Arithmetic classification of $E(1)$-local finite CW-complexes.

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Outline

Topological background, and some goals.

Classification of $E(1)$-local finite spectra.

F-crystals and L-functions for finite CW-complexes.

Recall that, for each prime p, there exists an infinite sequence of generalized homology theories

$$E(0), E(1), E(2), \dots,$$

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- $E(1) \cong \ell$, the Adams summand, so $E(1)$-localization is just localization at complex periodic K-theory and then localization at p.
- (Since the Quillen-Lichtenbaum conjecture is true, one knows that the $E(1)$-localization map $K_n(X)_{(p)} \rightarrow \pi_n(L_{E(1)}K(X))$ is an isomorphism for n greater than the étale cohomological dimension of X, when $1/p \in \Gamma(X)$. So algebraic K-theory is "mostly $E(1)$-local.")

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  the *chromatic tower,* for any space or spectrum X.
- Hopkins-Ravenel: $\operatorname{holim}_n L_{E(n)}X \cong X$ for finite p-local spectra X. (Many infinite spectra are still okay. S.: $\operatorname{holim}_n L_{E(n)}X \cong X$ for connective p-local spectra with BP-homology Cohen-Macaulay over the moduli stack of 1-dim'l formal groups. This includes all connective spectra with torsion-free homology. However, many infinite X are bad: S.: $\operatorname{holim}_n L_{E(n)}K(R)$ disagrees with $K(R)_{(p)}$ for commutative Noetherian rings R with a lower bound on their negative K-groups.)

- $E(n)_* \cong \mathbb{Z}_{(p)}[v_1, \dots, v_{n-1}, v_n^{\pm 1}]$, with $|v_i| = 2(p^i - 1)$.

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- Morava, Miller-Ravenel, ultimately Lubin-Tate: there exists a homology theory $(E_n)_*$, closely related to $E(n)_*$, with the property that $[E_n, E_n]^0 \cong W(\mathbb{F}_{p^n})[\mathrm{Aut}(F_n)]^\sigma$, with F_n a height n 1-dim'l formal group over $\mathbb{F}_{p^n}$.

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- Goerss-Hopkins-Miller, Davis-Lawson, others?: there exists a spectral sequence

$$H^*(\tilde{\mathrm{Aut}}(F_n); E_n^*(X)) \Rightarrow \pi_*(L_{K(n)}X \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)),$$

where $K(n)$ is yet another homology theory. (But $L_{E(n)}X$ can be recovered from $L_{K(0)}X, L_{K(1)}X, \ldots, L_{K(n)}X$.)

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In terms of the $E(1)$-local sphere: $\pi_n(L_{E(1)}S/p)$ is the p-factor in the denominator of $B_n B_{n+1}$, if you're willing to think of 0 as $0/1$, having denominator 1. Von Staudt-Clausen: B_{2n} is the product of all primes p such that $(p-1) \mid 2n$. And $\pi_n(L_{E(1)}S)$ is the p-factor in the denominator of $\zeta(-n)$, since $\zeta(-n) = -\frac{B_{n+1}}{n+1}$.

Some goals:

- Classify $E(1)$-local finite spectra, i.e., the $E(1)$-localizations of suspension spectra of finite CW-complexes. (Not quite the same thing, but almost, as finite $L_{E(1)}S$-modules!)

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- Generalize Adams' formula: let's try to describe homotopy groups of $L_{E(1)}X$ for *any* finite CW-complex X in terms of special values of zeta-functions, not just the case $X = S^0$.
- Do it in a way that could be generalized to $L_{E(2)}X, L_{E(3)}X, \ldots$.

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- Explain and clarify the appearances of the same objects in chromatic homotopy theory and the Langlands correspondences. (Some of this part is joint work with Morava.)
- Let's be jealous of the algebraic geometers and try to develop a theory of L-functions which count orders of homotopy groups, similar to how zeta-functions of varieties use (via Lefschetz) cohomology to count solutions in finite fields.

