Delta-discrete G–spectra and iterated homotopy fixed points

DANIEL G. DAVIS

Let G be a profinite group with finite virtual cohomological dimension and let X be a discrete G–spectrum. If H and K are closed subgroups of G, with $H \lhd K$, then, in general, the K/H–spectrum X^{hH} is not known to be a continuous K/H–spectrum, so that it is not known (in general) how to define the iterated homotopy fixed point spectrum $(X^{hH})^{hK/H}$. To address this situation, we define homotopy fixed points for delta-discrete G–spectra and show that the setting of delta-discrete G–spectra gives a good framework within which to work. In particular, we show that by using delta-discrete K/H–spectra, there is always an iterated homotopy fixed point spectrum, denoted $(X^{hH})^{h_\delta K/H}$, and it is just X^{hK}.

Additionally, we show that for any delta-discrete G–spectrum Y, there is an equivalence $(Y^{h_\delta H})^{h_\delta K/H} \simeq Y^{h_\delta K}$. Furthermore, if G is an *arbitrary* profinite group, there is a delta-discrete G–spectrum X_δ that is equivalent to X and, though X^{hH} is not even known in general to have a K/H–action, there is always an equivalence $\big((X_\delta)^{h_\delta H}\big)^{h_\delta K/H} \simeq (X_\delta)^{h_\delta K}$. Therefore, delta-discrete L–spectra, by letting L equal H, K, and K/H, give a way of resolving undesired deficiencies in our understanding of homotopy fixed points for discrete G–spectra.

55P42, 55P91

1 Introduction

1.1 An overview of iterated homotopy fixed points and the problem we consider in this paper

Let Spt be the stable model category of Bousfield–Friedlander spectra of simplicial sets. Also, given a profinite group G, let Spt_G be the model category of discrete G–spectra, in which a morphism f is a weak equivalence (cofibration) if and only if f is a weak equivalence (cofibration) in Spt (as explained by the author in [5, Section 3]). Given a fibrant replacement functor

$$(-)_{fG}\colon \mathrm{Spt}_G \to \mathrm{Spt}_G, \quad X \mapsto X_{fG},$$

so that there is a natural trivial cofibration $X \to X_{fG}$, with X_{fG} fibrant, in Spt_G, the G–homotopy fixed point spectrum X^{hG} is defined by

$$X^{hG} = (X_{fG})^G.$$

Let H and K be closed subgroups of G, with H normal in K and, as above, let X be a discrete G–spectrum. Then K/H is a profinite group, and it is reasonable to expect that X^{hH} is some kind of a continuous K/H–spectrum, so that the iterated homotopy fixed point spectrum $(X^{hH})^{hK/H}$ can be formed. Additionally, one might expect $(X^{hH})^{hK/H}$ to just be X^{hK}: following Dwyer and Wilkerson [14, page 434], when these two homotopy fixed point spectra are equivalent to each other, for all H, K and X, we say that homotopy fixed points for G have the *transitivity property*.

Under hypotheses on G and X that are different from those above, there are various cases where the iterated homotopy fixed point spectrum is well-behaved along the lines suggested above. For example, by [14, Lemma 10.5], when G is any discrete group, with $N \lhd G$ and $\mathcal{X}$ any G–space, the iterated homotopy fixed point space $(\mathcal{X}^{hN})^{hG/N}$ is always defined and is just $\mathcal{X}^{hG}$. Similarly, by Rognes [27, Theorem 7.2.3], if E is an S–module and $A \to B$ is a faithful E–local G–Galois extension of commutative S–algebras, where G is a stably dualizable group, then if N is an allowable normal subgroup of G, $(B^{hN})^{hG/N}$ is defined and is equivalent to B^{hG}.

Let k be a spectrum such that the Bousfield localization $L_k(-)$ is equivalent to $L_M L_T(-)$, where M is a finite spectrum and T is smashing, and let A be a k–local commutative S–algebra. If a spectrum E is a consistent profaithful k–local profinite G–Galois extension of A of finite vcd (the meaning of these terms is explained by Behrens and the author in [1]), then, by [1, Proposition 7.1.4, Theorem 7.1.6],

$$(E^{h_k N})^{h_k G/N} \simeq E^{h_k G},$$

for any closed normal subgroup N of G, where, for example, $(-)^{h_k G}$ denotes the k–local homotopy fixed points (as defined in [1, Section 6.1]).

Finally, let G be any compact Hausdorff group, let R be an orthogonal ring spectrum satisfying the assumptions of Fausk [15, Section 11.1, lines 1–3], and let $\mathcal{M}_R$ be the category of R–modules. Also, let M be any pro-G–R–module (so that, for example, the pro-spectrum M is a pro-R–module and a pro-orthogonal G–spectrum). By [15, Proposition 11.5], if N is any closed normal subgroup of G, there is an equivalence

$$(M^{h_G N})^{hG/N} \simeq M^{hG}$$

of pro-spectra in the Postnikov model structure on the category of pro-objects in $\mathcal{M}_R$. Here, $M^{h_G N}$ is the N–G–homotopy fixed point pro-spectrum of [15, Definition 11.3] and, as discussed in [15, page 165], there are cases when $M^{h_G N} \simeq M^{hN}$.

The above results show that when one works with the hypotheses of G is profinite and $X \in \mathrm{Spt}_G$ for the first time, it certainly is not unreasonable to hope that the expression $(X^{hH})^{hK/H}$ makes sense and that it fits into an equivalence $(X^{hH})^{hK/H} \simeq X^{hK}$. But it turns out that, in this setting, in general, these constructions are more subtle than the above results might suggest. For example, as explained by the author in [6, Section 5], X^{hH} is not even known to be a K/H–spectrum. However, when G has finite virtual cohomological dimension (*finite vcd*; that is, there exists an open subgroup U and a positive integer m such that the continuous cohomology $H^s_c(U;M) = 0$, whenever $s > m$ and M is a discrete U–module), then [6, Corollary 5.4] shows that X^{hH} is always weakly equivalent to a K/H–spectrum.

But, as explained by the author in detail in [7, Section 3], even when G has finite vcd, it is not known, in general, how to view X^{hH} as a continuous K/H–spectrum (in the sense of [5, 1]), so that it is not known how to form $(X^{hH})^{hK/H}$. For example, when $G = K = \mathbb{Z}/p \times \mathbb{Z}_q$, where p and q are distinct primes, and $H = \mathbb{Z}/p$, Ben Wieland found an example of a discrete G–spectrum Y such that Y^{hH} is not a discrete K/H–spectrum (an exposition of this example is given by Wieland and the author in [7, Appendix A]). More generally, it is not known if Y^{hH} is a continuous K/H–spectrum and there is no known construction of $(Y^{hH})^{hK/H}$.

By [7, Section 4], if G is any profinite group and X is a hyperfibrant discrete G–spectrum, then X^{hH} is always a discrete K/H–spectrum, and hence, $(X^{hH})^{hK/H}$ is always defined. However, it is not known if $(X^{hH})^{hK/H}$ must be equivalent to X^{hK}. Also, [7, Section 4] shows that if X is a totally hyperfibrant discrete G–spectrum, then $(X^{hH})^{hK/H}$ is X^{hK}. But, as implied by our remarks above regarding Y, it is not known that all the objects in Spt_G are hyperfibrant, let alone totally hyperfibrant.

The above discussion makes it clear that there are nontrivial gaps in our understanding of iterated homotopy fixed points in the world of Spt_G. To address these deficiencies, in this paper we define and study homotopy fixed points for *delta-discrete G–spectra*, $(-)^{h_\delta G}$, and within this framework, when G has finite vcd and $X \in \mathrm{Spt}_G$, we find that the iterated homotopy fixed point spectrum $(X^{hH})^{h_\delta K/H}$ is always defined and is equivalent to X^{hK}. In fact, when G has finite vcd, if Y is one of these delta-discrete G–spectra, then there is an equivalence

$$(Y^{h_\delta H})^{h_\delta K/H} \simeq Y^{h_\delta K}.$$

Before introducing this paper's approach to iterated homotopy fixed points in Spt_G in more detail, we quickly discuss some situations where the difficulties with iteration that were described earlier vanish, since it is helpful to better understand where the obstacles in "iteration theory" are.

In general, for any profinite group G and $X \in \mathrm{Spt}_G$,

$$(X^{h\{e\}})^{hK/\{e\}} = \big((X_{fG})^{\{e\}}\big)^{hK} \simeq X^{hK}$$

and, if H is open in K, then X_{fK} is fibrant in Spt_H and $(X_{fK})^H$ is fibrant in $\mathrm{Spt}_{K/H}$, so that

$$(X^{hH})^{hK/H} = ((X_{fK})^H)^{hK/H} \simeq ((X_{fK})^H)^{K/H} = (X_{fK})^K = X^{hK}$$

(see [1, Proposition 3.3.1] and [7, Theorem 3.4]). Thus, in general, the difficulties in forming the iterated homotopy fixed point spectrum occur only when H is a nontrivial non-open (closed normal) subgroup of K.

