

ITERATED HOMOTOPY FIXED POINTS FOR THE LUBIN-TATE SPECTRUM

DANIEL G. DAVIS¹

WITH AN APPENDIX BY DANIEL G. DAVIS² AND BEN WIELAND³

ABSTRACT. When G is a profinite group and H and K are closed subgroups, with H normal in K , it is not known, in general, how to form the iterated homotopy fixed point spectrum $(Z^{hH})^{hK/H}$, where Z is a continuous G -spectrum and all group actions are to be continuous. However, we show that, if $G = G_n$, the extended Morava stabilizer group, and $Z = \widehat{L}(E_n \wedge X)$, where \widehat{L} is Bousfield localization with respect to Morava K -theory, E_n is the Lubin-Tate spectrum, and X is any spectrum with trivial G_n -action, then the iterated homotopy fixed point spectrum can always be constructed. Also, we show that $(E_n^{hH})^{hK/H}$ is just E_n^{hK} , extending a result of Devinatz and Hopkins.

1. INTRODUCTION

Let G be a profinite group and let X be a continuous G -spectrum. Thus, X is the homotopy limit of a tower

$$X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots \leftarrow X_i \leftarrow \cdots$$

of discrete G -spectra that are fibrant as spectra (see Section 2 for a quick review of this notion that is developed in detail in [2]). Let H and K be closed subgroups of G , with H normal in K . Note that H is closed in K , and K , H , and the quotient K/H are all profinite groups.

It is easy to see that the fixed point spectrum X^H is a K/H -spectrum and

$$(X^H)^{K/H} = X^K.$$

Now, there is a model category \mathcal{T} of towers of discrete G -spectra and homotopy fixed points are the total right derived functor of

$$\lim_i (-)^G: \mathcal{T} \rightarrow \text{spectra}$$

(see [2, Section 4, Remark 8.4] for more detail). Thus, as noted in [5, Introduction], since homotopy fixed points are defined in terms of fixed points, it is reasonable to wonder (a) if the H -homotopy fixed point spectrum X^{hH} is a continuous K/H -spectrum; and (b) assuming that (a) holds, if there is an equivalence

$$(1.1) \quad (X^{hH})^{hK/H} \simeq X^{hK},$$

where $(X^{hH})^{hK/H}$ is the *iterated homotopy fixed point spectrum*.

¹The author was partially supported by an NSF VIGRE grant at Purdue University. Part of this paper was written during a visit to the Institut Mittag-Leffler (Djursholm, Sweden) and a year at Wesleyan University.

²The author was supported by a grant from the Louisiana Board of Regents Support Fund.

³The author was partially supported by an NSF Postdoctoral Fellowship.

It is not hard to see that, if H is open in K , then the isomorphism of (1.1) always holds; see Theorem 3.4 for the details. However, if H is not open in K , then the situation is much more complicated. In this case, it is not known, in general, how to construct X^{hH} as a K/H -spectrum, and, even when we can construct X^{hH} as a K/H -spectrum, it is not known, in general, how to view X^{hH} as a continuous K/H -spectrum. Thus, when H is not open in K , additional hypotheses are needed just to get past step (a) - see Sections 3 and 4 for a discussion of this point.

Now we consider the above issues in an example that is of interest in chromatic stable homotopy theory. Let $n \geq 1$, let p be a fixed prime, and let $K(n)$ be the n th Morava K -theory spectrum (so that $K(n)_* = \mathbb{F}_p[v_n^{\pm 1}]$, where $|v_n| = 2(p^n - 1)$). Then, let E_n be the Lubin-Tate spectrum: E_n is the $K(n)$ -local Landweber exact spectrum whose coefficients are given by $E_{n*} = W(\mathbb{F}_{p^n})[[u_1, \dots, u_{n-1}]]\langle u^{\pm 1} \rangle$, where $W(\mathbb{F}_{p^n})$ is the ring of Witt vectors of the field \mathbb{F}_{p^n} , each u_i has degree zero, and the degree of u is -2 .

Let S_n be the n th Morava stabilizer group (the automorphism group of the Honda formal group law Γ_n of height n over \mathbb{F}_{p^n}) and let

$$G_n = S_n \rtimes \text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$$

be the extended Morava stabilizer group, a profinite group. Then, given any closed subgroup K of G_n , let E_n^{dhK} be the construction, due to Devinatz and Hopkins, that is denoted E_n^{hK} in [8]. This work ([8]) shows that the spectra E_n^{dhK} behave like homotopy fixed point spectra, though they are not constructed with respect to a continuous K -action on E_n .

As in [2], let E_n^{hK} be the K -homotopy fixed point spectrum of E_n , formed with respect to the continuous K -action on E_n . In [4] (see [1, Theorem 8.2.1] for a proof), the author showed that

$$(1.2) \quad E_n^{hK} \simeq E_n^{dhK},$$

for all K closed in G_n . Now, as explained in [5, pg. 8], the profinite group K/H acts on E_n^{dhH} . Suppose that H is open in K , so that K/H is a finite group. Then, Devinatz and Hopkins (see [8, Theorem 4 and Section 7]) showed that the canonical map

$$E_n^{dhK} \rightarrow \text{holim}_{K/H} E_n^{dhH} = (E_n^{dhH})^{hK/H}$$

is a weak equivalence. Thus, by applying (1.2), we obtain that $(E_n^{hH})^{hK/H} \simeq E_n^{hK}$. Alternatively, as mentioned earlier, since E_n is a continuous K -spectrum and H is open in K , this same conclusion follows immediately from Theorem 3.4. Therefore, it is clear that, for E_n and the finite quotients K/H , the iterated homotopy fixed point spectrum behaves as desired - that is, the equivalence of (1.1) is valid.

However, when K/H is not finite, the papers [8, 5] do not construct $(E_n^{dhH})^{hK/H}$, and hence, they are not able to consider the question of whether this spectrum is just E_n^{dhK} . Thus, in this more complicated situation, we show how to construct $(E_n^{dhH})^{hK/H}$. More precisely, we prove the following result, where \widehat{L} denotes the Bousfield localization functor $L_{K(n)}$.

Theorem 1.3. *Let H and K be any closed subgroups of G_n , such that H is a normal subgroup of K , so that E_n^{dhH} carries the natural K/H -action. Also, let X be any spectrum (with no K/H -action). Then the spectrum $\widehat{L}(E_n^{dhH} \wedge X)$ is a*

continuous K/H -spectrum, with K/H acting diagonally on $\widehat{L}(E_n^{dhH} \wedge X)$ (X has the trivial K/H -action).

The theory of [2] (reviewed in Section 2), applied to Theorem 1.3, automatically yields

$$(\widehat{L}(E_n^{dhH} \wedge X))^{hK/H}$$

as the homotopy fixed points with respect to the continuous K/H -action. Running through the argument for Theorem 1.3 again, by omitting the $(- \wedge X)$ everywhere, gives the desired object $(E_n^{dhH})^{hK/H}$. By applying (1.2), we immediately find that the iterated homotopy fixed point spectrum $(E_n^{dhH})^{hK/H}$ is always defined.

In Section 6, we use the descent spectral sequence for $(\widehat{L}(E_n^{dhH} \wedge E_n))^{hK/H}$, where K/H is acting trivially on (the second) E_n , to prove the following result, which says that the iterated homotopy fixed point spectrum for E_n behaves as in (1.1), if one smashes with E_n before taking the K/H -homotopy fixed points.

Theorem 1.4. *If H and K are as in Theorem 1.3, then*

$$(\widehat{L}(E_n^{dhH} \wedge E_n))^{hK/H} \simeq \widehat{L}(E_n^{dhK} \wedge E_n).$$

In Section 7, we use (1.2) to prove that the iterated homotopy fixed point spectrum for E_n behaves as desired.

Theorem 1.5. *Let H and K be as above. Then there is an equivalence*

$$(E_n^{hH})^{hK/H} \simeq E_n^{hK}.$$

These iterated homotopy fixed point spectra, $(E_n^{hH})^{hK/H}$, play a useful role in chromatic homotopy theory. A major conjecture in this field is that $\pi_*(\widehat{L}(S^0))$ is a module of finite type over the p -adic integers \mathbb{Z}_p . An important way to tackle this conjecture is due to Devinatz (see [6, 7]), and a key part of his program is to find a closed subgroup H_0 of G_n and a finite spectrum Z that is not rationally acyclic, such that $\pi_*(E_n^{hH_0} \wedge Z)$ is of finite type and H_0 is a part of a chain

$$H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_t = G_n$$

of closed subgroups.

For suppose that, for some $i < t$, $\pi_*(E_n^{hH_i} \wedge Z)$ is of finite type. By [5, (0.1)] and Theorem 7.6, there is a strongly convergent descent spectral sequence

$$H_c^s(H_{i+1}/H_i; \pi_t(E_n^{hH_i} \wedge Z)) \Rightarrow \pi_{t-s}((E_n^{hH_i})^{hH_{i+1}/H_i} \wedge Z),$$

where the E_2 -term is continuous cohomology for profinite $\mathbb{Z}_p[[H_{i+1}/H_i]]$ -modules (defined just above (6.3)). As explained in [6, pg. 133], various properties of these cohomology groups and the spectral sequence imply that

$$\pi_*((E_n^{hH_i})^{hH_{i+1}/H_i} \wedge Z) \cong \pi_*(E_n^{hH_{i+1}} \wedge Z)$$

has finite type. Thus, induction shows that $\pi_*(E_n^{hG_n} \wedge Z) \cong \pi_*(\widehat{L}(Z))$ is of finite type, since $E_n^{hG_n} \simeq \widehat{L}(S^0)$ (by [8, Theorem 1(iii)], [1, Corollary 8.1.3]). In particular, since Z is not rationally acyclic, it follows that $\pi_*(\widehat{L}(S^0))$ would then have finite type (see [7, pg. 2]).

Other examples of the utility of the K/H -action on E_n^{hH} , with K/H an infinite profinite group, are in [8, Section 8] and [13, Section 2]: in the first case, the action helps (see [8, Proposition 8.1]) to construct an interesting element in $\pi_{-1}(\widehat{L}(S^0))$,

for all n and p ([8, Theorem 6]), and, in the second case, the action plays a role (see [13, Theorem 10]) in computing $\pi_*(L_2V(1))$, at the prime 3.

To study the iterated homotopy fixed points of E_n , we consider the notion of a hyperfibrant discrete G -spectrum, and we show that, for such a spectrum, the iterated homotopy fixed point spectrum is always defined in a natural way (see Definitions 4.1 and 4.5). Also, Lemma 4.9 shows that if a discrete G -spectrum X is a hyperfibrant discrete K -spectrum, then, for any H closed in G and normal in K , (1.1) is valid. Additionally, we show that, for a totally hyperfibrant discrete G -spectrum, (1.1) holds for all H and K closed in G , with H normal in K (see Definition 4.10).

Given distinct primes p and q , for $G = \mathbb{Z}/p \times \mathbb{Z}/q$, Ben Wieland has found an example of a discrete G -spectrum X that is not a hyperfibrant discrete G -spectrum. In the Appendix, Wieland and the author provide the details for this example. As explained in Section 3 (after Theorem 3.4), for discrete G -spectra that are not hyperfibrant, there are situations where it is not known how to define a continuous K/H -action on X^{hH} , so that it is also not known how to define the K/H -homotopy fixed points of X^{hH} , with respect to a continuous K/H -action.

At the end of Section 7, we point out that the results in this paper apply to certain other spectra that are like E_n , in that they replace the role of \mathbb{F}_{p^n} and the height n Honda formal group law Γ_n with certain other finite fields and height n formal group laws, respectively.

In [1, Theorem 7.1.1], Mark Behrens and the author show that if a spectrum E is a consistent profaithful k -local profinite G -Galois extension of A of finite vcd, then the map $\psi(E)_K^G$ of (3.5) is a k_* -equivalence, for all K closed in G , where $L_k(-) \simeq L_M L_T(-)$, with M a finite spectrum and T smashing, and A is a k -local commutative S -algebra. Behrens and the author use this result to show that if the closed subgroup K is normal in G , then $(E^{h_k K})^{h_k G/K} \simeq E^{h_k G}$ (see [1, Proposition 7.1.4, Theorem 7.1.6]), where, for example, $(-)^{h_k G}$ is the right derived functor of the fixed points $(-)^G$, with respect to the k -local model structure on discrete G -spectra (see [1, Section 6.1]).