In this talk we make progress toward all of these goals (or as much as we can describe in 40 minutes!), in the case $n = 1$ and $p > 2$, but our methods are designed to generalize to higher n...

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Morava, Miller-Ravenel:

$$H^*(\tilde{\mathrm{Aut}}(F_1); E_1^*(S/p)) \cong H^*(\mathrm{Aut}(F_1); \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathbb{F}_p[v_1^{\pm 1}].$$

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Since p acts nilpotently on $E_1^*(S/p)$, the fixed point spectral sequence converging to $\pi_* L_{K(1)} S/p$ coincides with the $E(1)$-local ANSS converging to $\pi_* L_{E(1)} S/p$. (Drawn on the chalkboard. No room for differentials!)

Then one runs a Bockstein spectral sequence for the extension

$$0 \to (E_1)_*(S) \to (E_1)_*(S) \to (E_1)_*(S/p) \to 0$$

to get $H^*(\tilde{\mathrm{Aut}}(F_1); E_1^*(S)) \Rightarrow \pi_*(L_{K(1)}S)$. (Drawn on the board. No room for differentials in the fixed point SS!)

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(Now draw the $E(1)$-local ANSS for S on the board too. Again no room for differentials.)
Miller-Ravenel-Wilson: if $p > 2$, all elements in positive degrees in $\pi_*(L_{E(1)}S)$ survive the homotopy chromatic spectral sequence, that is, the map $\pi_*S \to \pi_*(L_{E(1)}S)$ is surjective in positive degrees. (Compare the statement of the Q-L conjecture from earlier!)

So every element in $\pi_*(L_{E(1)}S)$ for $* > 0$ is not just an attaching map for a 2-cell $L_{E(1)}S$-module: even better, it is actually the $E(1)$-localization of an attaching map for an honest 2-cell CW-complex!

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Note that the cofiber of any element in the divided alpha family, i.e., any element on the $E(1)$-Adams 1-line, has torsion free homology. This will become important later on!

How do we get all the 3-cell complexes? Suppose we have a 2-cell complex

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How many ways are there to attach another cell to S/f by some map $\Sigma^j S \xrightarrow{g} S/f$?

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How many ways are there to attach another cell to S/f by some map $\Sigma^j S \xrightarrow{g} S/f$?
We have the exact sequence

$$[\Sigma^j S, S] \to [\Sigma^j S, S/f] \to [\Sigma^j S, \Sigma^{i+1} S],$$

so our map g either compresses to the 0-skeleton S in S/f, or it restricts nontrivially to the top cell $\Sigma^{i+1} S$.

The maps $\Sigma^j S \to S/f$ that compress to the 0-skeleton are in bijection with the cokernel of the multiplication-by-f map $[\Sigma^j S, \Sigma^i S] \to [\Sigma^j S, S]$, which is easy to compute after $E(1)$-localization, since almost all products in $\pi_*(L_{E(1)} S)$ are zero.

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The maps $\Sigma^j S \to S/f$ that restrict nontrivially to the top cell are in bijection with those maps in $[\Sigma^j S, \Sigma^{i+1} S]$ whose composition product with f, landing in $[\Sigma^j S, \Sigma^1 S]$, is zero. Also easy to compute after $E(1)$-localization because of products being almost all zero!

So suppose we attached a cell to S/f which restricted nontrivially to the top cell. We have a 3-cell complex $S/f, g$. How many ways can we attach a cell to $S/f, g$ to get a 4-cell complex?

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We have the exact sequence

$$[\Sigma^k S, S/f] \to [\Sigma^k S, S/f, g] \to [\Sigma^k S, \Sigma^{j+1} S]$$

so every attaching map $\Sigma^k S \to S/f, g$ either compresses to the previous skeleton S/f (a case we already understand), or it restricts nontrivially to the top cell $\Sigma^{j+1} S$.

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Now I owe you an explanation of what a Toda bracket is.

