphism in (11) (in the proof of Theorem 3.2) shows that $E_r(k)$ is isomorphic to the homotopy spectral sequence for

$$\operatorname*{holim}_{\Delta} \operatorname*{holim}_q \operatorname{Map}_c\big((G/K)^{\bullet+1}, L_k^q\big)^{G/K}.$$

It follows that $E_r(k)$ is isomorphic to spectral sequence $\mathcal{H}_r(k)$, which is defined to be the homotopy spectral sequence for

$$\operatorname*{holim}_{\Delta} \operatorname*{holim}_{q\geq 0} \mathrm{Map}_c\bigl((G/K)^{\bullet+1}, L^q_k\bigr)^{G/K} \simeq \Bigl(\operatorname*{holim}_{q\geq 0} L^q_k\Bigr)^{hG/K}$$

($\mathcal{H}_r(k)$ is an instance of the spectral sequence that is studied in [5, Theorem 8.8]), where the above equivalence uses that $L^q_k$ is a fibrant spectrum for each $q \geq 0$ and $G/K$ has finite virtual cohomological dimension, and the expression on the right-hand side above is the $G/K$-homotopy fixed points of the continuous $G/K$-spectrum $\mathrm{holim}_{q\geq 0} L^q_k$ (in the sense of [5]).

We want to compare spectral sequence $\mathcal{H}_r(k)$ with a certain other spectral sequence, and to accomplish this, we need to do some preliminary work. The first step is to note that for each $q \geq 0$, there is a canonical map $E'_{n,I_k} \to P^q E'_{n,I_k}$ in the category of profinite $G_n$-spectra, and hence, there is the induced $G/K$-equivariant map

$$\operatorname*{colim}_{j} \mathrm{Map}(EG, E'_{n,I_k})^{K_j} \to \operatorname*{colim}_{j} \mathrm{Map}(EG, P^q E'_{n,I_k})^{K_j}. \tag{17}$$

The source of the map in (17) is the first term in the following chain of equivalences:

$$\begin{aligned}\operatorname*{colim}_{j} \mathrm{Map}(EG, E'_{n,I_k})^{K_j} &= \operatorname*{colim}_{j} (E'_{n,I_k})^{hK_j} \simeq \operatorname*{colim}_{j} (E_n^{dhK_j} \wedge M_{I_k}) \\ &\simeq E_n^{dhK} \wedge M_{I_k},\end{aligned}$$

where the second equivalence follows from (16) and the last equivalence is due to [12, Proposition 6.3] (for the details, see [7, Lemma 7.1]). As recalled earlier, the spectrum $E_n^{dhK} \wedge M_{I_k}$ has finite homotopy groups, and hence, the source $\mathrm{colim}_j \mathrm{Map}(EG, E'_{n,I_k})^{K_j}$ of the map in (17) is a $\pi$-finite $G/K$-spectrum.

For each $q \geq 0$, there is a morphism

$$\operatorname*{colim}_{j} \mathrm{Map}(EG, E'_{n,I_k})^{K_j} \to L^q_k$$

of discrete $G/K$-spectra that is equal to the composition

$$\operatorname*{colim}_{j} \mathrm{Map}(EG, E'_{n,I_k})^{K_j} \to F^s_{G/K}\bigl(\operatorname*{colim}_{j} \mathrm{Map}(EG, E'_{n,I_k})^{K_j}\bigr) \to L^q_k,$$

where the first map in the above composition exists by Theorem 3.5, since the source is a $\pi$-finite $G/K$-spectrum, and the last map is obtained by applying $F^s_{G/K}$ to the map in (17). It follows that for each $q \geq 0$, there is an induced map

$$\mathrm{Map}_c\bigl((G/K)^{\bullet+1}, \operatorname*{colim}_{j} \mathrm{Map}(EG, E'_{n,I_k})^{K_j}\bigr)^{G/K} \to \mathrm{Map}_c\bigl((G/K)^{\bullet+1}, L^q_k\bigr)^{G/K}$$

of cosimplicial spectra, and hence, there is a map

$$\mathrm{Map}_c\bigl((G/K)^{\bullet+1}, \operatorname*{colim}_{j} \mathrm{Map}(EG, E'_{n,I_k})^{K_j}\bigr)^{G/K} \to \lim_{q\geq 0} \mathrm{Map}_c\bigl((G/K)^{\bullet+1}, L^q_k\bigr)^{G/K}$$

of cosimplicial spectra. Composition of the last map above with the canonical map from the limit to the homotopy limit yields a map

$$\mathrm{Map}_c\bigl((G/K)^{\bullet+1}, \operatorname*{colim}_{j} \mathrm{Map}(EG, E'_{n,I_k})^{K_j}\bigr)^{G/K} \to \operatorname*{holim}_{q\geq 0} \mathrm{Map}_c\bigl((G/K)^{\bullet+1}, L^q_k\bigr)^{G/K}$$

that induces a morphism from $\mathcal{D}_r(k)$, which is defined to be the homotopy spectral sequence for

$$\operatorname*{holim}_{\Delta} \mathrm{Map}_c\bigl((G/K)^{\bullet+1}, \operatorname*{colim}_{j} \mathrm{Map}(EG, E'_{n,I_k})^{K_j}\bigr)^{G/K},$$

to spectral sequence $\mathcal{H}_r(k)$.