We point out that, independently of our work, in [11, Sections 10.1, 10.2], Halvard Fausk considers the iterated homotopy fixed point pro-spectrum for pro- G -orthogonal spectra, and the associated descent spectral sequence, by making general use of Postnikov towers. However, our approach to iterated homotopy fixed points is different in technique from his: instead of Postnikov towers, we rely on the notion of hyperfibrancy, which, in the case of E_n , makes use of technical results about the Morava stabilizer group (see [8, the proof of Theorem 4.3]).

In [10, Lemma 10.5], it is shown that, if G is any discrete group, with K any normal subgroup, and, if X is a G -space, then X^{hK} is homotopy equivalent to $(X^{hK})^{hG/K}$. This result, described as verifying a “transitivity property” of homotopy fixed points, is, as far as the author has been able to determine, the first reference to iterated homotopy fixed points in the literature. Given an S -module F and a faithful F -local G -Galois extension $A \rightarrow B$ of commutative S -algebras, where G is a stably dualizable group, [20, Theorem 7.2.3] shows that, if K is an allowable normal subgroup of G , then $(B^{hK})^{hG/K} \simeq B^{hG}$.

We make a comment about notation in this paper. If a limit or colimit is indexed over N , as in $\lim_N G/N$, then the (co)limit is indexed over all open normal subgroups of G , unless stated otherwise.

Acknowledgements. I am grateful to Mark Behrens for a series of profitable conversations about iterated homotopy fixed points. I thank Tilman Bauer, Ethan Devinatz, and Paul Goerss for helpful discussions about this work. Also, I am grateful to the referee for many helpful comments that improved and sharpened the writing of this paper. Additionally, I thank Halvard Fausk, Rick Jardine, Ben Wieland, and an earlier referee for their comments. Both authors of the Appendix thank Paul Goerss and the referee for helpful remarks.

2. A SUMMARY OF THE THEORY OF CONTINUOUS G -SPECTRA AND THEIR HOMOTOPY FIXED POINTS

This section contains a quick review of material from [2] that is needed for our work in this paper. We begin with some terminology.

All of our spectra are Bousfield-Friedlander spectra of simplicial sets. Let G be a profinite group. A *discrete G -set* is a G -set S such that the action map $G \times S \rightarrow S$ is continuous, where S is regarded as a discrete space.

Definition 2.1. A *discrete G -spectrum* X is a G -spectrum such that each simplicial set X_k is a simplicial object in the category of discrete G -sets. Spt_G is the category of discrete G -spectra, where the morphisms are G -equivariant maps of spectra.

Theorem 2.2 ([2, Theorem 3.6]). *Spt_G is a model category, where f in Spt_G is a weak equivalence (cofibration) if and only if it is a weak equivalence (cofibration) of spectra.*

It is helpful to note that, by [2, Lemma 3.10], if X is fibrant in Spt_G , then X is fibrant as a spectrum. In this paper, we often take the smash product (in spectra) of a discrete G -spectrum X with a spectrum Y that has no G -action (and we never take the smash product in the case where Y has a non-trivial G -action). Then (following [2, Lemma 3.13]), since X^N , where N is an open normal subgroup of G , is a G/N -spectrum, the isomorphisms

$$X \wedge Y \cong (\text{colim}_N X^N) \wedge Y \cong \text{colim}_N (X^N \wedge Y)$$

show that $X \wedge Y$ is also a discrete G -spectrum.

Definition 2.3. If $X \in \text{Spt}_G$, then let $X \rightarrow X_{f,G} \rightarrow *$ be a trivial cofibration followed by a fibration, all in Spt_G . Then $X^{hG} = (X_{f,G})^G$.

When G is a finite group, this definition is equivalent to the classical definition of X^{hG} for a finite discrete group G (see [2, Section 5]).

Definition 2.4. A *tower of discrete G -spectra* $\{X_i\}$ is a diagram

$$X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \dots$$

in Spt_G , such that each X_i is fibrant as a spectrum.

Definition 2.5. If $\{X_i\}$ is a tower of discrete G -spectra, then $\text{holim}_i X_i$ is a *continuous G -spectrum*. Also, $\text{holim}_i (X_i)^{hG}$ is the *G -homotopy fixed point spectrum* of the continuous G -spectrum $\text{holim}_i X_i$. Given a tower $\{X_i\}$, we write $X = \text{holim}_i X_i$ and $X^{hG} = \text{holim}_i (X_i)^{hG}$.

We call the above construction “homotopy fixed points” because: (1) if G is a finite discrete group, then the above definition agrees with the classical definition

of X^{hG} (see [2, Lemma 8.2]); and (2) it is the total right derived functor of fixed points in the appropriate sense (see [2, Remark 8.4]).

Before going further, we note the following. Given a tower $\{X_i\}$ of discrete G -spectra, the continuous G -spectrum X and its homotopy fixed points are independent of the tower in the following sense. Suppose that $\{X_i\} \rightarrow \{Y_i\}$ is a map of towers of discrete G -spectra (that is, $\{Y_i\}$ is a tower of discrete G -spectra; for each i , there is a morphism $X_i \rightarrow Y_i$ in Spt_G ; and, for each $i > 0$, the diagram

$$\begin{array}{ccc} X_i & \longrightarrow & Y_i \\ \downarrow & & \downarrow \\ X_{i-1} & \longrightarrow & Y_{i-1} \end{array}$$

is commutative) such that each map $X_i \rightarrow Y_i$ is a weak equivalence in Spt_G . Then, since each $X_i \rightarrow Y_i$ is also a weak equivalence of spectra, with both the source and target fibrant spectra, the G -equivariant map $X = \text{holim}_i X_i \rightarrow \text{holim}_i Y_i = Y$ is a weak equivalence of spectra.

Additionally, let H be any closed subgroup of G . By using functorial fibrant replacement, each map $(X_i)_{f,H} \rightarrow (Y_i)_{f,H}$ is a weak equivalence in Spt_H , so that each map $(X_i)^{hH} \rightarrow (Y_i)^{hH}$ is a weak equivalence of spectra, with both the source and target fibrant spectra (by [2, Corollary 3.9]). Hence, the map

$$X^{hH} = \text{holim}_i (X_i)^{hH} \rightarrow \text{holim}_i (Y_i)^{hH} = Y^{hH}$$

is a weak equivalence.

Definition 2.6. We say that G has *finite virtual cohomological dimension* (*finite vcd*), and we write $\text{vcd}(G) < \infty$, if there exists an open subgroup U in G such that, for some positive integer m , $H_c^s(U; M) = 0$, for all $s > m$, whenever M is a discrete U -module.

It is useful to note that if G is a compact p -adic analytic group, then G has finite vcd (see the explanation just before Lemma 2.10 in [2]).

Before stating the next result, we need some notation. When X is a discrete set, let $\text{Map}_c(G, X)$ denote the discrete G -set of continuous maps $G \rightarrow X$, where the G -action is given by $(g \cdot f)(g') = f(g'g)$, for $g, g' \in G$. Now consider the functor

$$\Gamma_G: \text{Spt}_G \rightarrow \text{Spt}_G, \quad X \mapsto \Gamma_G(X) = \text{Map}_c(G, X),$$

where the action of G on $\text{Map}_c(G, X)$ is induced on the level of sets by the G -action on $\text{Map}_c(G, (X_k)_l)$, the set of l -simplices of the pointed simplicial set $\text{Map}_c(G, X_k)$, where $k, l \geq 0$.

Let Spt denote the category of spectra. Then the right adjoint to the forgetful functor $U: \text{Spt}_G \rightarrow \text{Spt}$ is the functor $\text{Map}_c(G, -)$. Also, as explained in [2, Definition 7.1], the functor Γ_G forms a triple and there is a cosimplicial discrete G -spectrum $\Gamma_G^\bullet X$. Given a tower $\{M_i\}$ of discrete G -modules, let $H_{\text{cont}}^s(G; \{M_i\})$ denote continuous cohomology in the sense of Jannsen (see [17]).

Theorem 2.7 ([2, Theorem 8.8]). *If G has finite vcd and $\{X_i\}$ is a tower of discrete G -spectra, then there is a conditionally convergent descent spectral sequence*

$$E_2^{s,t} \Rightarrow \pi_{t-s}(\text{holim}_i (X_i)^{hG}) = \pi_{t-s}(X^{hG}),$$

where $E_2^{s,t} = \pi^s \pi_t(\text{holim}_i(\Gamma_G^\bullet(X_i)_{f,G})^G)$. If the tower of abelian groups $\{\pi_t(X_i)\}$ satisfies the Mittag-Leffler condition for each $t \in \mathbb{Z}$, then

$$E_2^{s,t} \cong H_{\text{cont}}^s(G; \{\pi_t(X_i)\}).$$

3. ITERATED HOMOTOPY FIXED POINTS - A DISCUSSION

Let G be a profinite group, and let X be a continuous G -spectrum, so that $X = \text{holim}_i X_i$, where each X_i is a discrete G -spectrum that is fibrant as a spectrum. (Every discrete G -spectrum Z is a continuous G -spectrum, in the sense that there is a weak equivalence $Z \rightarrow \text{holim}_i Z_{f,G}$ that is given by the composition

$$Z \xrightarrow{\simeq} Z_{f,G} \xrightarrow{\cong} \lim_i Z_{f,G} \xrightarrow{\simeq} \text{holim}_i Z_{f,G},$$

which is G -equivariant, where $\text{holim}_i Z_{f,G}$ is a continuous G -spectrum and the limit and holim are each applied to a constant diagram.) In this section, we consider more carefully the iterated homotopy fixed point spectrum $(X^{hH})^{hK/H}$ and its relationship with X^{hK} .

The author thanks Mark Behrens and Paul Goerss for help with the next lemma and its proof.

Lemma 3.1. *Let G be a profinite group and let H be an open subgroup of G . If X is a fibrant discrete G -spectrum, then X is a fibrant discrete H -spectrum.*

Remark 3.2. If H is normal in G , then Lemma 3.1 is a consequence of [18, Remark 6.26], using the fact that the presheaf of spectra $\text{Hom}_G(-, X)$ is globally fibrant in the model category of presheaves of spectra on the site $G\text{-Sets}_{\text{df}}$ (for an explanation of this fact, see [2, Section 3]).

Proof of Lemma 3.1. Note that the forgetful functor $U: \text{Spt}_G \rightarrow \text{Spt}_H$ has a left adjoint $\text{Ind}_G^H: \text{Spt}_H \rightarrow \text{Spt}_G$, where, as a spectrum, $\text{Ind}_G^H(Z)$ can be identified with the finite wedge $\bigvee_{G/H} Z$ of copies of Z indexed by G/H .

Let X be a fibrant discrete G -spectrum. To show that X is fibrant in Spt_H , we will show that the forgetful functor U preserves all fibrations. To do this, it suffices to show that Ind_G^H preserves all weak equivalences and cofibrations. Thus, we only have to show that if $f: Y \rightarrow Z$ is a map of discrete H -spectra that is a weak equivalence (cofibration) of spectra, then $\text{Ind}_G^H(f)$ is also a weak equivalence (cofibration) of spectra.

If f is a weak equivalence, then, as a map of spectra, $\pi_*(\text{Ind}_G^H(f))$ is equivalent to $\bigoplus_{G/H} \pi_*(f)$, which is an isomorphism, so that $\text{Ind}_G^H(f)$ is a weak equivalence.

Now let f be a cofibration of spectra. Given $n \geq 0$, let, for example, Y_n be the n th pointed simplicial set of Y (a Bousfield-Friedlander spectrum of simplicial sets). Then $f_0: Y_0 \rightarrow Z_0$ and each induced map $(S^1 \wedge Z_n) \cup_{(S^1 \wedge Y_n)} Y_{n+1} \rightarrow Z_{n+1}$ are cofibrations of simplicial sets. It is easy to see that

$$(\text{Ind}_G^H(f))_0: \bigvee_{G/H} Y_0 \rightarrow \bigvee_{G/H} Z_0$$

is a cofibration of simplicial sets. Also, the composition

$$(S^1 \wedge \bigvee_{G/H} Z_n) \cup_{(S^1 \wedge \bigvee_{G/H} Y_n)} (\bigvee_{G/H} Y_{n+1}) \cong \bigvee_{G/H} ((S^1 \wedge Z_n) \cup_{(S^1 \wedge Y_n)} Y_{n+1}) \rightarrow \bigvee_{G/H} Z_{n+1}$$

is a cofibration of simplicial sets. Thus, $\text{Ind}_G^H(f)$ is a cofibration of spectra. \square

Let G and H be as in Lemma 3.1. If X is a discrete G -spectrum, then Lemma 3.1 implies that

$$(3.3) \quad X^{hH} = (X_{f,G})^H,$$

since $X \rightarrow X_{f,G}$ is a trivial cofibration in Spt_H .