Now let N be any nontrivial closed normal subgroup of G. There are cases where, thanks to a particular property that G has, the spectrum $(X^{hN})^{hG/N}$ is defined, with

$$(X^{hN})^{hG/N} \simeq X^{hG}. \tag{1.1}$$

To explain this, we assume that G is infinite:

- if G has finite cohomological dimension (that is, there exists a positive integer m such that $H^s_c(G;M) = 0$, whenever $s > m$ and M is a discrete G–module), then, by [1, Corollary 3.5.6], X^{hN} is a discrete G/N–spectrum, so that $(X^{hN})^{hG/N}$ is defined and, as hoped, the equivalence in (**1.1**) holds;
- the profinite group G is *just infinite* if N always has finite index in G (for more details about such groups, see Wilson [31]; this interesting class of profinite groups includes, for example, the finitely generated pro-p Nottingham group over the finite field $\mathbb{F}_{p^n}$, where p is any prime and $n \geq 1$, and the just infinite profinite branch groups (see Grigorchuk [18])), and thus, if G is just infinite, then N is always open in G, so that, as explained above, (**1.1**) is valid; and
- if G has the property that every nontrivial closed subgroup is open, then, by Morris, Oates-Williams and Thompson [26, Corollary 1] (see also Dikranjan [10]), G is topologically isomorphic to $\mathbb{Z}_p$, for some prime p, and, as before, (**1.1**) holds.

In addition to the above cases, there is a family of examples in chromatic stable homotopy theory where iteration works in the desired way. Let k be any finite field containing $\mathbb{F}_{p^n}$, where p is any prime and n is any positive integer. Given any height n formal group law Γ over k, let $E(k,\Gamma)$ be the Morava E–theory spectrum that satisfies

$$\pi_*(E(k,\Gamma)) = W(k)[\![u_1,\ldots,u_{n-1}]\!][u^{\pm 1}],$$

where $W(k)$ denotes the Witt vectors, the degree of u is -2 and the complete power series ring $W(k)[\![u_1,\ldots,u_{n-1}]\!] \cdot u^0$ is in degree zero (see Goerss and Hopkins [16, Section

7]). Also, let $G = S_n \rtimes \mathrm{Gal}(k/\mathbb{F}_p)$, the extended Morava stabilizer group: G is a compact p–adic analytic group, and hence, has finite vcd, and, by [16], G acts on $E(k,\Gamma)$. Then, by using Devinatz and Hopkins [9], Devinatz [8], Rognes [27], the work [1] by Behrens and the author, and the notion of total hyperfibrancy, [7] shows that $E(k,\Gamma)$ is a continuous G–spectrum, with

$$\big(E(k,\Gamma)^{hH}\big)^{hK/H} \simeq E(k,\Gamma)^{hK}$$

for all H and K (defined as usual). If F is any finite spectrum that is of type n, then, as in [5, Corollary 6.5], $E(k,\Gamma) \wedge F$ is a discrete G–spectrum and, again by the technique of [7],

$$\big((E(k,\Gamma) \wedge F)^{hH}\big)^{hK/H} \simeq (E(k,\Gamma) \wedge F)^{hK}.$$

We note that when $E(k,\Gamma) = E_n$, the Lubin–Tate spectrum, the text [7, page 2883] reviews some examples of how $\big(E(k,\Gamma)^{hH}\big)^{hK/H}$ plays a useful role in chromatic homotopy theory.

1.2 An introduction to the homotopical category of delta-discrete G–spectra and their homotopy fixed points

Now we explain the approach of this paper to iterated homotopy fixed points in more detail. Let G be an arbitrary profinite group and let $X \in \mathrm{Spt}_G$. Also, let $c(\mathrm{Spt}_G)$ be the category of cosimplicial discrete G–spectra (that is, the category of cosimplicial objects in Spt_G). If Z is a spectrum (which, in this paper, always means Bousfield–Friedlander spectrum), we let $Z_{k,l}$ denote the l–simplices of the kth simplicial set Z_k of Z. Then $\mathrm{Map}_c(G,X)$ is the discrete G–spectrum that is defined by

$$\mathrm{Map}_c(G,X)_{k,l} = \mathrm{Map}_c(G,X_{k,l}),$$

the set of continuous functions $G \to X_{k,l}$. The G–action on $\mathrm{Map}_c(G,X)$ is given by $(g \cdot f)(g') = f(g'g)$, for $g, g' \in G$ and $f \in \mathrm{Map}_c(G,X_{k,l})$. As explained by the author in [5, Definition 7.1], the functor

$$\mathrm{Map}_c(G,-)\colon \mathrm{Spt}_G \to \mathrm{Spt}_G, \quad X \mapsto \mathrm{Map}_c(G,X),$$

forms a triple and there is a cosimplicial discrete G–spectrum $\mathrm{Map}_c(G^\bullet,X)$, where, for each $[n] \in \Delta$, the n–cosimplices satisfy the isomorphism

$$\mathrm{Map}_c(G^\bullet,X)^n \cong \mathrm{Map}_c(G^{n+1},X).$$

Following [5, Remark 7.5], with X a discrete G–spectrum as above, let

$$\widehat{X} = \operatorname*{colim}_{N \lhd_o G} (X^N)_f,$$

a filtered colimit over the open normal subgroups of G. Here,

$$(-)_f \colon \mathrm{Spt} \to \mathrm{Spt}$$

denotes a fibrant replacement functor for the model category Spt. Notice that $\widehat{X}$ is a discrete G–spectrum and a fibrant object in Spt. Now let

$$X_\delta = \operatorname*{holim}_{\Delta} \mathrm{Map}_c(G^\bullet, \widehat{X}).$$

(In the subscript of "X_δ," instead of "Δ," we use its less obtrusive and lowercase counterpart.) In the definition of X_δ and everywhere else in this paper, the homotopy limit (as written in the definition) is formed in Spt (and not in Spt_G) and is defined in the explicit way of Hirschhorn [20, Definition 18.1.8]. As explained in Section 2.1, there is a natural G–equivariant map

$$\Psi \colon X \xrightarrow{\simeq} X_\delta$$

that is a weak equivalence in Spt. Since $\widehat{X}$ is a fibrant spectrum, $\mathrm{Map}_c(G^\bullet, \widehat{X})^n$ is a fibrant spectrum (by applying [5, Corollary 3.8, Lemma 3.10]), for each $[n] \in \Delta$.

The map Ψ and the properties (discussed above) possessed by its target X_δ motivate the following definition.

Definition 1.2 Let $X^\bullet$ be a cosimplicial discrete G–spectrum such that X^n is fibrant in Spt, for each $[n] \in \Delta$. Then the G–spectrum

$$\operatorname*{holim}_{\Delta} X^\bullet$$

is a *delta-discrete G–spectrum.* Delta-discrete G–spectra are the objects of the *category $\widehat{\mathrm{Spt}_G^\Delta}$ of delta-discrete G–spectra* (the "hat" in the notation $\widehat{\mathrm{Spt}_G^\Delta}$ is for the homotopy limit – a "completion" – that helps to build the objects in the category). If $\operatorname{holim}_\Delta X^\bullet$ and $\operatorname{holim}_\Delta Y^\bullet$ are arbitrary delta-discrete G–spectra and $f^\bullet \colon X^\bullet \to Y^\bullet$ is a map in $c(\mathrm{Spt}_G)$, then the induced map

$$f = \operatorname*{holim}_{\Delta} f^\bullet \colon \operatorname*{holim}_{\Delta} X^\bullet \to \operatorname*{holim}_{\Delta} Y^\bullet$$

defines the notion of morphism in the category $\widehat{\mathrm{Spt}_G^\Delta}$.

If f and h are morphisms in $\widehat{\mathrm{Spt}_G^\Delta}$, with associated maps $f^\bullet \colon X^\bullet \to Y^\bullet$ and $h^\bullet \colon Y^\bullet \to Z^\bullet$, respectively, in $c(\mathrm{Spt}_G)$ (thus, for example, $h = \operatorname{holim}_\Delta h^\bullet$), then

$$h \circ f := \operatorname*{holim}_{\Delta} (h^\bullet \circ f^\bullet) = \big(\operatorname*{holim}_{\Delta} h^\bullet\big) \circ \big(\operatorname*{holim}_{\Delta} f^\bullet\big).$$

We note that the objects and morphisms of $\widehat{\mathrm{Spt}_G^\Delta}$ are "formal constructions," in the sense that the objects $\operatorname{holim}_\Delta X^\bullet$ and $\operatorname{holim}_\Delta Y^\bullet$ are equal in $\widehat{\mathrm{Spt}_G^\Delta}$ if and only if $X^\bullet = Y^\bullet$ in

$c(\mathrm{Spt}_G)$ and two morphisms $\mathrm{holim}_\Delta f^\bullet$ and $\mathrm{holim}_\Delta h^\bullet$ are equal in $\widehat{\mathrm{Spt}_G^\Delta}$ if and only if $f^\bullet = h^\bullet$ in $c(\mathrm{Spt}_G)$. Thus, $\widehat{\mathrm{Spt}_G^\Delta}$ is defined so that given a morphism k in $\widehat{\mathrm{Spt}_G^\Delta}$, there is only one morphism $k^\bullet$ in $c(\mathrm{Spt}_G)$ "behind" it, with $k = \mathrm{holim}_\Delta k^\bullet$.

Remark 1.3 There are other possible ways to define "a category of delta-discrete G–spectra," but we use the above definition because (among other reasons) we think of $\widehat{\mathrm{Spt}_G^\Delta}$ as a category of G–spectra (see Definition 1.4 below) that captures the "direct image" of $c(\mathrm{Spt}_G)$ in a homotopically meaningful way.

The weak equivalence Ψ gives a natural way of associating a delta-discrete G–spectrum to every discrete G–spectrum. Also, we will see that X_δ plays a useful role in our work on iterated homotopy fixed points. In Corollary 2.4 and Remark 2.5, by using a variation of Ψ, we show that every discrete G–spectrum X can be built out of "smaller" delta-discrete G–spectra in a particularly nice and canonical way: for each $N \vartriangleleft_o G$, there is a "building block of X" that is an object in $\widehat{\mathrm{Spt}_G^\Delta}$, equivalent to the fixed points X^N, and is fibrant in $\mathrm{Spt}_{G/N}$.