Since $fg = 0$, if $gh = 0$ then the composite

$$\Sigma^{i+j+k} S \overset{\Sigma^{j+k} h}{\to} \Sigma^{i+j} S \xrightarrow{\Sigma^{j} g} \Sigma^{i} S \xrightarrow{f} S$$

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is nulhomotopic for *two different reasons.*
By the universal property of the suspension as the homotopy pushout of two maps to a point, that means the two nulhomotopies give rise to a map $\Sigma^{i+j+k+1} S \to S$. That's written $\langle f, g, h \rangle$ and called the Toda bracket.

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This means we need to know all the Toda brackets of nonnegatively-graded elements in $\pi_*(L_{E(1)} S)$.

Actually, it's convenient (and still reasonable) to ask for more than that. Suppose we attached a cell to S/f by a map $g : \Sigma^j S \to S/f$ that compressed into the 0-skeleton.

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We also want to allow maps like

$$\begin{bmatrix} \Sigma^{2p-3} p \\ \alpha_1 \end{bmatrix} : \Sigma^{2p-3} S \to \Sigma^{2p-3} S \vee S.$$

Thm. (folklore? Conjecture of Bruner?): Every finite CW-complex is stably homotopy equivalent to one of the form

$$\bigvee S/M_1, M_2, \dots M_n,$$

where each M_i is a matrix of elements in $\pi_*(S)$, and the total matric Toda bracket

$$\langle M_1, M_2, \dots, M_n \rangle$$

is defined and zero. (This forces every sub-bracket of it, including plain old products of consecutive elements, to be zero.)

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This isn't actually very hard to do (cleanest cases are on next slide). But it's messy and one wants a *unique* such way of writing each $E(1)$-local stable homotopy type, i.e., a canonical form for these strings of matrices. But I don't have this yet.

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Compared to Bousfield's description of finite modules over the $E(1)$-local sphere: disadvantages: messier statement, no uniqueness (yet). Advantages: one knows the attaching maps explicitly.

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Moss Convergence Thm., after an easy generalization to $E(1)$, $p > 2$: the (matric) Toda brackets in $\pi_*(L_{E(1)}S)$ are exactly the (matric) Massey products in $\mathrm{Ext}^{*,*}_{[E(1),E(1)]^*}(E(1)_*, E(1)_*)$.

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We compute all of these in the cobar DGA for the continuous linear dual of $[E(1), E(1)]^*$. The zero ones give attaching maps.

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(Some Toda brackets involving p and its powers, i.e., classes on the $E(1)$-local ANSS 0-line, are nonzero. We can make a general statement here but it's currently messy, so for now, let's just work with the matric Toda brackets only involving classes on the 1-line.)

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Thm. (S., easy): all the $E(1)$-local homotopy types defined in the theorem above (all entries on the ANSS 1-line) have torsion-free p-local homology, and every finite CW-complex with torsion-free p-local homology can be written as a matric Toda bracket of the form above.

One special class of $E(1)$-local homotopy types that we'll come back to later: for an odd prime p, let's call an $E(1)$-local homotopy type X *p-cyclic* if X can be written as a finite wedge of $E(1)$-local homotopy types constructed by matric Toda brackets of any length, such that:

- each matrix in the bracket is one-by-one, i.e., the matric Toda bracket is just an ordinary Toda bracket, and
- X has torsion-free p-local homology.
- We'll also say X is *p-simple* if each element in the bracket is in a degree divisible by p, i.e., a multiple of α_i where $p \mid 2(p-1)i - 1$.

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For example: S/α_1 is p-simple at every odd prime p, and $\mathbb{C}P^2$ is p-simple at every odd prime p.

One special class of $E(1)$-local homotopy types that we'll come back to later: for an odd prime p, let's call an $E(1)$-local homotopy type X *p-cyclic* if X can be written as a finite wedge of $E(1)$-local homotopy types constructed by matric Toda brackets of any length, such that:

- each matrix in the bracket is one-by-one, i.e., the matric Toda bracket is just an ordinary Toda bracket, and
- X has torsion-free p-local homology.
- We'll also say X is *p-simple* if each element in the bracket is in a degree divisible by p, i.e., a multiple of α_i where $p \mid 2(p-1)i-1$.