For the rest of this section, let H and K be closed subgroups of G , with H normal in K . The following result shows that, if H is open in K , then the equivalence of (1.1) always holds.

Theorem 3.4. *Let G be a profinite group with closed subgroups H and K , with H normal in K . Given $\{X_i\}$, a tower of discrete G -spectra, let $X = \text{holim}_i X_i$ be a continuous G -spectrum. If H is open in K , then there is a weak equivalence*

$$X^{hK} \rightarrow (X^{hH})^{hK/H}.$$

Proof. Since each X_i is a discrete H - and K -spectrum, X is a continuous H - and K -spectrum. By (3.3),

$$X^{hH} = \text{holim}_i (X_i)^{hH} = \text{holim}_i ((X_i)_{f,K})^H.$$

Since $\text{Hom}_K(-, (X_i)_{f,K})$ is a fibrant presheaf of spectra on the site $K\text{-Sets}_{df}$, the canonical map

$$(X_i)^{hK} = \text{Hom}_K(*, (X_i)_{f,K}) \rightarrow \text{holim}_{K/H} \text{Hom}_K(K/H, (X_i)_{f,K}) \cong \text{holim}_{K/H} ((X_i)_{f,K})^H$$

is a weak equivalence, by [18, Proposition 6.39]. By (3.3), there is the equality $\text{holim}_{K/H} ((X_i)_{f,K})^H = \text{holim}_{K/H} (X_i)^{hH}$, so that the previous sentence yields a weak equivalence

$$(X_i)^{hK} \rightarrow \text{holim}_{K/H} (X_i)^{hH}.$$

Note that both the source and target of this weak equivalence are fibrant spectra, since, for any profinite group L , the functor $(-)^L: \text{Spt}_L \rightarrow \text{Spt}$ preserves fibrant objects (see [2, Corollary 3.9]).

Thus, there is a weak equivalence

$$X^{hK} = \text{holim}_i (X_i)^{hK} \rightarrow \text{holim}_i \text{holim}_{K/H} (X_i)^{hH} \cong \text{holim}_{K/H} X^{hH}.$$

Since the identity map $X^{hH} \rightarrow X^{hH}$, where $X^{hH} = \text{holim}_i ((X_i)_{f,K})^H$ is, of course, a weak equivalence and a K/H -equivariant map, with X^{hH} fibrant, we can make the identification $\text{holim}_{K/H} X^{hH} = (X^{hH})^{hK/H}$, completing the proof. \square

It will be useful to define the following map. Given a discrete G -spectrum X , the fibrant replacement $X_{f,G} \rightarrow (X_{f,G})_{f,K}$ and the identity

$$\text{colim}_N X^{hNK} = \text{colim}_N (X_{f,G})^{NK},$$

which applies (3.3), induces the map

$$(3.5) \quad \psi(X)_K^G: \text{colim}_N X^{hNK} \cong (X_{f,G})^K \rightarrow ((X_{f,G})_{f,K})^K = X^{hK},$$

where the last equality is due to the fact that $X \rightarrow X_{f,G} \rightarrow (X_{f,G})_{f,K}$ is a trivial cofibration in Spt_K , with $(X_{f,G})_{f,K}$ fibrant in Spt_K . For convenience, we will sometimes write $\psi(X)_K^G$ as just ψ .

Now we consider the case when (1.1) is more difficult to understand; for the rest of this section, unless stated otherwise, we assume that H is not open in K . Also, since, when $H = \{e\}$, we have

$$(X^{h\{e\}})^{hK/\{e\}} = ((X_{f,G})^{\{e\}})^{hK} \simeq X^{hK},$$

we will assume that $H \neq \{e\}$. Additionally, we simplify the situation some by assuming that the continuous G -spectrum X is a discrete G -spectrum.

First of all, in this case, consider the statement that $X_{f,K}$ is a fibrant object in Spt_H . When H is open in K , this statement is Lemma 3.1, which we proved by showing that the forgetful functor $U: \text{Spt}_K \rightarrow \text{Spt}_H$ preserves fibrations, and we did this by using the fact that U is a right adjoint. However, in general, when H is not open in K , the forgetful functor U need not preserve limits, so that, in general, it need not be a right adjoint (see [1, Section 3.6]).

Ben Wieland has found an example of a profinite group G' , with finite vcd and a closed normal subgroup H' , and a discrete G' -spectrum Y , such that $Y_{f,G'}$ is not fibrant as a discrete H' -spectrum. For, if $Y_{f,G'}$ is fibrant as a discrete H' -spectrum, then the map $\psi(Y)_{H'}^{G'}$ of (3.5) is a weak equivalence. But $\psi(Y)_{H'}^{G'}$ is not a weak equivalence. The details of this example are in the Appendix. This example proves that, when H is closed and not open in K , a fibrant object in Spt_K is not necessarily fibrant in Spt_H .

The above considerations show that, in general, there is no reason to expect that $X_{f,K}$ is fibrant in Spt_H , and, without this fact, the above proof of Theorem 3.4 fails to work.

Now suppose that K has finite vcd. By [3, Theorem 4.2], there is a map

$$X \rightarrow \text{colim}_{N \triangleleft_o H} (\text{holim}_{\Delta} \Gamma_K^\bullet(X_{f,K}))^N$$

that is a weak equivalence in Spt_H , such that the target is a fibrant discrete H -spectrum. Then, by [3, Corollary 5.4], we can make the identification

$$X^{hH} = (\text{colim}_{N \triangleleft_o H} (\text{holim}_{\Delta} \Gamma_K^\bullet(X_{f,K}))^N)^H \cong (\text{holim}_{\Delta} \Gamma_K^\bullet(X_{f,K}))^H,$$

so that X^{hH} is the K/H -spectrum $(\text{holim}_{\Delta} \Gamma_K^\bullet(X_{f,K}))^H$.

Notice that K/H is an infinite profinite group: if not, then H is a closed subgroup of K with finite index, so H must be open in K , contrary to our assumption. To form the K/H -homotopy fixed points of X^{hH} , we need to know that X^{hH} can be regarded as either a discrete K/H -spectrum or as a homotopy limit of a tower of discrete K/H -spectra (or, more generally, as in [1], the homotopy limit of a cofiltered diagram of discrete K/H -spectra), for these are the cases where the K/H -homotopy fixed points are defined. However, in general, it is not known how to do this. For example, the Appendix shows that $Y^{hH'}$ can not be regarded as a discrete G'/H' -spectrum (if it could be, then $\pi_*(Y^{hH'})$ is a discrete G'/H' -module, which is false), so that, in general, it can not be assumed that X^{hH} is a discrete K/H -spectrum. Also, it is not known, in general, how to realize X^{hH} as the homotopy limit of a tower or cofiltered diagram of discrete K/H -spectra. (For example, letting \prod^* denote cosimplicial replacement,

$$\begin{aligned} X^{hH} &\cong \text{holim}_{\Delta} (\Gamma_K^\bullet(X_{f,K}))^H = \text{Tot}(\prod^*(\Gamma_K^\bullet(X_{f,K}))^H) \\ &\cong \lim_n \text{Tot}_n(\prod^*(\Gamma_K^\bullet(X_{f,K}))^H) \end{aligned}$$

presents X^{hH} as the limit of a tower of K/H -spectra, with each $(\Gamma_K^l(X_{f,K}))^H$ a discrete K/H -spectrum (by the discussion just above Lemma 4.6). However, $\text{Tot}_n(\prod^*(\Gamma_K^\bullet(X_{f,K}))^H)$ is not necessarily a discrete K/H -spectrum, since the infinite product $\prod^l(\Gamma_K^\bullet(X_{f,K}))^H$ need not be a discrete K/H -spectrum, so this attempt to present X^{hH} as a continuous K/H -spectrum fails.)

Let us consider in more detail whether or not X^{hH} can be regarded as a discrete K/H -spectrum. Note that

$$X^{hH} \cong \text{holim}_\Delta(\Gamma_K^\bullet(X_{f,K}))^H \cong \text{holim}_\Delta \text{colim}_{N \triangleleft_o K}(\Gamma_K^\bullet(X_{f,K}))^{NH}.$$

Also, there is a canonical K/H -equivariant map

$$\Psi(X)_H^K: \text{colim}_{N \triangleleft_o K} X^{hNH} \rightarrow X^{hH}$$

that is defined to be

$$\text{colim}_{N \triangleleft_o K} X^{hNH} \cong \text{colim}_{N \triangleleft_o K} \text{holim}_\Delta(\Gamma_K^\bullet(X_{f,K}))^{NH} \rightarrow \text{holim}_\Delta \text{colim}_{N \triangleleft_o K}(\Gamma_K^\bullet(X_{f,K}))^{NH}.$$

Notice that the canonical map $K/H \rightarrow K/(NH)$ makes the source of $\Psi(X)_H^K$ a discrete K/H -spectrum, since $\text{holim}_\Delta(\Gamma_K^\bullet(X_{f,K}))^{NH}$ is a $K/(NH)$ -spectrum and $K/(NH)$ is a finite group.

If $\Psi(X)_H^K$ is a weak equivalence, then X^{hH} can be identified with the discrete K/H -spectrum $\text{colim}_{N \triangleleft_o K} \text{holim}_\Delta(\Gamma_K^\bullet(X_{f,K}))^{NH}$, and, thus, in this case, $(X^{hH})^{hK/H}$ is defined (as $(\text{colim}_{N \triangleleft_o K} \text{holim}_\Delta(\Gamma_K^\bullet(X_{f,K}))^{NH})^{hK/H}$). However, once again, $\Psi(X)_H^K$ does not have to be a weak equivalence: if $\Psi(Y)_H^{G'}$ were a weak equivalence, then $\pi_0(Y^{hH'})$ must be a discrete G'/H' -module, and this is false.

Another way to express the failure of $\Psi(X)_H^K$ to always be a weak equivalence is by saying that the above holim and filtered colimit do not have to commute. However, there are conditions that guarantee that the holim and colimit do commute (up to weak equivalence). Let J be any closed subgroup of K . Then there is a homotopy spectral sequence $E_r^{*,*}(J)$ that has the form

$$E_2^{s,t}(J) = H_c^s(J; \pi_t(X)) \Rightarrow \pi_{t-s}(\text{holim}_\Delta(\Gamma_K^\bullet(X_{f,K}))^J)$$

(see the proof of [2, Lemma 7.12]). Notice that there is a map of spectral sequences

$$\text{colim}_{N \triangleleft_o K} E_r^{*,*}(NH) \rightarrow E_r^{*,*}(H),$$

such that

$$\text{colim}_{N \triangleleft_o K} E_2^{s,t}(NH) = \text{colim}_{N \triangleleft_o K} H_c^s(NH; \pi_t(X)) \cong H_c^s(H; \pi_t(X)) = E_2^{s,t}(H).$$

By [19, Proposition 3.3], if

- (i) there exists a fixed p such that $H_c^s(NH; \pi_t(X)) = 0$ for all $s > p$, all $t \in \mathbb{Z}$, and all $N \triangleleft_o K$; or
- (ii) there exists a fixed q such that $H_c^s(NH; \pi_t(X)) = 0$ for all $t > q$, all $s \geq 0$, and all N ,

then the colimit of spectral sequences has abutment equal to the colimit of the abutments, so that $\Psi(X)_H^K$ is a weak equivalence, and, hence, X^{hH} can, as above, be regarded as a discrete K/H -spectrum.

The two conditions above imply that X^{hH} can be identified with the discrete K/H -spectrum $\operatorname{colim}_{N \triangleleft_o K} X^{hNH}$ if, for example, one of the following conditions is satisfied:

- K has finite cohomological dimension (that is, there exists some p such that $H_c^s(K; M) = 0$ for all $s > p$ and all discrete K -modules M), so that there is a uniform bound on the cohomological dimension of all the NH ; or
- there exists some q such that $\pi_t(X) = 0$ for all $t > q$.