Definition 1.4 Let G–Spt denote the category of G–spectra and G–equivariant maps of spectra: more precisely, G–Spt is the diagram category of (covariant) functors $\{*_G\} \to \mathrm{Spt}$, where $\{*_G\}$ is the groupoid associated to G (as an abstract group). Since a delta-discrete G–spectrum can be regarded as a G–spectrum, there is the forgetful functor

$$\mathbb{U}_G \colon \widehat{\mathrm{Spt}_G^\Delta} \to G\text{–Spt}, \quad (X^\bullet, \operatorname*{holim}_\Delta X^\bullet) \mapsto \operatorname*{holim}_\Delta X^\bullet,$$

where, above, we are thinking of the objects in $\widehat{\mathrm{Spt}_G^\Delta}$ as being certain pairs in the category $c(\mathrm{Spt}_G) \times (G\text{–Spt})$.

Now we give several more useful definitions, including a definition of weak equivalence for the category of delta-discrete G–spectra, and we make some observations about this category that give it more homotopy-theoretic content.

Definition 1.5 Let $c(\mathrm{Spt})$ be the category of cosimplicial spectra. If $Z \in \mathrm{Spt}$, let $\mathrm{cc}^\bullet(Z)$ denote the constant cosimplicial object (in $c(\mathrm{Spt})$) on Z. Also, for $X \in \mathrm{Spt}_G$, let

$$\mathrm{cc}_G(X) = \operatorname*{holim}_\Delta \mathrm{cc}^\bullet(\widehat{X}).$$

Notice that $\mathrm{cc}_G(X) \in \widehat{\mathrm{Spt}_G^\Delta}$.

Definition 1.6 Let $f = \operatorname{holim}_{\Delta} f^\bullet \colon \operatorname{holim}_{\Delta} X^\bullet \to \operatorname{holim}_{\Delta} Y^\bullet$ be a morphism of delta-discrete G–spectra. If the morphism $f^\bullet \colon X^\bullet \to Y^\bullet$ in $c(\mathrm{Spt}_G)$ is an *objectwise weak equivalence* (that is, $f^n \colon X^n \to Y^n$ is a weak equivalence in Spt_G, for each $[n] \in \Delta$), then f is a *weak equivalence of delta-discrete G–spectra*. If f is a weak equivalence in $\widehat{\mathrm{Spt}_G^\Delta}$, then it is also a weak equivalence in Spt (since all the X^n and Y^n are fibrant spectra).

The category $c(\mathrm{Spt}_G)$ has an injective model structure where the weak equivalences are the objectwise weak equivalences. (This model structure and the one described in the next paragraph exist, for example, by the considerations of Section 3 and Lurie [24, Proposition A.2.8.2].) Since a model category is automatically a homotopical category (in the sense of Dwyer, Hirschhorn, Kan and Smith [12]), $c(\mathrm{Spt}_G)$ is a homotopical category, and this conclusion, coupled with the fact that $f = \operatorname{holim}_{\Delta} f^\bullet$ is a weak equivalence in $\widehat{\mathrm{Spt}_G^\Delta}$ if and only if the map $f^\bullet$ is a weak equivalence in $c(\mathrm{Spt}_G)$, implies that $\widehat{\mathrm{Spt}_G^\Delta}$ is a homotopical category.

We equip the diagram category G–Spt with the projective model structure so that a morphism in G–Spt is a weak equivalence (fibration) exactly when its underlying map in Spt is a weak equivalence (fibration). It follows that the functor $\mathbb{U}_G \colon \widehat{\mathrm{Spt}_G^\Delta} \to G\text{–Spt}$ is a homotopical functor; that is, $\mathbb{U}_G$ is a functor between homotopical categories that preserves weak equivalences. Also, given any map f in $\widehat{\mathrm{Spt}_G^\Delta}$, $\mathbb{U}_G(f)$ is a map between fibrant objects in G–Spt.

Now we define the key notion of homotopy fixed points for the category of delta-discrete G–spectra.

Definition 1.7 Given a delta-discrete G–spectrum $\operatorname{holim}_{\Delta} X^\bullet$, the *homotopy fixed point spectrum* $(\operatorname{holim}_{\Delta} X^\bullet)^{h_\delta G}$ is given by

$$(\operatorname{holim}_{\Delta} X^\bullet)^{h_\delta G} = \operatorname{holim}_{[n]\in\Delta} (X^n)^{hG},$$

where $(X^n)^{hG}$ is the homotopy fixed point spectrum of the discrete G–spectrum X^n. We use the phrase "delta-discrete homotopy fixed points" to refer to the general operation of taking the homotopy fixed points of a delta-discrete G–spectrum. Since a morphism $X^\bullet \to Y^\bullet$ in $c(\mathrm{Spt}_G)$ induces a morphism $\{(X^n)_{fG} \to (Y^n)_{fG}\}_{[n]\in\Delta}$ in $c(\mathrm{Spt}_G)$ and there is the functor $\operatorname{holim}_{\Delta}(-) \colon c(\mathrm{Spt}) \to \mathrm{Spt}$, delta-discrete homotopy fixed points give the functor

$$(-)^{h_\delta G} \colon \widehat{\mathrm{Spt}_G^\Delta} \to \mathrm{Spt}, \quad (X^\bullet, \operatorname{holim}_{\Delta} X^\bullet) \mapsto (\operatorname{holim}_{\Delta} X^\bullet)^{h_\delta G}.$$

Remark 1.8 We make a few comments to explain Remark 1.3 in more detail, by considering two (in the end, undesirable) ways to revise Definitions 1.2 and 1.6. Let f be a G–equivariant map of spectra, such that the source and target of f are delta-discrete G–spectra, and suppose that we define a morphism in $\widehat{\mathrm{Spt}_G^\Delta}$ to be such a map f with the property that $f = \mathrm{holim}_\Delta f^\bullet$ for *some* $f^\bullet$ in $c(\mathrm{Spt}_G)$ (thus, the map in $c(\mathrm{Spt}_G)$ behind f need not be unique). If we define f to be a weak equivalence in $\widehat{\mathrm{Spt}_G^\Delta}$ if and only if $f = \mathrm{holim}_\Delta h^\bullet$ for *some* weak equivalence $h^\bullet$ in $c(\mathrm{Spt}_G)$, then it is not hard to see that $\widehat{\mathrm{Spt}_G^\Delta}$ is not a homotopical category. Or, if we define f to be a weak equivalence in $\widehat{\mathrm{Spt}_G^\Delta}$ exactly when f is a weak equivalence in Spt, then $\widehat{\mathrm{Spt}_G^\Delta}$ is a homotopical category (since Spt is a homotopical category). But in both cases, the "functor" $(-)^{h_\delta G}$ above is not even a well-defined function on the morphisms of $\widehat{\mathrm{Spt}_G^\Delta}$.

1.3 A summary of the properties of the homotopy fixed points of delta-discrete G–spectra

In this paper, we show that the homotopy fixed points functor $(-)^{h_\delta G}\colon \widehat{\mathrm{Spt}_G^\Delta} \to \mathrm{Spt}$ has the following properties:

(a) when G is a finite group, the homotopy fixed points of a delta-discrete G–spectrum agree with the usual notion of homotopy fixed points for a finite group;

(b) for any profinite group G, the homotopy fixed points of delta-discrete G–spectra can be viewed as the total right derived functor of

$$\lim_\Delta (-)^G\colon c(\mathrm{Spt}_G) \to \mathrm{Spt},$$

where, as before, $c(\mathrm{Spt}_G)$ has the injective model category structure (defined in Section 3);

(c) the induced functor $\mathrm{Ho}\big((-)^{h_\delta G}\big)\colon \mathrm{Ho}\big(\widehat{\mathrm{Spt}_G^\Delta}\big) \to \mathrm{Ho}(\mathrm{Spt})$ on homotopy categories is the total right derived functor of the fixed points functor

$$(-)^G\colon \widehat{\mathrm{Spt}_G^\Delta} \to \mathrm{Spt}, \quad \operatorname*{holim}_\Delta X^\bullet \mapsto \big(\operatorname*{holim}_\Delta X^\bullet\big)^G;$$

(d) given $X \in \mathrm{Spt}_G$ and the delta-discrete G–spectrum X_δ associated to X, then, if G has finite vcd, there is a weak equivalence

$$X^{hL} \xrightarrow{\simeq} (X_\delta)^{h_\delta L}$$

in Spt, for every closed subgroup L of G;

(e) more generally, if G is any profinite group and $X \in \mathrm{Spt}_G$, then there is a G–equivariant map $X \xrightarrow{\simeq} \mathrm{cc}_G(X)$ that is a weak equivalence of spectra and a weak equivalence
$$X^{hG} \xrightarrow{\simeq} (\mathrm{cc}_G(X))^{h_\delta G};$$

(f) if f is a weak equivalence of delta-discrete G–spectra, then the induced map $(f)^{h_\delta G} = \mathrm{holim}_{[n]\in\Delta}(f^n)^{hG}$ is a weak equivalence in Spt (since each $(f^n)^{hG}$ is a weak equivalence between fibrant spectra), so that $(-)^{h_\delta G} \colon \widehat{\mathrm{Spt}_G^\Delta} \to \mathrm{Spt}$ is a homotopical functor; and

(g) if G has finite vcd, with closed subgroups H and K such that $H \lhd K$, and if Y is a delta-discrete G–spectrum, then $Y^{h_\delta H}$ is a delta-discrete K/H–spectrum and
$$(Y^{h_\delta H})^{h_\delta K/H} \simeq Y^{h_\delta K},$$
so that G–homotopy fixed points of delta-discrete G–spectra have the transitivity property.