The isomorphism between spectral sequences $\mathcal{H}_r(k)$ and $E_r(k)$, established earlier, implies that $\mathcal{H}_r(k)$ has $E_2$-term isomorphic to

$$E_2^{s,t}(k) = H_c^s\big(G/K; \pi_t\big((E'_{n,I_k})^{hK}\big)\big).$$

Also, as in [5, Theorem 7.9], spectral sequence $\mathcal{D}_r(k)$ has $E_2$-term equal to

$$H_c^s\big(G/K; \pi_t\big(\operatorname*{colim}_j \operatorname{Map}(EG, E'_{n,I_k})^{K_j}\big)\big),$$

with $\pi_t\big(\operatorname{colim}_j \operatorname{Map}(EG, E'_{n,I_k})^{K_j}\big)$ equal to a discrete $G/K$-module. Since

$$\operatorname*{colim}_j \operatorname{Map}(EG, E'_{n,I_k})^{K_j} \simeq E_n^{dhK} \wedge M_{I_k} \simeq (E'_{n,I_k})^{hK},$$

where the last equivalence above applies (16), and the finite profinite $G/K$-module $\pi_t\big((E'_{n,I_k})^{hK}\big)$ is automatically a discrete $G/K$-module, the $E_2$-terms of $\mathcal{D}_r(k)$ and $\mathcal{H}_r(k)$ are isomorphic. Therefore, spectral sequences $E_r(k)$ and $\mathcal{D}_r(k)$ are isomorphic to each other from the $E_2$-terms onward.

In [7, proof of Theorem 7.6], there is a descent spectral sequence that is referred to as $E_r^{*,*}(K, G, k)$ and which has the form

$$H_c^s(G/K; \pi_t(E_n^{h'K} \wedge M_{I_k})) \Rightarrow \pi_{t-s}(E_n^{h'G} \wedge M_{I_k}). \tag{18}$$

As done with the notation "$(E_n \wedge M_{I_k})_f$", if $Z$ denotes an arbitrary spectrum, we let $Z_f$ be a functorial fibrant replacement of $Z$ in the stable model category of spectra. It follows from [7, (6.2)] that spectral sequence $E_r^{*,*}(K, G, k)$ is the homotopy spectral sequence for

$$\operatorname*{holim}_\Delta \operatorname{Map}_c\big((G/K)^{\bullet+1}, \operatorname*{colim}_{N \lhd_o G_n} (E_n^{dhNK} \wedge M_{I_k})_f\big)^{G/K}.$$

Because

$$\operatorname*{colim}_j \operatorname{Map}(EG, E'_{n,I_k})^{K_j} \simeq E_n^{dhK} \wedge M_{I_k} \simeq \operatorname*{colim}_{N \lhd_o G_n} (E_n^{dhNK} \wedge M_{I_k})_f,$$

where the last equivalence follows from [12, Definition 1.5] (for some details, see [7, Lemma 5.2]), there is an isomorphism between spectral sequences $\mathcal{D}_r(k)$ and $E_r^{*,*}(K, G, k)$ (to see this, it is helpful to recall the definition of $\mathcal{D}_r(k)$).

Since the spectral sequence in (15) is the spectral sequence $\lim_k E_r(k)$, it is isomorphic to the spectral sequence $\lim_k \mathcal{D}_r(k)$, and hence, to the spectral sequence $\lim_k E_r^{*,*}(K, G, k)$, from the $E_2$-term onward. Since $\lim_k E_r^{*,*}(K, G, k)$ is the descent spectral sequence

$$H_c^s(G/K; \pi_t(E_n^{h'K})) \Rightarrow \pi_{t-s}(E_n^{h'G})$$

of (3) (see [7, proof of Theorem 7.6]), which is isomorphic to the strongly convergent spectral sequence

$$H_c^s(G/K; \pi_t(E_n^{dhK})) \Rightarrow \pi_{t-s}(E_n^{dhG})$$

of (2) from the $E_2$-term onward, it follows that the spectral sequence in (15) is isomorphic to spectral sequences (2) and (3) from the $E_2$-term onward, and its strong convergence follows from that of (2). $\square$

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