We conclude that when K has finite vcd, unless certain additional hypotheses are in place, it is not known how to show that X^{hH} is a continuous K/H -spectrum, so that, in such situations, it is not known how to form the iterated homotopy fixed point spectrum $(X^{hH})^{hK/H}$, with respect to the natural K/H -action.

The above discussion used the hypercohomology spectra $\operatorname{holim}_\Delta(\Gamma_K^\bullet(X_{f,K}))^J$, for various J closed in K . But suppose we are in a situation where these are not available to us. (If K has finite vcd, these are always available. If K does not have finite vcd, there are other conditions that permit the use of the hypercohomology spectra: these conditions are similar to those listed above and can be deduced from considering the proofs of Theorem 7.4 and Lemma 7.12 in [2].) Further, suppose that $X_{f,K}$ is not fibrant in Spt_H and that $(X_{f,K})^H \not\cong (X_{f,H})^H$, so that the discrete K/H -spectrum $(X_{f,K})^H = \operatorname{colim}_{N \triangleleft_o K} (X_{f,K})^{NH} = \operatorname{colim}_{N \triangleleft_o K} X^{hNH}$ fails to be a model for X^{hH} .

Under these assumptions, the only model for X^{hH} that is available to us is the definition $(X_{f,H})^H$, and, in this case, the abstract $X_{f,H}$ is not known to carry a K -action, so that $(X_{f,H})^H$ is not known to be a K/H -spectrum (with non-trivial K/H -action). In such a situation, apart from assigning X^{hH} the trivial K/H -action (which typically is not a natural thing to do), it is not known how to consider its K/H -homotopy fixed points.

4. HYPERFIBRANT AND TOTALLY HYPERFIBRANT DISCRETE G -SPECTRA

In this section, we consider a situation in which the iterated homotopy fixed point spectrum is always defined. As before, throughout this section, we let H and K be closed subgroups of the profinite group G , with H normal in K .

It is easy to see that the map $\psi: \operatorname{colim}_N X^{hNK} \rightarrow X^{hK}$ of (3.5) is a weak equivalence for all K open in G , since $X_{f,G}$ is always fibrant in Spt_K , for such K . If $X_{f,G}$ is fibrant in Spt_K for all closed K , then ψ is a weak equivalence for all closed K , but, in the Appendix, Wieland and the author prove that this does not always happen. This motivates the following definition.

Definition 4.1. Let X be a discrete G -spectrum. If ψ is a weak equivalence for all K closed in G , then X is a *hyperfibrant* discrete G -spectrum.

As alluded to above, there do exist profinite groups G for which there is a discrete G -spectrum that is not a hyperfibrant discrete G -spectrum: the map $\psi(\bigvee_{n \geq 0} \Sigma^n H(\mathbb{Z}/p[\mathbb{Z}/q^n]))_{\mathbb{Z}/p}^{\mathbb{Z}/p \times \mathbb{Z}_q}$, considered in the Appendix, is not a weak equivalence.

For the example below and the paragraph after it, we assume that G has finite vcd.

Example 4.2. Let X be any spectrum (with no G -action), so that $\operatorname{Map}_c(G, X)$, as defined in Section 2, is a discrete G -spectrum. The descent spectral sequence for

$\mathrm{Map}_c(G, X)^{hK}$ has the form

$$H_c^s(K; \mathrm{Map}_c(G, \pi_t(X))) \Rightarrow \pi_{t-s}(\mathrm{Map}_c(G, X)^{hK}).$$

Since $H_c^s(K; \mathrm{Map}_c(G, \pi_t(X))) = 0$, for $s > 0$,

$$\begin{aligned} \pi_*(\mathrm{Map}_c(G, X)^{hK}) &\cong H_c^0(K; \mathrm{Map}_c(G, \pi_*(X))) = \mathrm{Map}_c(G, \pi_*(X))^K \\ &\cong \mathrm{Map}_c(G/K, \pi_*(X)) \cong \pi_*(\mathrm{Map}_c(G, X)^K), \end{aligned}$$

and thus, $\mathrm{Map}_c(G, X)^{hK} \simeq \mathrm{Map}_c(G, X)^K$, for any K closed in G . Therefore, we have:

$$\begin{aligned} \mathrm{Map}_c(G, X)^{hK} &\simeq \mathrm{Map}_c(G, X)^K \cong \mathrm{colim}_N \mathrm{Map}_c(G, X)^{NK} \\ &\simeq \mathrm{colim}_N \mathrm{Map}_c(G, X)^{hNK}. \end{aligned}$$

Thus, for any spectrum X , $\mathrm{Map}_c(G, X)$ is a hyperfibrant discrete G -spectrum.

For any discrete G -spectrum X ,

$$X^{hK} \cong \mathrm{holim}_\Delta \mathrm{colim}_N (\Gamma_G^\bullet(X_{f,G}))^{NK},$$

and similarly,

$$\mathrm{colim}_N X^{hNK} = \mathrm{colim}_N \mathrm{holim}_\Delta (\Gamma_G^\bullet(X_{f,G}))^{NK}.$$

Thus, if ψ is a weak equivalence, for some K , then the canonical map (by a slight abuse of notation, we also call this map ψ)

$$(4.3) \quad \psi: \mathrm{colim}_N \mathrm{holim}_\Delta (\Gamma_G^\bullet(X_{f,G}))^{NK} \rightarrow \mathrm{holim}_\Delta \mathrm{colim}_N (\Gamma_G^\bullet(X_{f,G}))^{NK}$$

is a weak equivalence. Hence, if X is hyperfibrant, then, for all K , the colimit and the holim involved in defining ψ commute.

Lemma 4.4. *If X is a discrete G -spectrum and K is normal in L , where L is a closed subgroup of G , then $\mathrm{colim}_N X^{hNK}$ is a discrete L/K -spectrum.*

Proof. Note that the spectrum $X^{hNK} = (X_{f,G})^{NK}$ is an NL/NK -spectrum, where NL/NK is a finite group. The continuous map $L/K \rightarrow NL/NK$, given by $lK \mapsto lNK$, makes $(X_{f,G})^{NK}$ a discrete L/K -spectrum. Since the forgetful functor $\mathrm{Spt}_{L/K} \rightarrow \mathrm{Spt}$ is a left adjoint, colimits in $\mathrm{Spt}_{L/K}$ are formed in the category of spectra, so that $\mathrm{colim}_N (X_{f,G})^{NK}$ is a discrete L/K -spectrum. \square

Given a hyperfibrant discrete G -spectrum, there is a natural way to define the K/H -homotopy fixed points of X^{hH} .

Definition 4.5. Let X be a hyperfibrant discrete G -spectrum, so that the map $\psi(X)_H^G: \mathrm{colim}_N X^{hNH} \rightarrow X^{hH}$ is a weak equivalence. Notice that the source of this map, $\mathrm{colim}_N X^{hNH}$, is a discrete K/H -spectrum, and the closely related K/H -equivariant map ψ of (4.3), also from $\mathrm{colim}_N X^{hNH}$ to X^{hH} , is a weak equivalence. Thus, we can identify X^{hH} with the discrete K/H -spectrum $\mathrm{colim}_N X^{hNH}$, and, hence, the K/H -homotopy fixed points of X^{hH} are given by

$$(X^{hH})^{hK/H} = (\mathrm{colim}_N X^{hNH})^{hK/H}.$$

Let X be any discrete K -spectrum. As in the proof of Lemma 4.4, $\operatorname{colim}_N X^{NH}$, where the colimit is over all open normal subgroups of K , is a discrete K/H -spectrum, so that the isomorphism $X^H \cong \operatorname{colim}_N X^{NH}$ implies that X^H is also a discrete K/H -spectrum. Hence, there is a functor $(-)^H : \operatorname{Spt}_K \rightarrow \operatorname{Spt}_{K/H}$.

The author thanks Mark Behrens for help with the next lemma, which is basically a version of [18, Lemma 6.35], which is for simplicial presheaves.

Lemma 4.6. *The functor $(-)^H : \operatorname{Spt}_K \rightarrow \operatorname{Spt}_{K/H}$ preserves fibrant objects.*

Proof. It is easy to see that the functor $(-)^H$ has a left adjoint $t : \operatorname{Spt}_{K/H} \rightarrow \operatorname{Spt}_K$ that sends a discrete K/H -spectrum X to X , where now X is regarded as a discrete K -spectrum via the canonical map $K \rightarrow K/H$. It suffices to show that t preserves all weak equivalences and cofibrations, since this implies that $(-)^H$ preserves all fibrations.

If f is a weak equivalence (cofibration) in $\operatorname{Spt}_{K/H}$, then f is a K/H -equivariant map that is a weak equivalence (cofibration) of spectra. Since, as a map of spectra, $t(f) = f$, $t(f)$ is a weak equivalence (cofibration) of spectra. Because $t(f)$ is also K -equivariant, it is a weak equivalence (cofibration) in Spt_K . \square

This lemma implies the following result, giving another useful property of hyperfibrant discrete G -spectra.

Lemma 4.7. *If X is a hyperfibrant discrete G -spectrum and K is any closed normal subgroup of G , then $(X^{hK})^{hG/K} \simeq X^{hG}$.*

Proof. This result follows immediately from Lemma 4.9 below, by letting $K = G$ and $H = K$. \square

Now let X be a hyperfibrant discrete G -spectrum and suppose that the closed subgroup K is a proper subgroup of G . Let us consider if there is an equivalence $(X^{hH})^{hK/H} \simeq X^{hK}$. Notice that

$$(X^{hH})^{hK/H} = (\operatorname{colim}_N (X_{f,G})^{NH})^{hK/H} = (((X_{f,G})^H)_{f,K/H})^{K/H}$$

and

$$X^{hK} \xleftarrow{\simeq} \operatorname{colim}_N (X_{f,G})^{NK} = (X_{f,G})^K = ((X_{f,G})^H)^{K/H}.$$

Thus, $(X^{hH})^{hK/H} \simeq X^{hK}$ if the canonical map

$$(4.8) \quad ((X_{f,G})^H)^{K/H} \rightarrow (((X_{f,G})^H)_{f,K/H})^{K/H}$$

is a weak equivalence.

If $(X_{f,G})^H$ were fibrant in $\operatorname{Spt}_{K/H}$, then, in $\operatorname{Spt}_{K/H}$, the map

$$(X_{f,G})^H \rightarrow ((X_{f,G})^H)_{f,K/H}$$

is a weak equivalence between fibrant objects, so that the map in (4.8) is a weak equivalence (by [2, Corollary 3.9]). Thus, if $X_{f,G}$ is fibrant in Spt_K , then the preceding observation and Lemma 4.6 imply that $(X^{hH})^{hK/H} \simeq X^{hK}$. But, as we saw with $Y_{f,G'}$ not being fibrant in $\operatorname{Spt}_{H'}$, the forgetful functor $\operatorname{Spt}_G \rightarrow \operatorname{Spt}_K$ does not necessarily preserve fibrant objects. Also, we do not know of any general argument, for K not open in G , that would show that $(X_{f,G})^H$ is fibrant in $\operatorname{Spt}_{K/H}$. In conclusion, though we do not have an example of $(X^{hH})^{hK/H} \simeq X^{hK}$ failing to hold when X is a hyperfibrant discrete G -spectrum, we also do not know of any

general argument that shows that, when X is hyperfibrant, this equivalence must always hold. This leads us to the following considerations.

Lemma 4.9. *Let X be a discrete G -spectrum and let K be a closed subgroup of G . If X is a hyperfibrant discrete K -spectrum, then there is a weak equivalence $(X^{hH})^{hK/H} \simeq X^{hK}$, where H is any closed subgroup of G that is also normal in K .*

Proof. Since X is hyperfibrant in Spt_K , $\psi: \text{colim}_N X^{hNH} \rightarrow X^{hH}$ is a weak equivalence, where the colimit is over all open normal subgroups of K . Thus, by Definition 4.5,

$$(X^{hH})^{hK/H} = (\text{colim}_N X^{hNH})^{hK/H} = (\text{colim}_N (X_{f,K})^{NH})^{hK/H} \simeq ((X_{f,K})^H)^{hK/H}.$$

Since $X_{f,K}$ is fibrant in Spt_K , Lemma 4.6 implies that $(X_{f,K})^H$ is fibrant in $\text{Spt}_{K/H}$. Also, the identity map $(X_{f,K})^H \rightarrow (X_{f,K})^H$ is a trivial cofibration in $\text{Spt}_{K/H}$. Thus, we can let $(X_{f,K})^H = ((X_{f,K})^H)_{f,K/H}$, and hence,

$$((X_{f,K})^H)^{hK/H} = (((X_{f,K})^H)_{f,K/H})^{K/H} = ((X_{f,K})^H)^{K/H} = (X_{f,K})^K = X^{hK}.$$

□

This lemma implies that if the discrete G -spectrum X is hyperfibrant as a discrete K -spectrum, for all K closed in G , then not only is the iterated homotopy fixed point spectrum always defined, but (1.1) is always true. Thus, we make the following definition.