In the above list of properties, (a) is Theorem 5.2, (b) is justified in Theorem 3.4, (c) is proven in Section 4, (d) is Lemma 2.9, (e) is verified right after the proof of Theorem 5.2, and (g) is obtained in Theorem 6.4 (and the three paragraphs that precede it).

Notice that properties (b) and (c) above show that the *homotopy* fixed points of delta-discrete G–spectra are the total right derived functor of fixed points in two different and necessary senses. Also, (e) shows that, for any G and any $X \in \mathrm{Spt}_G$, the delta-discrete G–spectrum $\mathrm{cc}_G(X)$ is equivalent to X and their homotopy fixed points are the same. Thus, the category of delta-discrete G–spectra and the homotopy fixed points $(-)^{h_\delta G}$ "include" and generalize the category of discrete G–spectra and the homotopy fixed points $(-)^{hG}$. Therefore, properties (a) – (g) show that the homotopy fixed points of a delta-discrete G–spectrum are a good notion that does indeed deserve to be called "homotopy fixed points."

Now suppose that G has finite vcd and, as usual, let $X \in \mathrm{Spt}_G$. As above, let H and K be closed subgroups of G, with H normal in K. In Lemma 2.10, we show that, by making a canonical identification, X^{hH} is a delta-discrete K/H–spectrum. Thus, it is natural to form the iterated homotopy fixed point spectrum $(X^{hH})^{h_\delta K/H}$ and, by Theorem 2.11, there is a weak equivalence
$$X^{hK} \xrightarrow{\simeq} (X^{hH})^{h_\delta K/H}.$$

In this way, we show that when G has finite vcd, by using delta-discrete K/H–spectra, there is a sense in which the iterated homotopy fixed point spectrum can always be formed and the transitivity property holds.

More generally, in Theorem 7.3, we show that for *any* G, though it is not known if X^{hH} always has a K/H–action (as mentioned earlier), there is an equivalence

$$\textbf{(1.9)} \qquad \big((X_\delta)^{h_\delta H}\big)^{h_\delta K/H} \simeq (X_\delta)^{h_\delta K}.$$

Thus, for any G, by using $(-)_\delta$ and $(-)^{h_\delta H}$, the delta-discrete homotopy fixed points for discrete G–spectra are – *in general* – transitive. Also, there is a map

$$\rho(X)_H \colon X^{hH} \to (X_\delta)^{h_\delta H}$$

that relates the "discrete homotopy fixed points" X^{hH} to $(X_\delta)^{h_\delta H}$, and this map is a weak equivalence whenever the map $\operatorname{colim}_{U \lhd_o H} \Psi^U$ is a weak equivalence (see Theorem 7.2 and the discussion that precedes it).

For any G and X, since the G–equivariant map $\Psi \colon X \to X_\delta$ is a weak equivalence, X can always be regarded as a delta-discrete G–spectrum, and thus, X can be thought of as having two types of homotopy fixed points, X^{hH} and $(X_\delta)^{h_\delta H}$, and, though its discrete homotopy fixed points X^{hH} are not known to always be well-behaved with respect to iteration, there is a reasonable alternative, the delta-discrete homotopy fixed points $(X_\delta)^{h_\delta H}$, which, thanks to (**1.9**), are always well-behaved.

As the reader might have noticed, given the work of [7] (as discussed earlier) and that of this paper, when G has finite vcd and X is a hyperfibrant discrete G–spectrum, there are two different ways to define an iterated homotopy fixed point spectrum: as $(X^{hH})^{hK/H}$ and as $(X^{hH})^{h_\delta K/H}$. Though we are not able to show that these two objects are always equivalent, in Theorem 8.2, we show that if the canonical map $(X_{fK})^H \to ((X_{fK})_{fH})^H$ is a weak equivalence, then there is a weak equivalence $(X^{hH})^{hK/H} \xrightarrow{\simeq} (X^{hH})^{h_\delta K/H}$.

In the last section of this paper, Section 9, we show in two different, but interrelated ways that, for arbitrary G, the delta-discrete homotopy fixed point spectrum is always equivalent to a discrete homotopy fixed point spectrum. In each case, the equivalence is induced by a map between a discrete G–spectrum and a delta-discrete G–spectrum that need *not* be a weak equivalence. In Remark 9.7, we note several consequences of this observation for the categories $c(\mathrm{Spt}_G)$ and $c(\mathrm{Spt})$, when each is equipped with the injective model structure.

We close this section by mentioning that, given a discrete G–spectrum X, the cosimplicial discrete G–spectrum $\operatorname{Map}_c(G^\bullet, X)$ (defined in Section 1.2) is an example of a Godement resolution: this perspective on the diagram $\operatorname{Map}_c(G^\bullet, X)$ and the relevant sheaf theory is explained in more detail in Thomason [29, 1.31–1.33], Mitchell [25, Section 3.2] and [5, pages 332–333, 340]. Thus, the useful G–spectrum X_δ, the "primordial delta-discrete G–spectrum," is the homotopy limit of a Godement resolution, so that this paper gives additional examples of the well-known utility of Godement resolutions.

Acknowledgements

With regard to the topic of homotopy limits, which appear frequently in this paper, I thank Kathryn Hess and Michael Shulman for helpful discussions and Chris Douglas for several useful comments. Also, I am grateful to the referee for a variety of helpful remarks, especially with regard to my initial treatment of homotopy limits and several aspects of the category of delta-discrete G–spectra; these remarks helped me to improve the original version of this paper in various ways. The author was partially supported by a grant (# LEQSF(2008–11)-RD-A-27) from the Louisiana Board of Regents.

2 The new framework for iterated homotopy fixed points of discrete G–spectra

Let G be an arbitrary profinite group and X a discrete G–spectrum. In Section 2.1 below, we review the author's construction in [6] of a natural G–equivariant map $\Psi\colon X \to X_\delta$ that is a weak equivalence in Spt, giving a natural way of associating a delta-discrete G–spectrum to each object of Spt_G. Also, in Section 2.2, we show that when G has finite vcd, then by using the framework of delta-discrete K/H–spectra, there is a sense in which homotopy fixed points of discrete G–spectra satisfy transitivity.

2.1 The canonical association of a certain delta-discrete G–spectrum to a discrete G–spectrum

Let G be any profinite group and let $X \in \mathrm{Spt}_G$. There is a G–equivariant monomorphism

$$i\colon X \to \mathrm{Map}_c(G, X)$$

that is defined, on the level of sets, by $i(x)(g) = g \cdot x$, where $x \in X_{k,l}$ and $g \in G$. Notice that i induces a G–equivariant map

$$\widetilde{i_X}\colon\ \operatorname*{holim}_{\Delta} \mathrm{cc}^\bullet(X) \to \operatorname*{holim}_{\Delta} \mathrm{Map}_c(G^\bullet, X).$$

There is a natural G–equivariant map

$$\psi\colon X \cong \operatorname*{colim}_{N \lhd_o G} X^N \to \operatorname*{colim}_{N \lhd_o G} (X^N)_f = \widehat{X}$$

to the discrete G–spectrum $\widehat{X}$. We would like to know that ψ is a weak equivalence in Spt_G; however, the validity of this is not obvious, since $\{X^N\}_{N \lhd_o G}$ is not known to be a diagram of fibrant spectra, and hence, we cannot use the fact that filtered colimits preserve weak equivalences between fibrant spectra. Nevertheless, the following lemma shows that it is still the case that ψ is a weak equivalence in Spt_G (the author stated this without proof in [5, Remark 7.5] and [6, Definition 3.4]).

Lemma 2.1 *If X is a discrete G–spectrum, then the map $\psi\colon X \to \widehat{X}$ is a weak equivalence in* Spt_G.

Proof Since X is a discrete G–spectrum,

$$\mathrm{Hom}_G(-,X)\colon (G\!-\!\mathbf{Sets}_{df})^{\mathrm{op}} \to \mathrm{Spt}, \quad C \mapsto \mathrm{Hom}_G(C,X)$$

is a presheaf of spectra on the site $G\!-\!\mathbf{Sets}_{df}$ of finite discrete G–sets (for more detail, we refer the reader to [5, Section 3] and Jardine [22, Sections 2.3, 6.2]). Here, the l–simplices of the kth simplicial set $\mathrm{Hom}_G(C,X)_k$ are given by $\mathrm{Hom}_G(C,X_{k,l})$. Also, the composition

$$(-)_f \circ \mathrm{Hom}_G(-,X)\colon (G\!-\!\mathbf{Sets}_{df})^{\mathrm{op}} \to \mathrm{Spt}, \quad C \mapsto (\mathrm{Hom}_G(C,X))_f$$

is a presheaf of spectra and there is a map of presheaves

$$\widehat{\psi}\colon \mathrm{Hom}_G(-,X) \to (-)_f \circ \mathrm{Hom}_G(-,X)$$

that comes from the natural transformation $\mathrm{id}_{\mathrm{Spt}} \to (-)_f$.