For example: S/α_1 is p-simple at every odd prime p, and $\mathbb{C}P^2$ is p-simple at every odd prime p.
(The p-simplicity condition will guarantee, later on, that in certain products of L-functions, factors in the numerators of special values of one L-function do not cancel factors in the denominators of special values of the other L-functions in the product. One can get away without p-simplicity but a certain theorem later on in the talk is then much harder to state.)

Outline

Topological background, and some goals.

Classification of $E(1)$-local finite spectra.

F-crystals and L-functions for finite CW-complexes.

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People write $W(\mathbb{F}_q)$ for the Witt ring of $\mathbb{F}_q$. This is the universal complete local characteristic 0 commutative ring with residue field $\mathbb{F}_q$, but it's also just $\hat{\mathbb{Z}}_p[\zeta_{p^n-1}]$.

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The Frobenius action on $\mathbb{F}_q$ lifts to a Frobenius action on $W(\mathbb{F}_q)$. If $x \in W(\mathbb{F}_q)$, I'll write x^σ for the Frobenius on x.

By an *F-crystal* (over $W(\mathbb{F}_q)$, which is suppressed from the notation) one means a $W(\mathbb{F}_q)$-module M and a map of abelian groups $F : M \to M$, such that:

- M is free and finite-dimensional over $W(\mathbb{F}_{p^n})$.
- F is semilinear, that is, $F(\omega x) = \omega^\sigma Fx$.
- M/FM is finite length (every chain of submodules is finite).

Here is a slide about how the p-adic geometers use these things. I don’t use this slide for the rest of the talk, so don’t be too afraid to tune me out for the duration of this slide.

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Fontaine-Messing: If X is an abelian variety with good reduction at p then the Galois action on crystalline H^1 gives the same p-adic Galois representation as Fontaine's functor applied to $crys(\mathbb{G}_X)$.

Now there's the first Morava E-homology, E_1. We're going to use this. In general:

- $\pi_*(E_n) \cong W(\mathbb{F}_{p^n})[[u_1, \ldots, u_{n-1}]][u^{\pm 1}]$ with $|u_i| = 0$ and $|u| = 2$.

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- E_n is a BP-module spectrum, with $BP_* \to (E_n)_*$ sending v_i to $u_i u^{p^i}$ for $i = 1, \ldots, n - 1$, and sending v_n to u^{p^n}, and sending all the higher v generators to zero.

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- Lubin-Tate, Morava: $\pi_0(E_n)$ is the classifying ring of deformations of a certain height n one-dim'l formal group over $\mathbb{F}_{p^n}$ to characteristic zero.
- Goerss-Hopkins-Miller: E_n is a commutative ring spectrum and the strict automorphism group $\mathrm{Aut}(F_n)$ acts on E_n and the induced action on $(E_n)_*$ agrees with the action coming from the deformation problem.

The pro-p-group $\mathrm{Aut}(F_n)$ is a pro-p-Sylow subgroup of the group of units of

$$W(\mathbb{F}_{p^n})\langle S\rangle/(S^n = p, \omega^\sigma S - S\omega).$$

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Thm. (S., easy): if X is a finite CW-complex with torsion-free p-local homology, then $(E_1)^n(X)$ is an F-crystal for each n, with $F : (E_1)^n(X) \to (E_1)^n(X)$ given by $F(x) = (1 + S)(x) - x$.

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Thm. (S., easy): if X is a finite CW-complex with torsion-free p-local homology, then $(E_1)^n(X)$ is an F-crystal for each n, with $F : (E_1)^n(X) \to (E_1)^n(X)$ given by $F(x) = (1 + S)(x) - x$. (You can do this at higher heights too but you have to work harder. S.: for X as above, $(E_2)^n(X)$ is a (ϕ, N)-module, i.e., an F-crystal equipped with a monodromy differential operator N. Here $\phi = F$ above, while $N = -\log(1 + u_1)\frac{\delta}{\delta u_1}$. These are even better than F-crystals because, by Kedlaya's theory of slopes, one can use them to get F-crystals with Hodge filtration that satisfy the admissibility criterion for Fontaine's functor. Still trying to prove Kedlaya's slope 0 condition for these (ϕ, N)-modules!)