Definition 4.10. Let X be a discrete G -spectrum. If X is a hyperfibrant discrete K -spectrum, for all K closed in G , then X is a *totally hyperfibrant* discrete G -spectrum. Therefore, if X is a totally hyperfibrant discrete G -spectrum, then $(X^{hH})^{hK/H} \simeq X^{hK}$, for any H and K closed in G , with H normal in K .

Example 4.11. For any spectrum X , $\text{Map}_c(G, X)$ is a totally hyperfibrant discrete G -spectrum. To see this, let K be any closed subgroup of G , and let L be any closed subgroup of K . Since L is closed in G , $\text{Map}_c(G, X)^{hL}$ is defined. Also, if N is an open normal subgroup of K , then NL is closed in K , since K is a profinite group and NL is an open subgroup of K . Thus, NL is a closed subgroup of G , so that $\text{Map}_c(G, X)^{hNL}$ is defined. Then, by Example 4.2 and by taking the colimit over all open normal subgroups N of K , we have:

$$\text{colim}_N \text{Map}_c(G, X)^{hNL} \simeq \text{colim}_N \text{Map}_c(G, X)^{NL} \cong \text{Map}_c(G, X)^L \simeq \text{Map}_c(G, X)^{hL}.$$

5. ITERATED HOMOTOPY FIXED POINTS FOR E_n

In this section, we show that it is always possible to construct the spectrum $(E_n^{dhH})^{hK/H}$. As mentioned in the Introduction, applying (1.2) allows us to identify E_n^{dhH} with E_n^{hH} , yielding the iterated homotopy fixed point spectrum $(E_n^{hH})^{hK/H}$. We begin with some notation.

We use $(-)_f$ to denote functorial fibrant replacement in the category of spectra. Let $E(n)$ be the Johnson-Wilson spectrum, with $E(n)_* = \mathbb{Z}_{(p)}[v_1, \dots, v_{n-1}][v_n^{\pm 1}]$, where the degree of each v_i is $2(p^i - 1)$. Let

$$M_0 \leftarrow M_1 \leftarrow M_2 \leftarrow \dots$$

be a tower of generalized Moore spectra such that $\widehat{L}(S^0) \simeq \text{holim}_i(L_n M_i)_{\mathfrak{f}}$, where L_n is Bousfield localization with respect to $E(n)$. This tower of finite type n spectra exists by [16, Proposition 4.22]. Let $\{U_j\}_{j \geq 0}$ be a descending chain of open normal subgroups of G_n , with $\bigcap_j U_j = \{e\}$ (see [8, (1.4)]), so that $G_n \cong \lim_j G_n/U_j$. Also, let H and K be closed subgroups of G_n , with H normal in K .

We recall the following useful result from [15, Section 2].

Lemma 5.1. *If Z is an $E(n)$ -local spectrum, then there is an equivalence*

$$\widehat{L}Z \simeq \text{holim}_i(Z \wedge M_i)_{\mathfrak{f}}.$$

By [8, Definition 1.5], $E_n^{dhH} \simeq \widehat{L}(\text{colim}_j E_n^{dhU_j H})$, and, by [16, Lemma 7.2], there is an equivalence $\widehat{L}(\text{colim}_j E_n^{dhU_j H}) \wedge M_i \simeq L_n(\text{colim}_j E_n^{dhU_j H}) \wedge M_i$. Thus, since each $E_n^{dhU_j H}$ is $E(n)$ -local, we immediately obtain the following result.

Lemma 5.2. *For each i , $E_n^{dhH} \wedge M_i \simeq \text{colim}_j(E_n^{dhU_j H} \wedge M_i)$.*

The following definition, which we recall from [2], defines a useful spectrum.

Definition 5.3. Let $F_n = \text{colim}_j E_n^{dhU_j}$, a discrete G_n -spectrum.

Remark 5.4. Using [2, Corollary 9.8] and (1.2), Lemma 5.2 implies that

$$\begin{aligned} (F_n \wedge M_i)^{hH} &\simeq E_n^{dhH} \wedge M_i \simeq \text{colim}_j(E_n^{dhU_j H} \wedge M_i) \\ &\simeq \text{colim}_j(F_n \wedge M_i)^{hU_j H} \simeq \text{colim}_N(F_n \wedge M_i)^{hNH}, \end{aligned}$$

where the last colimit is over all open normal subgroups of G_n . Therefore, since Lemma 5.2 is valid if H is replaced with any closed subgroup of G_n , $F_n \wedge M_i$ is a hyperfibrant discrete G_n -spectrum.

For the rest of this section, we let X be any spectrum, with no K/H -action.

Lemma 5.5. *The spectrum $\text{colim}_j(E_n^{dhU_j H} \wedge M_i \wedge X)_{\mathfrak{f}}$ is a discrete K/H -spectrum.*

Proof. Note that $E_n^{dhU_j H}$ is a $U_j K/U_j H$ -spectrum. Thus, $E_n^{dhU_j H} \wedge M_i \wedge X$ is a $U_j K/U_j H$ -spectrum. By functoriality, $(E_n^{dhU_j H} \wedge M_i \wedge X)_{\mathfrak{f}}$ is also a $U_j K/U_j H$ -spectrum. The argument is completed as in the proof of Lemma 4.4. \square

Corollary 5.6. *The spectrum*

$$\widehat{L}(E_n^{dhH} \wedge X) \simeq \text{holim}_i(\text{colim}_j(E_n^{dhU_j H} \wedge M_i \wedge X)_{\mathfrak{f}})$$

is a continuous K/H -spectrum. In particular,

$$E_n^{dhH} \simeq \text{holim}_i(\text{colim}_j(E_n^{dhU_j H} \wedge M_i)_{\mathfrak{f}})$$

is a continuous K/H -spectrum.

Proof. By Lemma 5.1,

$$\begin{aligned} \widehat{L}(E_n^{dhH} \wedge X) &\simeq \text{holim}_i(E_n^{dhH} \wedge M_i \wedge X)_{\mathfrak{f}} \\ &\simeq \text{holim}_i(\text{colim}_j(E_n^{dhU_j H} \wedge M_i \wedge X)_{\mathfrak{f}}). \end{aligned}$$

Since $(E_n^{dhU_j H} \wedge M_i \wedge X)_{\mathfrak{f}}$ is fibrant, so is $\text{colim}_j(E_n^{dhU_j H} \wedge M_i \wedge X)_{\mathfrak{f}}$, so that the homotopy limit is of a diagram of fibrant spectra, as required. \square

Remark 5.7. We can just as well phrase Corollary 5.6 as saying that

$$\widehat{L}(E_n^{dhH} \wedge X) \simeq \operatorname{holim}_i ((\operatorname{colim}_j E_n^{dhU_jH}) \wedge M_i \wedge X)_{f,K/H}.$$

In particular, note that $E_n^{dhH} \simeq \operatorname{holim}_i ((\operatorname{colim}_j E_n^{dhU_jH}) \wedge M_i)_{f,K/H}$. Now recall that $E_n \simeq \operatorname{holim}_i (F_n \wedge M_i)_{f,G_n}$. Thus, $\operatorname{colim}_j E_n^{dhU_jH}$ is playing the same role as F_n and is the analogue of F_n .

Note that, since $\widehat{L}(E_n^{dhH} \wedge X)$ is a continuous K/H -spectrum (that is, we identify this spectrum with $\operatorname{holim}_i (\operatorname{colim}_j (E_n^{dhU_jH} \wedge M_i \wedge X)_{\mathfrak{f}})$), Definition 2.5 gives

$$(\widehat{L}(E_n^{dhH} \wedge X))^{hK/H} = \operatorname{holim}_i ((\operatorname{colim}_j (E_n^{dhU_jH} \wedge M_i \wedge X)_{\mathfrak{f}})^{hK/H}).$$

6. THE DESCENT SPECTRAL SEQUENCE FOR $\widehat{L}(E_n^{dhH} \wedge X)$

Let H and K be closed subgroups of G_n , with H normal in K . By [9, Theorem 9.6], if G is a p -adic analytic group, U a closed subgroup of G , and N a closed normal subgroup of G , then U and G/N , with the quotient topology, are p -adic analytic groups. Thus, we obtain the following useful result.

Lemma 6.1. *The profinite group K/H is a compact p -adic analytic group and $\operatorname{vcd}(K/H) < \infty$.*

Since, by Corollary 5.6, $\widehat{L}(E_n^{dhH} \wedge X)$ is a continuous K/H -spectrum, for any spectrum X with trivial K/H -action, Theorem 2.7 gives a descent spectral sequence

$$(6.2) \quad E_2^{s,t} \Rightarrow \pi_{t-s}((\widehat{L}(E_n^{dhH} \wedge X))^{hK/H}),$$

where

$$E_2^{s,t} = \pi^s \pi_t(\operatorname{holim}_i (\Gamma_{K/H}^\bullet(\operatorname{colim}_j (E_n^{dhU_jH} \wedge M_i \wedge X)_{\mathfrak{f}}))^{K/H}).$$

Let $H_{\text{cts}}^s(G; M)$ denote the continuous cohomology of continuous cochains of G , with coefficients in the topological G -module M (see [22, Section 2]). Given a tower $\{M_i\}$ of discrete G -modules, [17, Theorem 2.2] implies that, if the tower of abelian groups $\{M_i\}$ satisfies the Mittag-Leffler condition, then

$$H_{\text{cont}}^s(G; \{M_i\}) \cong H_{\text{cts}}^s(G; \lim_i M_i),$$

for all $s \geq 0$. Therefore, if the tower of abelian groups $\{\pi_t(E_n^{dhH} \wedge M_i \wedge X)\}_i$ satisfies the Mittag-Leffler condition for each $t \in \mathbb{Z}$, then

$$E_2^{s,t} \cong H_{\text{cont}}^s(K/H; \{\pi_t(\widehat{L}(E_n^{dhH} \wedge M_i \wedge X))\}) \cong H_{\text{cts}}^s(K/H; \pi_t(\widehat{L}(E_n^{dhH} \wedge X))).$$

Now let X be a finite spectrum. Then $\widehat{L}(E_n^{dhH} \wedge X) \simeq E_n^{dhH} \wedge X$ and the hypotheses of the preceding sentence are satisfied, so that

$$E_2^{s,t} \cong H_{\text{cts}}^s(K/H; \pi_t(E_n^{dhH} \wedge X)).$$

Additionally, $\pi_t(E_n^{dhH} \wedge X) \cong \lim_i \pi_t(E_n^{dhH} \wedge M_i \wedge X)$ is a profinite $\mathbb{Z}_p[[K/H]]$ -module, and, in this case, the continuous cohomology has more structure. In general, if M is a profinite $\mathbb{Z}_p[[K/H]]$ -module with $M = \lim_\alpha M_\alpha$, where each M_α is a finite discrete $\mathbb{Z}_p[[K/H]]$ -module, then

$$H_{\text{cts}}^s(K/H; M) \cong \lim_\alpha H_c^s(K/H; M_\alpha) =: H_c^s(K/H; M),$$

for all $s \geq 0$, where the rightmost term can also be defined as the Ext group $\mathbf{Ext}_{\mathbb{Z}_p[[K/H]]}^s(\mathbb{Z}_p, M)$ (see [5, pg. 137] and [21, (3.7.10)] for more detail). Putting the above facts together, we obtain that spectral sequence (6.2) has the form

$$(6.3) \quad H_c^s(K/H; \pi_t(E_n^{dhH} \wedge X)) \Rightarrow \pi_{t-s}((E_n^{dhH} \wedge X)^{hK/H}).$$

Recall from [5] that there is a Lyndon-Hochschild-Serre spectral sequence

$$(6.4) \quad H_c^s(K/H; \pi_t(E_n^{dhH} \wedge X)) \Rightarrow \pi_{t-s}(E_n^{dhK} \wedge X),$$

where X is a finite spectrum. The fact that spectral sequences (6.3) and (6.4) have identical E_2 -terms suggests that there is an equivalence $(E_n^{dhH})^{hK/H} \simeq E_n^{dhK}$. In the next result, we use descent spectral sequence (6.2), when $X = E_n$, to show that this equivalence holds after smashing with E_n , before taking K/H -homotopy fixed points. First of all, we define the relevant map.