Since the map $\widehat{\psi}(C)\colon \mathrm{Hom}_G(C,X) \to (\mathrm{Hom}_G(C,X))_f$ is a weak equivalence for every $C \in G\!-\!\mathbf{Sets}_{df}$, the map $\pi_t\big(\widehat{\psi}\big)$ of presheaves is an isomorphism for every integer t. Therefore, for every integer t, $\widetilde{\pi}_t\big(\widehat{\psi}\big)$, the map of sheaves associated to $\pi_t\big(\widehat{\psi}\big)$, is an isomorphism, so that the map $\widehat{\psi}$ is a local stable equivalence, and hence, a stalkwise weak equivalence. Thus, the map $\mathrm{colim}_{N \lhd_o G}\, \widehat{\psi}(G/N)$ is a weak equivalence of spectra, and therefore, the map

$$\operatorname*{colim}_{N \lhd_o G} X^N \cong \operatorname*{colim}_{N \lhd_o G} \mathrm{Hom}_G(G/N,X) \xrightarrow{\simeq} \operatorname*{colim}_{N \lhd_o G} (\mathrm{Hom}_G(G/N,X))_f \cong \operatorname*{colim}_{N \lhd_o G} (X^N)_f$$

is a weak equivalence, giving the desired conclusion. □

The following definition is from [6, page 145].

Definition 2.2 Given any profinite group G and any $X \in \mathrm{Spt}_G$, the composition

$$X \xrightarrow{\psi} \widehat{X} \xrightarrow{\cong} \lim_{\Delta} \mathrm{cc}^{\bullet}(\widehat{X}) \xrightarrow{\xi} \operatorname*{holim}_{\Delta} \mathrm{cc}^{\bullet}(\widehat{X}) \xrightarrow{\widetilde{i}_{\widehat{X}}} \operatorname*{holim}_{\Delta} \mathrm{Map}_c(G^{\bullet},\widehat{X}) = X_\delta$$

of natural maps, where the map ξ is the usual one (for example, see Hirschhorn [20, Example 18.3.8, (2)]), defines the natural G–equivariant map

$$\Psi\colon X \to X_\delta$$

of spectra.

The next result was obtained in [6, page 145]; however, we give the proof below for completeness and because it is a key result.

Lemma 2.3 *If G is any profinite group and X is a discrete G–spectrum, then the natural map $\Psi\colon X \xrightarrow{\simeq} X_\delta$ is a weak equivalence of spectra.*

Proof Following [6, page 145], there is a homotopy spectral sequence

$$E_2^{s,t} = \pi^s\big(\pi_t(\mathrm{Map}_c(G^\bullet, \widehat{X}))\big) \Rightarrow \pi_{t-s}(X_\delta).$$

Let $\pi_t(\mathrm{Map}_c(G^*, \widehat{X}))$ be the canonical cochain complex associated to the cosimplicial abelian group $\pi_t(\mathrm{Map}_c(G^\bullet, \widehat{X}))$. As in [5, proof of Theorem 7.4], there is an exact sequence

$$0 \to \pi_t(\widehat{X}) \to \mathrm{Map}_c(G^*, \pi_t(\widehat{X})) \cong \pi_t(\mathrm{Map}_c(G^*, \widehat{X})),$$

so that $E_2^{0,t} \cong \pi_t(\widehat{X}) \cong \pi_t(X)$, where the last isomorphism is by Lemma 2.1, and $E_2^{s,t} = 0$, when $s > 0$. Thus, the above spectral sequence collapses, giving the desired result. □

If $N \lhd_o G$, then the canonical epimorphism $G \to G/N$ makes every delta-discrete G/N–spectrum a delta-discrete G–spectrum. This observation, combined with the next result, shows that every discrete G–spectrum X is a filtered colimit in Spt of "smaller" delta-discrete G–spectra (more precisely, of a diagram of "smaller" objects in the category $\widehat{\mathrm{Spt}_G^\Delta}$) in a canonical way. (In the proof of Corollary 2.4, we show that each term of the aforementioned colimit is equivalent, for some $N \lhd_o G$, to $(X^N)_f$, and hence, to X^N, justifying our use of the adjective "smaller" in the preceding sentence.)

Corollary 2.4 *If G is any profinite group and $X \in \mathrm{Spt}_G$, then there is a map*

$$\widetilde{\Psi}\colon X \xrightarrow{\simeq} \operatorname*{colim}_{N \lhd_o G} \operatorname*{holim}_{\Delta} \mathrm{Map}_c\big((G/N)^\bullet, (X^N)_f\big)$$

that is a weak equivalence in Spt_G.

Proof Given any $N \lhd_o G$, then, by slightly modifying Definition 2.2 and applying the argument in the proof of Lemma 2.3, there is a natural G/N–equivariant map

$$\Psi_N\colon (X^N)_f \xrightarrow{\cong} \lim_{\Delta} \mathrm{cc}^\bullet\big((X^N)_f\big) \xrightarrow{\xi} \operatorname*{holim}_{\Delta} \mathrm{cc}^\bullet\big((X^N)_f\big) \xrightarrow{\widetilde{i}_{(X^N)_f}} \operatorname*{holim}_{\Delta} \mathrm{Map}_c\big((G/N)^\bullet, (X^N)_f\big)$$

that is a weak equivalence in Spt. Notice that the target of Ψ_N is a delta-discrete G/N–spectrum.

Now suppose that $N < N'$ are open normal subgroups of G. Also, let Y_N and $Y_{N'}$ be a G/N–spectrum and a G/N'–spectrum, respectively, and suppose that $Y_{N'} \to Y_N$ is a G/N–equivariant map of spectra, where $Y_{N'}$ is a G/N–spectrum, thanks to the canonical map $G/N \to G/N'$. Then this input data yields the composition

$$\mathrm{Map}_c(G/N', Y_{N'}) \to \mathrm{Map}_c(G/N, Y_{N'}) \to \mathrm{Map}_c(G/N, Y_N)$$

(the middle term can be formed because $Y_{N'}$ is a G/N–spectrum): this composition is again a G/N–equivariant map from a G/N'–spectrum to a G/N–spectrum. Applying this conclusion iteratively to the G/N–equivariant map $(X^{N'})_f \to (X^N)_f$ (which is from the filtered diagram $\{(X^N)_f\}_{N \lhd_o G}$ that is used to form $\widehat{X}$) gives a filtered diagram $\{\mathrm{Map}_c\big((G/N)^\bullet, (X^N)_f\big)\}_{N \lhd_o G}$ of cosimplicial discrete G–spectra, and hence, we can form the G–equivariant map

$$\operatorname*{colim}_{N \lhd_o G} \Psi_N : \widehat{X} = \operatorname*{colim}_{N \lhd_o G} (X^N)_f \to \operatorname*{colim}_{N \lhd_o G} \operatorname*{holim}_{\Delta} \mathrm{Map}_c\big((G/N)^\bullet, (X^N)_f\big),$$

with target equal to a discrete G–spectrum.

Since each map Ψ_N is a weak equivalence between fibrant spectra, the filtered colimit $\operatorname*{colim}_{N \lhd_o G} \Psi_N$ is also a weak equivalence of spectra. Then the desired weak equivalence $\widetilde{\Psi}$ is given by the composite

$$X \xrightarrow[\psi]{\simeq} \widehat{X} \xrightarrow[\operatorname*{colim}_{N \lhd_o G} \Psi_N]{\simeq} \operatorname*{colim}_{N \lhd_o G} \operatorname*{holim}_{\Delta} \mathrm{Map}_c\big((G/N)^\bullet, (X^N)_f\big)$$

of weak equivalences. □

Remark 2.5 Let G and X be as in Corollary 2.4: we point out a nice property of this result's "colimit presentation" of X. Let N be any open normal subgroup of G. There are isomorphisms

$$\begin{aligned}\operatorname*{holim}_{\Delta} \mathrm{Map}_c\big((G/N)^\bullet, (X^N)_f\big) &\cong \operatorname*{colim}_{L \lhd_o G/N} \Big(\operatorname*{holim}_{\Delta} \mathrm{Map}_c\big((G/N)^\bullet, (X^N)_f\big)\Big)^L \\ &\cong \operatorname*{holim}_{\Delta}{}^{G/N} \mathrm{Map}_c\big((G/N)^\bullet, (X^N)_f\big)\end{aligned}$$

of G/N–spectra, where the last term above is the homotopy limit in the simplicial model category $\mathrm{Spt}_{G/N}$, as defined in Hirschhorn [20, Definition 18.1.8]; the first isomorphism is due to the fact that any G/N–spectrum automatically belongs to $\mathrm{Spt}_{G/N}$; and the second isomorphism applies [6, Theorem 2.3]. By repeated application of [5, Corollary 3.8, Lemma 3.10], $\mathrm{Map}_c\big((G/N)^\bullet, (X^N)_f\big)$ is a fibrant discrete G/N–spectrum in each codegree, so that by [20, Theorem 18.5.2, (2)], the last term above, $\operatorname*{holim}_{\Delta}{}^{G/N} \mathrm{Map}_c\big((G/N)^\bullet, (X^N)_f\big)$, is a fibrant discrete G/N–spectrum. Therefore, in the statement of Corollary 2.4, each spectrum $\operatorname{holim}_{\Delta} \operatorname{Map}_c\big((G/N)^\bullet, (X^N)_f\big)$ is a fibrant discrete G/N–spectrum (which is stronger than being a fibrant object in the model category G/N–Spt).

2.2 Iteration for Spt_G by using $\widehat{\mathrm{Spt}_Q^{\Delta}}$, as $Q = K/H$ varies

Now we consider the iterated homotopy fixed points of discrete G–spectra by using the setting of delta-discrete K/H–spectra. Here, as in the Introduction, H and K are closed subgroups of G, with $H \lhd K$. We begin with a definition and some preliminary observations.