Now suppose M is an F-crystal. We define a p-local L-factor from M:

$$L_p(s, M) = \frac{1}{\det(1 - p^{-s}(F|_{M \otimes_{W(\mathbb{F}_{p^n})} K}))},$$

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And, if X is a finite CW-complex with torsion-free homology, we define a global L-function by its Euler product:

$$L(s, X) = \prod_p L_p(s, (E_1)^0(X)).$$

(There's a different E_1 for each p. The product is over all of them.)

Thm. (S.): let P be a finite set of primes including 2. If X is p-simple at each prime p not in P, then $L(s, X)$ is holomorphic for Re s greater than one plus the dimension of the top cell of X, and has a meromorphic continuation to the complex plane. Furthermore, for any integer n greater than the dimension of the top cell of X, the denominator of $L(-n, X)$ is equal to the order of $\pi_n(L_{KU}X[P^{-1}])$, modulo P-primary factors.

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(One can also make a similar statement for n between zero and the dimension of the top cell, but the theorem has exceptions then, for example, since some homotopy groups are infinite in that range.) This theorem has teeth, because:
Thm. (S., easy): every finite CW-complex which is p-cyclic for all but finitely many primes p is also p-simple for all but finitely many primes p. So the above theorem applies to many CW-complexes with torsion-free homology, e.g. any 2-cell complex with torsion-free homology, or in general any finite CW-complex built by attaching one cell at a time which restricts nontrivially to the top cell at each stage.

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The L-function doesn't even make sense for spaces with torsion in their homology. But an L-function-theoretic description is still possible for at least some CW-complexes with torsion in their homology, you just have to start with a torsion-free homology complex and keep track of cells where you plan to kill a power of p. Then you can get a theorem analogous to the description of $\pi_* L_{E(1)} S/p$ at the beginning of the talk: if X has p-power-torsion in its homology, the orders of the groups aren't just special values of an L-function, but they are computable from special values of an L-function for a torsion-free homology complex on whose cells you killed some powers of p to get X.

Some comments to end with:

- Most of the new theorems in this talk have been about $E(1)$-local stable homotopy types of spaces with torsion-free p-local homology. Much of it can be generalized to spaces with at least certain kinds of p-torsion in homology, with work and extra structure to keep track of. But much of that case can be handled by old (but relatively unexploited!) methods.

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- Most of the new theorems in this talk have been about $E(1)$-local stable homotopy types of spaces with torsion-free p-local homology. Much of it can be generalized to spaces with at least certain kinds of p-torsion in homology, with work and extra structure to keep track of. But much of that case can be handled by old (but relatively unexploited!) methods. Ravenel, 1970s: for $p > 2$, $E(0)$-acyclic $E(1)$-local stable homotopy types of finite complexes, i.e., $E(1)$-local stable homotopy types of spaces whose homology is entirely p-power-torsion, are equivalent to Iwasawa modules, i.e., $\mathbb{Z}/(2p-2)\mathbb{Z}$-graded p-power-torsion discrete $\hat{\mathbb{Z}}_p[[t]]$-modules.

- So between results of this talk and Iwasawa theory, in principle one can understand the $E(1)$-local stable homotopy types of spaces whose homology is either all p-power-torsion or p-power-torsion-free.

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$$\begin{bmatrix} \Sigma^{2p-3} p \\ \alpha_1 \end{bmatrix} : \Sigma^{2p-3} S \to \Sigma^{2p-3} S \vee S,$$

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- So we want to generalize our methods further and further and further: handle $E(1)$-local homotopy types with p-power torsion in homology, handle higher heights (draw some of this on the chalkboard), handle small primes ($p \leq n + 1$) where $E(n)$-local ANSS vanishing lines are trickier...