Let

$$C_i = \operatorname{colim}_j (E_n^{dhU_jH} \wedge M_i \wedge E_n)_{\mathfrak{f}}.$$

Then there is a map

$$\theta: \widehat{L}(E_n^{dhK} \wedge E_n) \rightarrow (\widehat{L}(E_n^{dhH} \wedge E_n))^{hK/H},$$

which is defined by making the identifications

$$\widehat{L}(E_n^{dhK} \wedge E_n) = \operatorname{holim}_i (\operatorname{colim}_j (E_n^{dhU_jK} \wedge M_i \wedge E_n)_{\mathfrak{f}})$$

and

$$(\widehat{L}(E_n^{dhH} \wedge E_n))^{hK/H} = \operatorname{holim}_i (((C_i)_{\mathfrak{f}, K/H})^{K/H}),$$

and by taking the composite of the canonical map

$$\operatorname{holim}_i (\operatorname{colim}_j (E_n^{dhU_jK} \wedge M_i \wedge E_n)_{\mathfrak{f}}) \rightarrow \lim_{K/H} (\operatorname{holim}_i C_i),$$

the map

$$\lim_{K/H} (\operatorname{holim}_i C_i) \xrightarrow{\cong} \operatorname{holim}_i \lim_{K/H} C_i \xrightarrow{\cong} \operatorname{holim}_i ((C_i)^{K/H}),$$

and the map

$$\operatorname{holim}_i ((C_i)^{K/H}) \rightarrow \operatorname{holim}_i (((C_i)_{\mathfrak{f}, K/H})^{K/H}).$$

To avoid confusion, we point out that, in the expression $(\widehat{L}(E_n^{dhH} \wedge E_n))^{hK/H}$, the lone E_n has trivial K/H -action, as stated just after Lemma 6.1.

Theorem 6.5. *The map*

$$(6.6) \quad \theta: \widehat{L}(E_n^{dhK} \wedge E_n) \xrightarrow{\simeq} (\widehat{L}(E_n^{dhH} \wedge E_n))^{hK/H}$$

is a weak equivalence.

Proof. By [8, Proposition 6.3],

$$\pi_t(E_n^{dhH} \wedge E_n \wedge M_i) \cong \operatorname{Map}_c(G_n/H, \pi_t(E_n \wedge M_i)).$$

Thus, since $\{\pi_t(E_n \wedge M_i)\}$ satisfies the Mittag-Leffler condition for every integer t , the tower $\{\pi_t(E_n^{dhH} \wedge E_n \wedge M_i)\}$ does too, because $\operatorname{Map}_c(G_n/H, -): \mathbf{Ab} \rightarrow \mathbf{Ab}$, where \mathbf{Ab} is the category of abelian groups, is an exact and additive functor. Therefore, descent spectral sequence (6.2) takes the form

$$E_2^{s,t} = H_{\text{cts}}^s(K/H; \operatorname{Map}_c(G_n/H, \pi_t(E_n))) \Rightarrow \pi_{t-s}((\widehat{L}(E_n^{dhH} \wedge E_n))^{hK/H}).$$

Since the continuous epimorphism $\pi: G_n/H \rightarrow G_n/K$ has a continuous section $\sigma: G_n/K \rightarrow G_n/H$, such that $\sigma(eK) = eH$, there is a homeomorphism

$$h: K/H \times G_n/K \rightarrow G_n/H, \quad (kH, gK) \mapsto kH \cdot \sigma(gK)$$

(see [5, proof of Lemma 3.15]). Notice that h is K/H -equivariant, with K/H acting on the source by acting only on K/H , since

$$k'H \cdot (kH, gK) = (k'kH, gK) \mapsto k'kH \cdot \sigma(gK) = k'H \cdot (kH \cdot \sigma(gK)).$$

Thus,

$$\mathrm{Map}_c(G_n/H, \pi_t(E_n)) \cong \mathrm{Map}_c(K/H, \mathrm{Map}_c(G_n/K, \pi_t(E_n)))$$

is an isomorphism of topological K/H -modules. Therefore,

$$E_2^{s,t} \cong \lim_i H_c^s(K/H; \mathrm{Map}_c(K/H, \mathrm{Map}_c(G_n/K, \pi_t(E_n \wedge M_i))))$$

which is 0, for $s > 0$, and equals $\mathrm{Map}_c(G_n/K, \pi_t(E_n))$, when $s = 0$. Thus,

$$\pi_*((\widehat{L}(E_n^{dhH} \wedge E_n))^{hK/H}) \cong \mathrm{Map}_c(G_n/K, \pi_*(E_n)) \cong \pi_*(\widehat{L}(E_n^{dhK} \wedge E_n)).$$

□

Remark 6.7. In Theorem 7.3, by using (1.2), we will show that $(E_n^{hH})^{hK/H} \simeq E_n^{hK}$. Therefore, using (1.2) again, (6.6) implies that taking K/H -homotopy fixed points commutes with smashing with E_n :

$$(\widehat{L}(E_n^{hH} \wedge E_n))^{hK/H} \simeq \widehat{L}(E_n^{hK} \wedge E_n).$$

This is interesting because such a commutation need not hold in general, unless one is smashing with a finite spectrum. However, (6.6) is not surprising, because it is known that, for all H ,

$$\widehat{L}(E_n^{dhH} \wedge E_n) \simeq (\widehat{L}(E_n \wedge E_n))^{hH},$$

where on the right-hand side, the second E_n has the trivial H -action. This last equivalence follows from the fact that

$$\pi_*(\widehat{L}(E_n^{dhH} \wedge E_n)) \cong \mathrm{Map}_c(G_n, E_{n*})^H \cong \pi_*((\widehat{L}(E_n \wedge E_n))^{hH}),$$

where the second isomorphism is obtained from the descent spectral sequence for $(\widehat{L}(E_n \wedge E_n))^{hH}$ and the fact that $\pi_*(\widehat{L}(E_n \wedge E_n)) \cong \mathrm{Map}_c(G_n, E_{n*})$ (see the last paragraph of [2]).

7. A PROOF OF (1.1), IN THE CASE OF E_n , AND SOME CONSEQUENCES

In [4] (see [1, Theorem 8.2.1] for a more efficient proof), the author showed that, as stated in (1.2), E_n^{hK} can be identified with E_n^{dhK} , for all closed subgroups K of G_n . In this section, we use this result to show that (1.1) holds when X equals E_n .

Lemma 7.1. *Let K be any closed subgroup of G_n and let L be any closed subgroup of K . Then the canonical map $\mathrm{colim}_N(E_n^{dhNL} \wedge M_i) \rightarrow E_n^{dhL} \wedge M_i$, where the colimit is over all open normal subgroups N of K , is a weak equivalence.*

Proof. Both L and NL are closed subgroups of G_n , so that E_n^{dhL} and E_n^{dhNL} are defined. Since both the source and target of the map are E_n -local, it suffices to show that the associated map

$$\mathrm{colim}_N(E_n^{dhNL} \wedge E_n \wedge M_i) \rightarrow E_n^{dhL} \wedge E_n \wedge M_i$$

is a weak equivalence. For any integer t , by using [8, Proposition 6.3] and the fact that $\text{Map}_c(G_n, \pi_t(E_n \wedge M_i))$ is a discrete K -module (since it is a discrete G_n -module), we have:

$$\begin{aligned} \pi_t(\text{colim}_N(E_n^{dhNL} \wedge E_n \wedge M_i)) &\cong \text{colim}_N \text{Map}_c(G_n, \pi_t(E_n \wedge M_i))^{NL} \\ &\cong \text{Map}_c(G_n, \pi_t(E_n \wedge M_i))^L \\ &\cong \pi_t(E_n^{dhL} \wedge E_n \wedge M_i), \end{aligned}$$

completing the proof. \square

Corollary 7.2. *For each i , the spectrum $F_n \wedge M_i$ is a totally hyperfibrant discrete G_n -spectrum.*

Proof. Let K be any closed subgroup of G_n and let L be any closed subgroup of K . Then, by [2, Corollary 9.8] and Lemma 7.1,

$$\begin{aligned} (F_n \wedge M_i)^{hL} &\simeq E_n^{hL} \wedge M_i \simeq E_n^{dhL} \wedge M_i \simeq \text{colim}_N(E_n^{dhNL} \wedge M_i) \\ &\simeq \text{colim}_N(E_n^{hNL} \wedge M_i) \simeq \text{colim}_N(F_n \wedge M_i)^{hNL}, \end{aligned}$$

where the colimit is over all open normal subgroups N of K . \square

Now let H be any closed subgroup of G_n that is also normal in K . Since $E_n^{hH} \simeq E_n^{dhH}$, we identify these two spectra, so that the construction of $(E_n^{dhH})^{hK/H}$ automatically yields the iterated homotopy fixed point spectrum $(E_n^{hH})^{hK/H}$.

Theorem 7.3. *For any H and K , there is an equivalence*

$$(E_n^{hH})^{hK/H} \simeq E_n^{hK}.$$

Proof. By applying Lemma 4.9 to Corollary 7.2, we have that

$$((F_n \wedge M_i)^{hH})^{hK/H} \simeq (F_n \wedge M_i)^{hK}.$$

Thus, we obtain:

$$\begin{aligned} (E_n^{hH})^{hK/H} &= (\text{holim}_i (F_n \wedge M_i)^{hH})^{hK/H} = \text{holim}_i ((F_n \wedge M_i)^{hH})^{hK/H} \\ &\simeq \text{holim}_i (F_n \wedge M_i)^{hK} = E_n^{hK}. \end{aligned}$$

\square

Let X be any finite spectrum and continue to let H and K be closed subgroups of G_n , with $H \triangleleft K$. Also, recall from Lemma 6.1 that, for any closed subgroups J and L in G_n , with $J \triangleleft L$, the profinite group L/J has finite vcd. Then, by [2, Remark 7.16], L/J -homotopy fixed points commute with smashing with X . We use the equivalence $E_n^{hJ} \simeq E_n^{dhJ}$ as needed. Thus, (6.3) gives a descent spectral sequence that has the form

$$(7.4) \quad H_c^s(K/H; \pi_t(E_n^{hH} \wedge X)) \Rightarrow \pi_{t-s}((E_n^{hH})^{hK/H} \wedge X).$$

By [5, (0.1)] (see (6.4)), there is a strongly convergent Adams-type spectral sequence that has the form

$$(7.5) \quad H_c^s(K/H; \pi_t(E_n^{hH} \wedge X)) \Rightarrow \pi_{t-s}(E_n^{hK} \wedge X).$$

The following result shows that these two spectral sequences are isomorphic to each other.

Theorem 7.6. *The descent spectral sequence of (7.4) is isomorphic to the strongly convergent spectral sequence of (7.5) from the E_2 -terms onward.*

Proof. By Theorem 7.3, the abutment of (7.4) can be written as $\pi_{t-s}(E_n^{hK} \wedge X)$. Then, it is not hard to see that spectral sequence (7.4) is an inverse limit over $i \geq 0$ of conditionally convergent descent spectral sequences $E_r^{*,*}(H, K, i)$ that have the form

$$E_2^{s,t}(H, K, i) = H_c^s(K/H; \pi_t(E_n^{hH} \wedge M_i \wedge X)) \Rightarrow \pi_{t-s}(E_n^{hK} \wedge M_i \wedge X)$$

(e.g., see the proof of [14, Proposition 7.4]). Similarly, spectral sequence (7.5) is an inverse limit over $i \geq 0$ of strongly convergent Adams-type spectral sequences $\mathbb{E}_r^{*,*}(H, K, i)$ that have the same form as $E_r^{*,*}(H, K, i)$:

$$\mathbb{E}_2^{s,t}(H, K, i) = H_c^s(K/H; \pi_t(E_n^{hH} \wedge M_i \wedge X)) \Rightarrow \pi_{t-s}(E_n^{hK} \wedge M_i \wedge X).$$

Hence, to prove the theorem, it suffices to show that there is an isomorphism $E_r^{*,*}(H, K, i) \cong \mathbb{E}_r^{*,*}(H, K, i)$ of spectral sequences, from the E_2 -terms onward, for each i .