Definition 2.6 If $Y^\bullet$ is a cosimplicial discrete G–spectrum, such that Y^n is fibrant in Spt_G, for each $[n] \in \Delta$, then we call $Y^\bullet$ a *cosimplicial fibrant discrete G–spectrum.*

The next result is from [6, proof of Theorem 3.5].

Lemma 2.7 *If G is any profinite group and $X \in \mathrm{Spt}_G$, then the cosimplicial discrete G–spectrum $\operatorname{Map}_c(G^\bullet, \widehat{X})$ is a cosimplicial fibrant discrete L–spectrum, for every closed subgroup L of G.*

For the rest of this section, we assume that G has finite vcd and, as usual, X is a discrete G–spectrum. As in Lemma 2.7, let L be any closed subgroup of G. Then, by [5, Remark 7.13] and [6, Definition 5.1, Theorem 5.2] (the latter citation sets the former on a stronger footing), there is an identification

$$X^{hL} = \operatorname{holim}_{\Delta} \operatorname{Map}_c(G^\bullet, \widehat{X})^L. \tag{2.8}$$

(Notice that, under this identification, $X^{hL} \cong (X_\delta)^L$.)

Lemma 2.9 *For each L, there is a weak equivalence $X^{hL} \xrightarrow{\simeq} (X_\delta)^{h_\delta L}$.*

Proof By Lemma 2.7, the fibrant replacement map

$$\operatorname{Map}_c(G^\bullet, \widehat{X})^n \xrightarrow{\simeq} (\operatorname{Map}_c(G^\bullet, \widehat{X})^n)_{fL}$$

is a weak equivalence between fibrant objects in Spt_L, for each $[n] \in \Delta$, so that

$$(\operatorname{Map}_c(G^\bullet, \widehat{X})^n)^L \xrightarrow{\simeq} ((\operatorname{Map}_c(G^\bullet, \widehat{X})^n)_{fL})^L = (\operatorname{Map}_c(G^\bullet, \widehat{X})^n)^{hL}$$

is a weak equivalence between fibrant objects in Spt. Thus, there is a weak equivalence

$$X^{hL} = \operatorname{holim}_{\Delta} \operatorname{Map}_c(G^\bullet, \widehat{X})^L \xrightarrow{\simeq} (\operatorname{holim}_{\Delta} \operatorname{Map}_c(G^\bullet, \widehat{X}))^{h_\delta L} = (X_\delta)^{h_\delta L}. \qquad \square$$

Now we are ready for the task at hand. It will be useful to recall that in [7, Lemma 4.6], the author showed that the functor

$$(-)^H \colon \mathrm{Spt}_K \to \mathrm{Spt}_{K/H}, \quad Y \mapsto Y^H$$

preserves fibrant objects. Also, since $\mathrm{Map}_c(G^\bullet, \widehat{X})^H$ is a cosimplicial discrete K/H–spectrum that is fibrant in Spt in each codegree, we can immediately conclude the following.

Lemma 2.10 *The spectrum* $X^{hH} = \operatorname{holim}_{\Delta} \mathrm{Map}_c(G^\bullet, \widehat{X})^H$ *is a delta-discrete* K/H*–spectrum.*

The above result implies that

$$(X^{hH})^{h_\delta K/H} = \operatorname*{holim}_{[n]\in\Delta} \big((\mathrm{Map}_c(G^\bullet, \widehat{X})^n)^H\big)^{hK/H}.$$

Theorem 2.11 *There is a weak equivalence* $X^{hK} \xrightarrow{\simeq} (X^{hH})^{h_\delta K/H}$.

Proof Since $\mathrm{Map}_c(G^\bullet, \widehat{X})$ is a cosimplicial fibrant discrete K–spectrum, the diagram $\mathrm{Map}_c(G^\bullet, \widehat{X})^H$ is a cosimplicial fibrant discrete K/H–spectrum. Hence, each fibrant replacement map

$$(\mathrm{Map}_c(G^\bullet, \widehat{X})^n)^H \xrightarrow{\simeq} ((\mathrm{Map}_c(G^\bullet, \widehat{X})^n)^H)_{fK/H}$$

is a weak equivalence between fibrant objects in $\mathrm{Spt}_{K/H}$, so that the induced map

$$\textbf{(2.12)} \qquad \operatorname*{holim}_{[n]\in\Delta} \big((\mathrm{Map}_c(G^\bullet, \widehat{X})^n)^H\big)^{K/H} \xrightarrow{\simeq} \operatorname*{holim}_{[n]\in\Delta} \big(((\mathrm{Map}_c(G^\bullet, \widehat{X})^n)^H)_{fK/H}\big)^{K/H}$$

is a weak equivalence. The weak equivalence in (2.12) is exactly the weak equivalence

$$X^{hK} = \operatorname*{holim}_{[n]\in\Delta} (\mathrm{Map}_c(G^\bullet, \widehat{X})^n)^K \xrightarrow{\simeq} \operatorname*{holim}_{[n]\in\Delta} \big((\mathrm{Map}_c(G^\bullet, \widehat{X})^n)^H\big)^{hK/H} = (X^{hH})^{h_\delta K/H}. \quad \square$$

Lemma 2.10 and Theorem 2.11 show that when G has finite vcd, the iterated homotopy fixed point spectrum $(X^{hH})^{h_\delta K/H}$ is always defined and it is just X^{hK}.

Remark 2.13 Let Y be the discrete $(\mathbb{Z}/p \times \mathbb{Z}_q)$–spectrum discussed in Section 1.1 (and in more detail by Wieland and the author in [7, Appendix A]). Though the $\mathbb{Z}_q$–spectrum $Y^{h\mathbb{Z}/p}$ is not a *discrete* $\mathbb{Z}_q$–spectrum and there is no known construction of $\big(Y^{h\mathbb{Z}/p}\big)^{h\mathbb{Z}_q}$, $Y^{h\mathbb{Z}/p}$ *is* a delta-discrete $\mathbb{Z}_q$–spectrum, by Lemma 2.10, and hence, $\big(Y^{h\mathbb{Z}/p}\big)^{h_\delta \mathbb{Z}_q}$ can be formed, and it is just $Y^{h(\mathbb{Z}/p\times\mathbb{Z}_q)}$, by Theorem 2.11.

3 The homotopy fixed points of delta-discrete G–spectra as a total right derived functor

Let G be any profinite group. In this section, we show that (the output of) the functor $(-)^{h_\delta G} \colon \widehat{\mathrm{Spt}_G^\Delta} \to \mathrm{Spt}$ can be viewed as the total right derived functor of

$$\lim_{\Delta} (-)^G \colon c(\mathrm{Spt}_G) \to \mathrm{Spt}, \quad X^\bullet \mapsto \lim_{[n]\in\Delta} (X^n)^G,$$

where $c(\mathrm{Spt}_G)$ has the injective model category structure (defined below). We will obtain this result as a special case of a more general result. Thus, we let $\mathcal{C}$ denote any small category and we use " $Z^\bullet$ " and " $X^\bullet$ " to denote $\mathcal{C}$–shaped diagrams in Spt and Spt_G, respectively, since we are especially interested in the case when $\mathcal{C} = \Delta$.

Recall that Spt is a combinatorial model category (for the definition of this notion, we refer the reader to the helpful expositions in Dugger [11, Section 2] and Lurie [24, Section A.2.6 (and Definition A.1.1.2)]); this well-known fact is stated explicitly in Rosický [28, page 459]. Therefore, [24, Proposition A.2.8.2] implies that $\mathrm{Spt}^{\mathcal{C}}$, the category of functors $\mathcal{C} \to \mathrm{Spt}$, has an injective model category structure: more precisely, $\mathrm{Spt}^{\mathcal{C}}$ has a model structure in which a map $f^\bullet \colon Z^\bullet \to W^\bullet$ in $\mathrm{Spt}^{\mathcal{C}}$ is a weak equivalence (cofibration) if and only if, for each $C \in \mathcal{C}$, the map $f^C \colon Z^C \to W^C$ is a weak equivalence (cofibration) in Spt.

Similar comments apply to Spt_G. For example, in the proof of [1, Theorem 2.2.1], Behrens and the author apply Hovey [21, Definition 3.3] to obtain the model structure on Spt_G, and thus, Spt_G is a cellular model category. Hence, Spt_G is cofibrantly generated. Also, the category Spt_G is equivalent to the category of sheaves of spectra on the site $G-\mathbf{Sets}_{df}$, as explained in [5, Section 3], and thus, Spt_G is a locally presentable category, since standard arguments show that such a category of sheaves is locally presentable. (For example, see the general comment about such categories in Toën and Vezzosi [30, page 2]. The basic ideas are contained in the proof of the fact that a Grothendieck topos is locally presentable (see, for example, Borceux [3, Proposition 3.4.16]), and hence, the category of sheaves of sets on the aforementioned site is locally presentable.)

From the above considerations, we conclude that Spt_G is a combinatorial model category. Therefore, as before, [24, Proposition A.2.8.2] implies that the category $(\mathrm{Spt}_G)^{\mathcal{C}}$ of $\mathcal{C}$–shaped diagrams in Spt_G has an injective model structure in which a map $h^\bullet$ is a weak equivalence (cofibration) if and only if each map h^C is a weak equivalence (cofibration) in Spt_G.

It will be useful to note that, by [24, Remark A.2.8.5], if $f^\bullet$ is a fibration in $\mathrm{Spt}^{\mathcal{C}}$, then f^C is a fibration in Spt, for every $C \in \mathcal{C}$. Similarly, if $h^\bullet$ is a fibration in $(\mathrm{Spt}_G)^{\mathcal{C}}$, then each map h^C is a fibration in Spt_G.

