THE G_n-ACTION ON E_n IN THE STABLE CATEGORY

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Abstract. It is a well-known fact that, by Brown representability, the extended Morava stabilizer group G_n acts on the Lubin-Tate spectrum E_n, *in the stable category.* Though it is not hard to prove this, as far as the author knows, the proof is not written down in the literature. Thus, he wrote out the details for himself, and is making it available, in case it can be helpful to others.

1. Introduction

Let S_n be the nth Morava stabilizer group, and let $G_n = S_n \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$. Let E_n be the Lubin-Tate spectrum with

$$E_{n*} = W(\mathbb{F}_{p^n})[[u_1, ..., u_{n-1}]][u^{\pm 1}],$$

where the degree of u is -2 and the complete power series ring over the Witt vectors is in degree zero.

As a part of his change of rings theorem [8], Jack Morava pointed out that, by the deformation theory of formal group laws, the profinite group G_n acts continuously on the graded profinite coefficient ring E_{n*}. It is a well-known fact that, by Brown representability, one can conclude that the action on the level of topological algebra is actually induced by an action of G_n on E_n in the stable homotopy category; that is, there is a well-defined action through homotopy classes of maps.

In the early 1990's, Mike Hopkins and Haynes Miller showed that G_n actually acts on the spectrum E_n on the point-set level, in the category of A_∞-ring spectra [9]. They showed that E_n is an A_∞-ring spectrum and G_n acts on the spectrum through A_∞-ring spectrum maps. This point-set level action induces the homotopy action of G_n on E_n.

Subsequently, Paul Goerss and Hopkins showed that E_n is a commutative S-algebra, and the G_n-action is by maps of commutative S-algebras ([5], [6], [4]). Then, using the beautiful and deep paper [4], by Ethan Devinatz and Hopkins, the author showed in his thesis [2] that G_n actually acts continuously on E_n, on the point-set level: there is a model of E_n that is an inverse limit of G_n-spectra of simplicial sets, such that each set constituting these spectra is a discrete G_n-set.

As far as the author knows, the proof that the G_n-action on $\pi_*(E_n)$ implies a homotopy action of G_n on E_n is not written down anywhere in the literature. Thus, I wrote down the details for myself, and am making them available in case it can be useful to others who are beginning to study the beautiful story of the relationship betwen G_n and E_n. I thank Paul Goerss for explaining to me how Brown representability yields the homotopy action.

2. G_n acts on E_n in Ho(Sp), by ring spectrum maps

The precise statement we will prove is that the G_n-action on E_{n*} implies that there is a unique action of G_n on E_n by ring spectrum maps in the stable homotopy category ([3, pg. 767], [4, pg. 9]).

The stable homotopy category of spectra is a monogenic Brown category, satisfying a particular form of Brown representability (see [1, pp. 342-344] and [7, pp. 6, 54-61]). For us, this means the following. A natural transformation $E_*(-) \longrightarrow F_*(-)$ of homology theories on the category of finite spectra is induced by a map $E \longrightarrow F$, in the stable category, of spectra. The map $E \longrightarrow F$ is unique if the set $\mathcal{P}(E, F)$ of phantom maps is zero. Also, $\mathcal{P}(E, F) = 0$ if and only if $[E, F]$ is Hausdorff.

Now consider the fact that the G_n-action on E_{n*} makes $E_{n*}(X)$ a natural G_n-module. This means that, given any map $X \longrightarrow X'$ of finite spectra, there is a commutative diagram

$$\begin{array}{ccc} E_{n*}(X) & \xrightarrow{g} & E_{n*}(X) \\ \downarrow & & \downarrow \\ E_{n*}(X') & \xrightarrow{g} & E_{n*}(X'), \end{array}$$

where g is induced by the action of $g \in G_n$. Thus, $g\colon E_{n*}(-) \longrightarrow E_{n*}(-)$ is a natural transformation of homology theories induced by a map $g\colon E_n \longrightarrow E_n$ of spectra. Letting $E_n = \operatorname{colim}_\alpha E_\alpha$ be a colimit of finite subspectra over a directed set $\{\alpha\}$ implies that

$$[E_n, E_n] \cong \varprojlim_\alpha \pi_0(E_n \wedge DE_\alpha) \cong \varprojlim_{\alpha, I} \pi_0(E_n \wedge M_I \wedge DE_\alpha)$$

is a profinite and hence, Hausdorff, topological space. Thus, there are no phantom maps $E_n \longrightarrow E_n$ and the map g is unique. This shows that there is a unique action of G_n on E_n in the stable category that induces the action of G_n on E_{n*}.

Now we explain why this action is by maps of ring spectra. Since E_n is a ring spectrum, there are the unit and multiplication maps $\eta\colon S^0 \to E_n$ and $\mu\colon E_n \wedge E_n \to E_n$, respectively, in the stable category. Now consider the map $g\colon E_n \to E_n$. The action of G_n on E_{n*} is by ring homomorphisms so that $g\colon E_{n*} \to E_{n*}$ sends η to itself. Since this homomorphism is induced by $g\colon E_n \to E_n$, the following diagram commutes:

$$\begin{array}{ccc} S^0 & \xrightarrow{\eta} & E_n \\ & \searrow^{\eta} & \downarrow g \\ & & E_n. \end{array}$$

The pairing $E_{n*}(X) \otimes_{E_{n*}} E_{n*}(Y) \to E_{n*}(X \wedge Y)$ is G_n-equivariant, where the tensor product is given the diagonal action. Thus, given the maps $S^0 \to E_n \wedge X$ and $S^0 \to E_n \wedge Y$ in the stable category, the following diagram is commutative:

(2.1)

$$\begin{array}{ccccccc} S^0 & \longrightarrow & E_n \wedge X \wedge E_n \wedge Y & \longrightarrow & E_n \wedge E_n \wedge X \wedge Y & \xrightarrow{\mu \wedge 1} & E_n \wedge X \wedge Y \\ \downarrow & & & & & & \downarrow g \wedge 1 \\ E_n \wedge X \wedge E_n \wedge Y & \xrightarrow{g \wedge 1 \wedge g \wedge 1} & E_n \wedge X \wedge E_n \wedge Y & \longrightarrow & E_n \wedge E_n \wedge X \wedge Y & \xrightarrow{\mu \wedge 1} & E_n \wedge X \wedge Y. \end{array}$$

As a map between spectra, $g\mu\colon E_n \wedge E_n \to E_n$ gives a natural transformation of homology theories $(E_n \wedge E_n)_*(-) \to E_{n*}(-)$. Thus, any map $W \to Z$ of finite spectra gives a commutative diagram

$$\begin{array}{ccc} (E_n \wedge E_n)_*(W) & \longrightarrow & E_{n*}(W) \\ \downarrow & & \downarrow \\ (E_n \wedge E_n)_*(Z) & \longrightarrow & E_{n*}(Z). \end{array} \tag{2.2}$$

Setting $X = S^0$, and $Y = W$ and then Z, in 2.1 implies that 2.2 is also induced by $\mu(g \wedge g)$. Since the natural transformations $(E_n \wedge E_n)_*(-) \to E_{n*}(-)$ coming from $g\mu$ and $\mu(g \wedge g)$ give the same collection of commutative squares, they are the same natural transformation. Since the same argument as above shows that $[E_n \wedge E_n, E_n]$ is a profinite and hence, Hausdorff, topological space, $\mathcal{P}(E_n \wedge E_n, E_n) = 0$ and the natural transformation is represented by a unique map. Thus, $g\mu = \mu(g \wedge g)$, and $g\colon E_n \to E_n$ is a map of ring spectra.

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