Notice that the limit $\lim_j U_j K / U_j H$ presents K/H as a profinite group and there is an isomorphism

$$\begin{aligned} \operatorname{colim}_j \mathbb{E}_2^{s,t}(U_j H, U_j K, i) &= \operatorname{colim}_j H^s(U_j K / U_j H; \pi_t(E_n^{hU_j H} \wedge M_i \wedge X)) \\ &\cong H_c^s(K/H; \pi_t(E_n^{hH} \wedge M_i \wedge X)), \end{aligned}$$

where the isomorphism applies Lemma 5.2. Thus, noting that each spectral sequence $\mathbb{E}_r^{*,*}(U_j H, U_j K, i)$ has abutment $\pi_*(E_n^{hU_j K} \wedge M_i \wedge X)$ and using Lemma 5.2 again, it is easy to see that there is an isomorphism

$$(7.7) \quad \mathbb{E}_r^{*,*}(H, K, i) \cong \operatorname{colim}_j \mathbb{E}_r^{*,*}(U_j H, U_j K, i)$$

of spectral sequences. Now, since each $U_j K / U_j H$ is a finite group, [5, Theorem A.1] shows that there is an isomorphism

$$\mathbb{E}_r^{*,*}(U_j H, U_j K, i) \cong E_r^{*,*}(U_j H, U_j K, i)$$

of spectral sequences, for all j . Thus,

$$\mathbb{E}_r^{*,*}(H, K, i) \cong \operatorname{colim}_j E_r^{*,*}(U_j H, U_j K, i) \cong E_r^{*,*}(H, K, i),$$

where the last isomorphism is verified in the same way that (7.7) is. \square

The results in this paper apply to other spectra that are closely related to E_n . Let k be any finite field containing \mathbb{F}_{p^n} . By [12, Section 7], for any height n formal group law Γ over k , there is a commutative S -algebra $E(k, \Gamma)$ with

$$E(k, \Gamma)_* = \mathbb{W}(k)[[u_1, \dots, u_{n-1}]] [u^{\pm 1}],$$

graded in the same way that E_{n*} is. Ethan Devinatz has informed the author that all the results of [8] and [5] go through as is, with E_n replaced with $E(k, \Gamma)$ and G_n replaced with $G(k) = S_n \rtimes \operatorname{Gal}(k/\mathbb{F}_p)$, a compact p -adic analytic group. Thus, [8] implies that (a) there is an isomorphism

$$\pi_*(\widehat{L}(E(k, \Gamma)^{dhK} \wedge E(k, \Gamma))) \cong \operatorname{Map}_c(G(k), \pi_*(E(k, \Gamma)))^K;$$

and (b) $E(k, \Gamma)^{dhK} \simeq \widehat{L}(\operatorname{colim}_j E(k, \Gamma)^{dhU_j K})$, where $\{U_j\}$ is a nested sequence of open normal subgroups of $G(k)$, as before.

By (b), there is a continuous $G(k)$ -spectrum

$$E(k, \Gamma) \simeq \operatorname{holim}_i((\operatorname{colim}_j E(k, \Gamma)^{dhU_j}) \wedge M_i),$$

so that, for any closed subgroup K of $G(k)$, $E(k, \Gamma)^{hK}$ exists. Then

$$E(k, \Gamma)^{hK} \simeq E(k, \Gamma)^{dhK},$$

by using the arguments of [1, Theorem 8.2.1] and [4]. Also, (a) and (b) imply that $(\operatorname{colim}_j E(k, \Gamma)^{dhU_j}) \wedge M_i$ is a totally hyperfibrant discrete $G(k)$ -spectrum, so that

$$(E(k, \Gamma)^{hH})^{hK/H} \simeq E(k, \Gamma)^{hK},$$

for any H closed in $G(k)$ and normal in K .

A. APPENDIX: AN EXAMPLE OF A DISCRETE G -SPECTRUM THAT IS NOT HYPERFIBRANT

DANIEL G. DAVIS, BEN WIELAND

Let G be a profinite group. In this appendix, we give an example, due to the second author, of a discrete G -spectrum X that is not a hyperfibrant discrete G -spectrum (for the meaning of this term, see Definition 4.1). All spectra are Bousfield-Friedlander spectra of simplicial sets.

Let p and q be distinct primes, let $\mathbb{Z}_q = \lim_{n \geq 0} \mathbb{Z}/q^n$, and let

$$G = \mathbb{Z}/p \times \mathbb{Z}_q.$$

For each $n \geq 0$, let \mathbb{Z}/q^n act on itself by translation. Denote by $\mathbb{Z}/p[\mathbb{Z}/q^n]$ the free (\mathbb{Z}/p) -module on this set and by $H(\mathbb{Z}/p[\mathbb{Z}/q^n])$ the Eilenberg-MacLane spectrum of that abelian group. By the functoriality of these constructions, this is a (\mathbb{Z}/q^n) -spectrum, for each n . By letting \mathbb{Z}/p act trivially on $H(\mathbb{Z}/p[\mathbb{Z}/q^n])$, $H(\mathbb{Z}/p[\mathbb{Z}/q^n])$ is a (discrete) $(\mathbb{Z}/p \times \mathbb{Z}/q^n)$ -spectrum. Let G act through the canonical surjection $G \rightarrow \mathbb{Z}/p \times \mathbb{Z}/q^n$, making $H(\mathbb{Z}/p[\mathbb{Z}/q^n])$ a discrete G -spectrum. By letting G act trivially on S^n , the spectrum $\Sigma^n H(\mathbb{Z}/p[\mathbb{Z}/q^n])$ is also a discrete G -spectrum. Let

$$H_n = \Sigma^n H(\mathbb{Z}/p[\mathbb{Z}/q^n])$$

and set

$$X = \bigvee_{n \geq 0} H_n.$$

Since the colimit in the category of discrete G -spectra is formed in the category of spectra, X is a discrete G -spectrum. We will show that X is not a hyperfibrant discrete G -spectrum. In particular, we will show that the map $\psi(X)_{\mathbb{Z}/p}^G$ of (3.5) is not a weak equivalence.

It is well-known that $\operatorname{cd}_p(\mathbb{Z}/p) = \infty$. But, since $\operatorname{cd}_q(\mathbb{Z}_q) = 1$ and \mathbb{Z}_q is open in G , it follows that G has finite virtual cohomological dimension. Thus, if K is a closed subgroup of G , then, by [3, Theorem 5.2], there is an identification

$$X^{hK} = \operatorname{holim}_{\Delta} (\Gamma_G^\bullet(X_{f,G}))^K.$$

There is a canonical map

$$\psi: \operatorname{colim}_{N \triangleleft_o G} X^{hN\mathbb{Z}/p} = \operatorname{colim}_{N \triangleleft_o G} \operatorname{holim}_{\Delta} (\Gamma_G^\bullet(X_{f,G}))^{N\mathbb{Z}/p} \rightarrow \operatorname{holim}_{\Delta} (\Gamma_G^\bullet(X_{f,G}))^{\mathbb{Z}/p} = X^{h\mathbb{Z}/p}.$$

Since $\mathbb{Z}/p \triangleleft G$, $N\mathbb{Z}/p \triangleleft G$, for each N . Hence, the above identification implies that $X^{hN\mathbb{Z}/p}$ is a $(G/(N\mathbb{Z}/p))$ -spectrum, so that $X^{hN\mathbb{Z}/p}$ is also a discrete

G -spectrum, since $G/(N\mathbb{Z}/p)$ is finite. This implies that $X^{hN\mathbb{Z}/p}$ is a discrete \mathbb{Z}_q -spectrum, so that $\text{colim}_{N \triangleleft_o G} X^{hN\mathbb{Z}/p}$, the source of ψ , is also a discrete \mathbb{Z}_q -spectrum. Notice that $X^{h\mathbb{Z}/p}$ is a $(G/(\mathbb{Z}/p))$ -spectrum; that is, the target of ψ is a \mathbb{Z}_q -spectrum. Also, it is easy to see that the map ψ is \mathbb{Z}_q -equivariant.

Now suppose that X is a hyperfibrant discrete G -spectrum. Thus, ψ is a weak equivalence, so that $\pi_*(\psi)$ is a \mathbb{Z}_q -equivariant isomorphism between \mathbb{Z}_q -modules, such that the source is a *discrete* \mathbb{Z}_q -module. This implies that $\pi_*(X^{h\mathbb{Z}/p})$ must also be a discrete \mathbb{Z}_q -module. Hence, to show that X is not a hyperfibrant discrete G -spectrum, it suffices to show that $\pi_*(X^{h\mathbb{Z}/p})$ is not a discrete \mathbb{Z}_q -module. In particular, we only have to show that $\pi_0(X^{h\mathbb{Z}/p})$ fails to be a discrete \mathbb{Z}_q -module.

We now compute $\pi_0(X^{h\mathbb{Z}/p})$ explicitly as a \mathbb{Z}_q -module. If K is any group and Z is a K -spectrum, let $Z \rightarrow Z_{\mathfrak{f}}$ be a weak equivalence to a fibrant spectrum; by the functoriality of fibrant replacement, $Z_{\mathfrak{f}}$ is a K -spectrum and the weak equivalence is K -equivariant. If Z is any spectrum, we let $\text{Sets}(K, Z)$ be the spectrum that is defined by $\text{Sets}(K, Z)_l = \text{Sets}(K, Z_l)$, for each $l \geq 0$, where $\text{Sets}(K, Z_l)$ is the pointed simplicial set with k -simplices equal to $\text{Sets}(K, Z_{l,k})$, for all $k \geq 0$. Also, let EK_{\bullet} be the usual free contractible simplicial K -set, with $EK_k = K^{k+1}$, so that $\text{Sets}(EK_{\bullet}, Z_{\mathfrak{f}})$ is a cosimplicial spectrum. Then

$$X^{h\mathbb{Z}/p} = \text{holim}_{\Delta} \text{Sets}(E(\mathbb{Z}/p)_{\bullet}, X_{\mathfrak{f}})^{\mathbb{Z}/p} = \text{holim}_{[k] \in \Delta} \text{Sets}((\mathbb{Z}/p)^{k+1}, X_{\mathfrak{f}})^{\mathbb{Z}/p},$$

where the (\mathbb{Z}/p) -fixed points are taken with respect to the (\mathbb{Z}/p) -action on the spectrum $\text{Sets}((\mathbb{Z}/p)^{k+1}, X_{\mathfrak{f}})$ that is given by conjugation, so that, given $h \in \mathbb{Z}/p$, $f \in \text{Sets}((\mathbb{Z}/p)^{k+1}, X_{\mathfrak{f}})$, and $(h_i)_i \in (\mathbb{Z}/p)^{k+1}$,

$$(h \cdot f)((h_i)_i) = h \cdot f((h^{-1}h_i)_i) = f((h^{-1}h_i)_i),$$

where the last equality is due to the fact that $X_{\mathfrak{f}}$ has the trivial (\mathbb{Z}/p) -action (since X does). (Here, and elsewhere, we write maps in terms of elements because of the simplicial *sets* that constitute all of our spectra.) Additionally, $X^{h\mathbb{Z}/p}$ is a \mathbb{Z}_q -spectrum, where the \mathbb{Z}_q -action is defined by

$$(g \cdot f)((h_i)_i) = g \cdot f((h_i)_i),$$

for $g \in \mathbb{Z}_q$ and $f \in \text{Sets}((\mathbb{Z}/p)^{k+1}, X_{\mathfrak{f}})^{\mathbb{Z}/p}$. Notice that, given $h \in \mathbb{Z}/p$,

$$(h \cdot (g \cdot f))((h_i)_i) = g \cdot f(h^{-1} \cdot (h_i)_i) = g \cdot (h^{-1} \cdot f((h_i)_i)) = (g \cdot f)((h_i)_i),$$

so that $g \cdot f \in \text{Sets}((\mathbb{Z}/p)^{k+1}, X_{\mathfrak{f}})^{\mathbb{Z}/p}$, as required.