The functor
$$\lim_{\mathcal{C}}\,(-)^G\colon (\mathrm{Spt}_G)^{\mathcal{C}} \to \mathrm{Spt}, \quad X^\bullet \mapsto \lim_{C\in\mathcal{C}}\,(X^C)^G$$
is right adjoint to the functor $\underline{t}\colon \mathrm{Spt} \to (\mathrm{Spt}_G)^{\mathcal{C}}$ that sends an arbitrary spectrum Z to the constant $\mathcal{C}$–shaped diagram $\underline{t}(Z)$ on the discrete G–spectrum $t(Z)$, where (as in [5, Corollary 3.9])
$$t\colon \mathrm{Spt} \to \mathrm{Spt}_G, \quad Z \mapsto t(Z) = Z$$
is the functor that equips Z with the trivial G–action. If f is a weak equivalence (cofibration) in Spt, then $t(f)$ is a weak equivalence (cofibration) in Spt_G, and hence, $\underline{t}(f)$ is a weak equivalence (cofibration) in $(\mathrm{Spt}_G)^{\mathcal{C}}$. This observation immediately gives the following result.

Lemma 3.1 *The functors* $(\underline{t}, \lim_{\mathcal{C}}(-)^G)$ *are a Quillen pair for* $(\mathrm{Spt}, (\mathrm{Spt}_G)^{\mathcal{C}})$.

We let
$$(-)_{\mathcal{F}ib}\colon (\mathrm{Spt}_G)^{\mathcal{C}} \to (\mathrm{Spt}_G)^{\mathcal{C}}, \quad X^\bullet \mapsto (X^\bullet)_{\mathcal{F}ib}$$
denote a fibrant replacement functor, such that there is a morphism $X^\bullet \to (X^\bullet)_{\mathcal{F}ib}$ in $(\mathrm{Spt}_G)^{\mathcal{C}}$ that is a natural trivial cofibration, with $(X^\bullet)_{\mathcal{F}ib}$ fibrant, in $(\mathrm{Spt}_G)^{\mathcal{C}}$. Then Lemma 3.1 implies the existence of the total right derived functor
$$\mathbf{R}\big(\lim_{\mathcal{C}}\,(-)^G\big)\colon \mathrm{Ho}\big((\mathrm{Spt}_G)^{\mathcal{C}}\big) \to \mathrm{Ho}(\mathrm{Spt}), \quad X^\bullet \mapsto \lim_{\mathcal{C}}\,((X^\bullet)_{\mathcal{F}ib})^G.$$

Now we prove the key result that will allow us to relate Definition 1.7 to the total right derived functor $\mathbf{R}\big(\lim_{\Delta}\,(-)^G\big)$.

Theorem 3.2 *Given* $X^\bullet$ *in* $(\mathrm{Spt}_G)^{\mathcal{C}}$, *the canonical map*
$$\Big(\mathbf{R}\big(\lim_{\mathcal{C}}\,(-)^G\big)\Big)\big(X^\bullet\big) = \lim_{\mathcal{C}}\,((X^\bullet)_{\mathcal{F}ib})^G \xrightarrow{\simeq} \operatorname*{holim}_{\mathcal{C}}\,((X^\bullet)_{\mathcal{F}ib})^G$$
is a weak equivalence of spectra.

Proof Let $t^{\mathcal{C}}\colon \mathrm{Spt}^{\mathcal{C}} \to (\mathrm{Spt}_G)^{\mathcal{C}}$ be the functor that sends $Z^\bullet$ to $t^{\mathcal{C}}(Z^\bullet) = Z^\bullet$, where each $(t^{\mathcal{C}}(Z^\bullet))(C) = Z^C$ is regarded as having the trivial G–action. Then $t^{\mathcal{C}}$ preserves weak equivalences and cofibrations, so that the right adjoint of $t^{\mathcal{C}}$, the fixed points functor $(-)^G\colon (\mathrm{Spt}_G)^{\mathcal{C}} \to \mathrm{Spt}^{\mathcal{C}}$, preserves fibrant objects. Thus, the diagram $((X^\bullet)_{\mathcal{F}ib})^G$ is fibrant in $\mathrm{Spt}^{\mathcal{C}}$.

Let $Z^\bullet$ be a fibrant object in $\mathrm{Spt}^{\mathcal{C}}$. Then, to complete the proof, it suffices to know that the canonical map

$$\lambda\colon \lim_{\mathcal{C}} Z^\bullet \to \operatorname*{holim}_{\mathcal{C}} Z^\bullet$$

is a weak equivalence: this conclusion follows from Theorem 3.5 below. $\square$

Given $X^\bullet \in (\mathrm{Spt}_G)^{\mathcal{C}}$, we let $((X^\bullet)_{\mathcal{Fib}})^C$ denote the value of $(X^\bullet)_{\mathcal{Fib}}\colon \mathcal{C} \to \mathrm{Spt}_G$ on the object $C \in \mathcal{C}$.

Lemma 3.3 *If $X^\bullet$ is an object in $(\mathrm{Spt}_G)^{\mathcal{C}}$, then there is a weak equivalence*

$$\operatorname*{holim}_{C\in\mathcal{C}} (X^C)^{hG} \xrightarrow{\simeq} \operatorname*{holim}_{C\in\mathcal{C}} \big(((X^\bullet)_{\mathcal{Fib}})^C\big)^G.$$

Proof Let $(X^\bullet)_{fG}$ be the object in $(\mathrm{Spt}_G)^{\mathcal{C}}$ that is equal to the composition of functors

$$(-)_{fG} \circ (X^\bullet)\colon \mathcal{C} \to \mathrm{Spt}_G, \quad C \mapsto (X^C)_{fG}.$$

Since $X^\bullet \to (X^\bullet)_{fG}$ is a trivial cofibration in $(\mathrm{Spt}_G)^{\mathcal{C}}$, the fibrant object $(X^\bullet)_{\mathcal{Fib}}$ induces a weak equivalence

$$\ell^\bullet\colon (X^\bullet)_{fG} \xrightarrow{\simeq} (X^\bullet)_{\mathcal{Fib}}$$

in $(\mathrm{Spt}_G)^{\mathcal{C}}$. Therefore, since $(X^C)_{fG}$ and $((X^\bullet)_{\mathcal{Fib}})^C$ are fibrant discrete G–spectra, for each $C \in \mathcal{C}$, there is a weak equivalence

$$\operatorname*{holim}_{C\in\mathcal{C}} (\ell^C)^G\colon \operatorname*{holim}_{C\in\mathcal{C}} (X^C)^{hG} = \operatorname*{holim}_{C\in\mathcal{C}} ((X^C)_{fG})^G \xrightarrow{\simeq} \operatorname*{holim}_{C\in\mathcal{C}} \big(((X^\bullet)_{\mathcal{Fib}})^C\big)^G. \qquad \square$$

By letting $\mathcal{C} = \Delta$, Theorem 3.2 and Lemma 3.3 immediately yield the next result, which allows us to conclude that the *homotopy* fixed points functor for delta-discrete G–spectra, $(-)^{h_\delta G}$, can indeed be regarded as the total right derived functor of fixed points, in the appropriate sense.

Theorem 3.4 *If $\operatorname*{holim}_{\Delta} X^\bullet$ is a delta-discrete G–spectrum, then there is a zigzag*

$$(\operatorname*{holim}_{\Delta} X^\bullet)^{h_\delta G} \xrightarrow{\simeq} \operatorname*{holim}_{\Delta} ((X^\bullet)_{\mathcal{Fib}})^G \xleftarrow{\simeq} \big(\mathbf{R}\big(\lim_{\Delta} (-)^G\big)\big)(X^\bullet)$$

of weak equivalences in Spt.

Now we give Theorem 3.5, which was used in the proof of Theorem 3.2 and will also be helpful in the proof of Theorem 9.9. Though we are not able to point to a place where Theorem 3.5 appears in the literature, it is well-known and the proof below, which is not hard, uses a well-established special case and follows a script suggested by the "projective arguments" of Jardine [23, page 79: "**hom**(, Y) satisfies

F3"] and Casacuberta and Chorny [4, Lemma 1.2]. Since the result is quite useful (see Remark 3.6), we felt it was worthwhile to run through a proof. The argument below that $\mathrm{Map}(N, M^\bullet)$ is fibrant in $\mathcal{S}^{\mathcal{C}}$ follows an argument in Lurie [24, proof of Proposition A.3.3.12] (which is written in an enriched setting).

Theorem 3.5 *Let $\mathcal{C}$ be a small category and let $\mathcal{M}$ be a simplicial model category with all small limits and colimits. Suppose that the injective model structure exists for the diagram category $\mathcal{M}^{\mathcal{C}}$. If $M^\bullet$ is fibrant in $\mathcal{M}^{\mathcal{C}}$ (when it is equipped with the injective model structure), then the canonical map*

$$\lambda\colon \lim_{\mathcal{C}} M^\bullet \xrightarrow{\simeq} \operatorname*{holim}_{\mathcal{C}} M^\bullet$$

is a weak equivalence in $\mathcal{M}$, where $\operatorname*{holim}_{\mathcal{C}} M^\bullet$ is defined as in Hirschhorn [20, Definition 18.1.8].