For each n , $\pi_*(H_n)$ is $\mathbb{Z}/p[\mathbb{Z}/q^n]$ in degree n and zero elsewhere, so that the G -equivariant map

$$\phi: X = \bigvee_{n \geq 0} H_n \rightarrow \prod_{n \geq 0} H_n,$$

with G acting diagonally on the target, is a weak equivalence. Since ϕ induces the composition $X_{\mathfrak{f}} \rightarrow (\prod_{n \geq 0} H_n)_{\mathfrak{f}} \rightarrow (\prod_{n \geq 0} (H_n)_{\mathfrak{f}})_{\mathfrak{f}}$, which is G -equivariant and a weak equivalence, there are weak equivalences

$$\begin{aligned} X^{h\mathbb{Z}/p} &= \text{holim}_{[k] \in \Delta} \text{Sets}((\mathbb{Z}/p)^{k+1}, X_{\mathfrak{f}})^{\mathbb{Z}/p} \xrightarrow{\cong} \text{holim}_{[k] \in \Delta} \text{Sets}((\mathbb{Z}/p)^{k+1}, (\prod_{n \geq 0} (H_n)_{\mathfrak{f}})_{\mathfrak{f}})^{\mathbb{Z}/p} \\ &\xrightarrow{\cong} \text{holim}_{[k] \in \Delta} \text{Sets}((\mathbb{Z}/p)^{k+1}, \prod_{n \geq 0} (H_n)_{\mathfrak{f}})^{\mathbb{Z}/p} \\ &\cong \prod_{n \geq 0} \text{holim}_{[k] \in \Delta} \text{Sets}((\mathbb{Z}/p)^{k+1}, (H_n)_{\mathfrak{f}})^{\mathbb{Z}/p} = \prod_{n \geq 0} H_n^{h\mathbb{Z}/p}. \end{aligned}$$

The maps in the above zigzag are \mathbb{Z}_q -equivariant, so that there are \mathbb{Z}_q -equivariant isomorphisms

$$\pi_0(X^{h\mathbb{Z}/p}) \cong \pi_0(\prod_{n \geq 0} H_n^{h\mathbb{Z}/p}) \cong \prod_{n \geq 0} \pi_0(H_n^{h\mathbb{Z}/p}),$$

with \mathbb{Z}_q acting diagonally on $\prod_{n \geq 0} \pi_0(H_n^{h\mathbb{Z}/p})$.

For each $n \geq 0$, there is a descent spectral sequence

$$E_2^{s,t} = H^s(\mathbb{Z}/p; \pi_t(H_n)) \Rightarrow \pi_{t-s}(H_n^{h\mathbb{Z}/p}),$$

where, for all s, t , $E_2^{s,t}$ is $H^s(\mathbb{Z}/p; \mathbb{Z}/p[\mathbb{Z}/q^n])$ when $t = n$ and zero otherwise. Thus,

$$\pi_t(H_n^{h\mathbb{Z}/p}) \cong \begin{cases} H^{n-t}(\mathbb{Z}/p; \mathbb{Z}/p[\mathbb{Z}/q^n]), & \text{if } t \leq n; \\ 0, & \text{if } t > n. \end{cases}$$

In particular,

$$\pi_0(H_n^{h\mathbb{Z}/p}) \cong H^n(\mathbb{Z}/p; \mathbb{Z}/p[\mathbb{Z}/q^n]) \cong \bigoplus_{\mathbb{Z}/q^n} H^n(\mathbb{Z}/p; \mathbb{Z}/p) \cong \bigoplus_{\mathbb{Z}/q^n} \mathbb{Z}/p,$$

where the second isomorphism uses the fact that $\mathbb{Z}/p[\mathbb{Z}/q^n]$ is a free (\mathbb{Z}/p) -module, with each copy of \mathbb{Z}/p having the trivial (\mathbb{Z}/p) -action. Therefore,

$$(A.1) \quad \pi_0(X^{h\mathbb{Z}/p}) \cong \prod_{n \geq 0} \mathbb{Z}/p[\mathbb{Z}/q^n].$$

By naturality of the above constructions, it is immediate that (A.1) is a \mathbb{Z}_q -equivariant isomorphism of \mathbb{Z}_q -modules, where the action on the right-hand side is defined as follows: given $g \in \mathbb{Z}_q$ and an element $((r_{j_n} j_n)_{j_n})_{n \geq 0}$ in $\prod_{n \geq 0} \mathbb{Z}/p[\mathbb{Z}/q^n]$, where $(r_{j_n} j_n)_{j_n} \in \mathbb{Z}/p[\mathbb{Z}/q^n]$,

$$g \cdot ((r_{j_n} j_n)_{j_n})_{n \geq 0} = ((r_{j_n} g j_n)_{j_n})_{n \geq 0}.$$

(In the notation $(r_{j_n} j_n)_{j_n}$, each r_{j_n} is an element in \mathbb{Z}/p and j_n runs through the elements of \mathbb{Z}/q^n .) It only remains to show that $\prod_{n \geq 0} \mathbb{Z}/p[\mathbb{Z}/q^n]$ is not a discrete \mathbb{Z}_q -module.

Consider the injection of sets

$$i: \mathbb{Z}_q = \lim_{n \geq 0} \mathbb{Z}/q^n \rightarrow \prod_{n \geq 0} \mathbb{Z}/p[\mathbb{Z}/q^n], \quad (\alpha_n)_{n \geq 0} \mapsto ((i(\alpha_n, j_n) j_n)_{j_n})_{n \geq 0},$$

where $\alpha_n \in \mathbb{Z}/q^n$ and

$$i(\alpha_n, j_n) = \begin{cases} 1, & \text{if } j_n = \alpha_n; \\ 0, & \text{if } j_n \neq \alpha_n. \end{cases}$$

It is easy to see that i is \mathbb{Z}_q -equivariant, so that we can regard the \mathbb{Z}_q -set \mathbb{Z}_q as a \mathbb{Z}_q -subset of $\prod_{n \geq 0} \mathbb{Z}/p[\mathbb{Z}/q^n]$.

Now suppose that $\prod_{n \geq 0} \mathbb{Z}/p[\mathbb{Z}/q^n]$ is a discrete \mathbb{Z}_q -module. Let \mathbb{G} be the profinite group \mathbb{Z}_q and let the \mathbb{G} -set \mathbb{Z}_q have the discrete topology. Then the composition

$$\mathbb{G} \times \mathbb{Z}_q \xrightarrow{\text{id} \times i} \mathbb{G} \times (\prod_{n \geq 0} \mathbb{Z}/p[\mathbb{Z}/q^n]) \xrightarrow{\text{action}} \prod_{n \geq 0} \mathbb{Z}/p[\mathbb{Z}/q^n]$$

is continuous, and, since the image of this map is inside \mathbb{Z}_q , the induced action map $\mathbb{G} \times \mathbb{Z}_q \rightarrow \mathbb{Z}_q$ is continuous, so that \mathbb{Z}_q is a discrete \mathbb{G} -set. But this is a contradiction, since \mathbb{Z}_q is a profinite \mathbb{G} -space and not a discrete \mathbb{G} -set. Therefore, $\prod_{n \geq 0} \mathbb{Z}/p[\mathbb{Z}/q^n]$ is not a discrete \mathbb{Z}_q -module.

This completes the proof that X is not a hyperfibrant discrete G -spectrum. The same construction also works in the case that $p = q$, but the analysis is a little more complicated. This realizes a suggestion of the referee of this paper, who had

suggested to the first author, independently of the second author's work, that there might exist a non-hyperfibrant discrete $(\mathbb{Z}/p \times \mathbb{Z}_p)$ -spectrum.

REFERENCES

- [1] Mark Behrens and Daniel G. Davis. The homotopy fixed point spectra of profinite Galois extensions. Accepted by *Trans. Amer. Math. Soc.*, available online at the arXiv: math.AT/0808.1092, 2008.
- [2] Daniel G. Davis. Homotopy fixed points for $L_{K(n)}(E_n \wedge X)$ using the continuous action. *J. Pure Appl. Algebra*, 206(3):322–354, 2006.
- [3] Daniel G. Davis. Explicit fibrant replacement for discrete G -spectra. *Homology, Homotopy Appl.*, 10(3):137–150, 2008.
- [4] Daniel G. Davis. The Lubin-Tate spectrum and its homotopy fixed point spectra. Thesis, 106 pp., Northwestern University, May 9, 2003, available at www.ucs.louisiana.edu/~dxd0799.
- [5] Ethan S. Devinatz. A Lyndon-Hochschild-Serre spectral sequence for certain homotopy fixed point spectra. *Trans. Amer. Math. Soc.*, 357(1):129–150 (electronic), 2005.
- [6] Ethan S. Devinatz. Homotopy groups of homotopy fixed point spectra associated to E_n . In *Proceedings of the Nishida Fest, Kinoshita, 2003*, in *Geom. Topol. Monogr.*, vol. 10, pages 131–145. Mathematical Sciences Publishers, Berkeley, 2007.
- [7] Ethan S. Devinatz. Towards the finiteness of $\pi_*(L_{K(n)}(S^0))$. *Adv. Math.*, 219:1656–1688, 2008.
- [8] Ethan S. Devinatz and Michael J. Hopkins. Homotopy fixed point spectra for closed subgroups of the Morava stabilizer groups. *Topology*, 43(1):1–47, 2004.
- [9] J. D. Dixon, M. P. F. du Sautoy, A. Mann, and D. Segal. *Analytic pro- p groups*. Cambridge University Press, Cambridge, second edition, 1999.
- [10] W. G. Dwyer and C. W. Wilkerson. Homotopy fixed-point methods for Lie groups and finite loop spaces. *Ann. of Math. (2)*, 139(2):395–442, 1994.
- [11] Halvard Fausk. Equivariant homotopy theory for pro-spectra. *Geom. Topol.*, 12(1):103–176, 2008.
- [12] P. G. Goerss and M. J. Hopkins. Moduli spaces of commutative ring spectra. In *Structured ring spectra*, volume 315 of *London Math. Soc. Lecture Note Ser.*, pages 151–200. Cambridge Univ. Press, Cambridge, 2004.
- [13] Paul Goerss, Hans-Werner Henn, and Mark Mahowald. The homotopy of $L_2V(1)$ for the prime 3. In *Categorical decomposition techniques in algebraic topology (Isle of Skye, 2001)*, volume 215 of *Progr. Math.*, pages 125–151. Birkhäuser, Basel, 2004.
- [14] Michael J. Hopkins, Mark Mahowald, and Hal Sadofsky. Constructions of elements in Picard groups. In *Topology and representation theory (Evanston, IL, 1992)*, pages 89–126. Amer. Math. Soc., Providence, RI, 1994.
- [15] Mark Hovey. Bousfield localization functors and Hopkins' chromatic splitting conjecture. In *The Čech centennial (Boston, MA, 1993)*, pages 225–250. Amer. Math. Soc., Providence, RI, 1995.
- [16] Mark Hovey and Neil P. Strickland. Morava K -theories and localisation. *Mem. Amer. Math. Soc.*, 139(666):viii+100, 1999.
- [17] Uwe Jannsen. Continuous étale cohomology. *Math. Ann.*, 280(2):207–245, 1988.
- [18] J. F. Jardine. *Generalized étale cohomology theories*. Birkhäuser Verlag, Basel, 1997.
- [19] Stephen A. Mitchell. Hypercohomology spectra and Thomason's descent theorem. In *Algebraic K-theory (Toronto, ON, 1996)*, pages 221–277. Amer. Math. Soc., Providence, RI, 1997.
- [20] John Rognes. Galois extensions of structured ring spectra. In *Galois extensions of structured ring spectra/Stably dualizable groups*, in *Mem. Amer. Math. Soc.*, vol. 192 (898), pages 1–97. 2008.
- [21] Peter Symonds and Thomas Weigel. Cohomology of p -adic analytic groups. In *New horizons in pro- p groups*, pages 349–410. Birkhäuser Boston, Boston, MA, 2000.
- [22] John Tate. Relations between K_2 and Galois cohomology. *Invent. Math.*, 36:257–274, 1976.

E-mail address: dgdavis@louisiana.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF LOUISIANA AT LAFAYETTE, LAFAYETTE, LA
70504

E-mail address: `wieland@math.brown.edu`

DEPARTMENT OF MATHEMATICS, BROWN UNIVERSITY, PROVIDENCE, RI 02912