Proof Since the functor $\mathcal{M} \to \mathcal{M}^{\mathcal{C}}$ that sends an object M to the constant $\mathcal{C}$–shaped diagram on M preserves weak equivalences and cofibrations, its right adjoint $\lim_{\mathcal{C}}(-)\colon \mathcal{M}^{\mathcal{C}} \to \mathcal{M}$ is a right Quillen functor. Thus, $\lim_{\mathcal{C}} M^\bullet$ is fibrant in $\mathcal{M}$.

Given $C \in \mathcal{C}$, the evaluation functor $\mathrm{ev}_C\colon \mathcal{M}^{\mathcal{C}} \to \mathcal{M}$ that sends a diagram $N^\bullet$ to N^C has a left adjoint F^C such that for each map f in $\mathcal{M}$, $\mathrm{ev}_D(F^C(f))$ is a coproduct of copies of f (indexed by the set $\mathcal{C}(C, D)$), for each $D \in \mathcal{C}$, and hence, F^C preserves cofibrations and trivial cofibrations. Thus, ev_C is a right Quillen functor, implying that M^C is fibrant in $\mathcal{M}$, for each $C \in \mathcal{C}$. Therefore, we can conclude that $\operatorname{holim}_{\mathcal{C}} M^\bullet$ is fibrant in $\mathcal{M}$.

Because λ is a map between fibrant objects, to show that it is a weak equivalence, it suffices to show that in $\mathcal{S}$, the model category of simplicial sets, the induced map

$$\lambda^N\colon \lim_{\mathcal{C}} \mathrm{Map}(N, M^\bullet) \cong \mathrm{Map}(N, \lim_{\mathcal{C}} M^\bullet) \to \mathrm{Map}(N, \operatorname*{holim}_{\mathcal{C}} M^\bullet) \cong \operatorname*{holim}_{\mathcal{C}} \mathrm{Map}(N, M^\bullet)$$

is a weak equivalence for every cofibrant object N in $\mathcal{M}$. Given M and M' in $\mathcal{M}$, we are using $\mathrm{Map}(M, M')$ to denote the usual mapping space in $\mathcal{S}$.

By Goerss and Jardine [17, page 407: proof of Lemma 2.11], to show that λ^N is a weak equivalence, we only need to show that $\mathrm{Map}(N, M^\bullet)$ is fibrant in $\mathcal{S}^{\mathcal{C}}$, when it is equipped with the injective model structure. Thus, it is enough to show that in $\mathcal{S}^{\mathcal{C}}$ the diagram $\mathrm{Map}(N, M^\bullet)$ has the right lifting property with respect to every trivial cofibration $i\colon K^\bullet \to L^\bullet$. Then by adjunction, we only need to verify that in $\mathcal{M}^{\mathcal{C}}$ the diagram $M^\bullet$ has the right lifting property with respect to $N \otimes i\colon N \otimes K^\bullet \to N \otimes L^\bullet$, where, for example, $N \otimes K^\bullet$ is the $\mathcal{C}$–shaped diagram $\{N \otimes K^C\}_{C \in \mathcal{C}}$. Since N is cofibrant and i is a trivial cofibration, $N \otimes i$ is a trivial cofibration in $\mathcal{M}^{\mathcal{C}}$, and hence, $M^\bullet$ has the desired lifting property. □

Remark 3.6 Suppose that all the hypotheses of Theorem 3.5 are satisfied. Then by the opening observation in its proof, we can form the total right derived functor

$$\mathbf{R}\big(\lim_{\mathcal{C}}(-)\big)\colon \mathrm{Ho}\big(\mathcal{M}^{\mathcal{C}}\big) \to \mathrm{Ho}(\mathcal{M}), \quad N^\bullet \mapsto \lim_{\mathcal{C}}(N^\bullet)_{\mathcal{F}ib},$$

for some $N^\bullet \to (N^\bullet)_{\mathcal{F}ib}$, a trivial cofibration to a fibrant target, in $\mathcal{M}^{\mathcal{C}}$. Thus, there is a weak equivalence

$$\big(\mathbf{R}\big(\lim_{\mathcal{C}}(-)\big)\big)\big(N^\bullet\big) = \lim_{\mathcal{C}}(N^\bullet)_{\mathcal{F}ib} \xrightarrow{\simeq} \operatorname{holim}_{\mathcal{C}}(N^\bullet)_{\mathcal{F}ib},$$

reprising the verification that two well-known models for the "homotopically sound homotopy limit of $N^\bullet$" – the source and target of the above weak equivalence – are equivalent to each other. This observation is not original and we refer the interested reader to Dwyer, Hirschhorn, Kan and Smith [12, Sections 20, 21, Chapter VIII] for a more general perspective on homotopy limits.

4 Homotopy fixed points for delta-discrete G–spectra are the right approximation of fixed points

In this section, we continue to let G be any profinite group and we let

$$(-)^G\colon \widehat{\mathrm{Spt}_G^\Delta} \to \mathrm{Spt}, \quad \operatorname{holim}_\Delta X^\bullet \mapsto \big(\operatorname{holim}_\Delta X^\bullet\big)^G$$

be the fixed points functor. In the previous section, we showed that given a delta-discrete G–spectrum, its homotopy fixed point spectrum is the output of the total right derived functor $\mathbf{R}\big(\lim_\Delta(-)^G\big)$, a functor out of the category $\mathrm{Ho}\big(c(\mathrm{Spt}_G)\big)$. But since the source and target categories for the homotopy fixed points functor $(-)^{h_\delta G}\colon \widehat{\mathrm{Spt}_G^\Delta} \to \mathrm{Spt}$ are homotopical categories and $(-)^{h_\delta G}$ is a homotopical functor, there is something else that we expect of $(-)^{h_\delta G}$: it ought to be a homotopical functor that is "closest to $(-)^G$ from the right" in a homotopy-theoretic sense. In this section, we show that this is indeed true in a precise sense: to do this, we freely use the language of homotopical categories, as in Dwyer, Hirschhorn, Kan and Smith [12].

In this section only, due to its frequent usage, we sometimes use $h\big(X^\bullet\big)$ to denote the object $\operatorname{holim}_\Delta X^\bullet$ in $\widehat{\mathrm{Spt}_G^\Delta}$. Also, it is helpful to recall that, as objects in $\widehat{\mathrm{Spt}_G^\Delta}$, $h\big(X^\bullet\big) = h\big(Y^\bullet\big)$ if and only if $X^\bullet = Y^\bullet$ in $c(\mathrm{Spt}_G)$, so that it is sometimes useful to think of $h\big(X^\bullet\big)$ as the pair $\big(X^\bullet, h\big(X^\bullet\big)\big)$. One can make a similar statement about morphisms in $\widehat{\mathrm{Spt}_G^\Delta}$.

Since $\big(\operatorname{holim}_{\Delta} X^\bullet\big)^G \cong \operatorname{holim}_{\Delta} (X^\bullet)^G$, we take the barely noticeable step of identifying the fixed points functor $(-)^G \colon \widehat{\mathrm{Spt}_G^\Delta} \to \mathrm{Spt}$ with the functor $\widehat{(\text{-})^G} \colon \widehat{\mathrm{Spt}_G^\Delta} \to \mathrm{Spt}$ that sends $\operatorname{holim}_{\Delta} X^\bullet$ to $\operatorname{holim}_{\Delta} (X^\bullet)^G$. Thus, we henceforth use the notation

$$(-)^G \colon \widehat{\mathrm{Spt}_G^\Delta} \to \mathrm{Spt}, \quad \operatorname{holim}_{\Delta} X^\bullet \mapsto \operatorname{holim}_{[n]\in\Delta} (X^n)^G$$

for the fixed points functor.

Let $\widehat{\mathrm{id}} : \widehat{\mathrm{Spt}_G^\Delta} \to \widehat{\mathrm{Spt}_G^\Delta}$ be the identity functor and notice that the functor

$$\widehat{(\text{-})_{fG}} \colon \widehat{\mathrm{Spt}_G^\Delta} \to \widehat{\mathrm{Spt}_G^\Delta}, \quad h\big(X^\bullet\big) \mapsto h\big((X^\bullet)_{fG}\big) = \operatorname{holim}_{[n]\in\Delta} (X^n)_{fG}$$

is a homotopical functor. (Above, each $(X^n)_{fG}$ is a fibrant spectrum, by [5, Lemma 3.10], so that $h\big((X^\bullet)_{fG}\big)$ is a delta-discrete G–spectrum, as required.) As mentioned in Section 1.1, given $X \in \mathrm{Spt}_G$, there is a natural trivial cofibration $X \to X_{fG}$ in Spt_G, and hence, there is a natural transformation

$$\eta_{(\text{-})} \colon \widehat{\mathrm{id}} \to \widehat{(\text{-})_{fG}}, \quad \eta_{h(X^\bullet)} \colon \operatorname{holim}_{\Delta} X^\bullet \to \operatorname{holim}_{\Delta} (X^\bullet)_{fG}.$$

Since each map $X^n \to (X^n)_{fG}$ is a weak equivalence in Spt_G, the map $\eta_{h(X^\bullet)}$ is a weak equivalence in $\widehat{\mathrm{Spt}_G^\Delta}$, so that $\eta_{(\text{-})}$ is a natural weak equivalence (in the sense of [12, 33.1, (iv)]). These observations show that the pair $\big(\widehat{(\text{-})_{fG}}, \eta_{(\text{-})}\big)$ is a right deformation of $\widehat{\mathrm{Spt}_G^\Delta}$.

Lemma 4.1 *The pair $\big(\widehat{(\text{-})_{fG}}, \eta_{(\text{-})}\big)$ is a right $(-)^G$–deformation of $\widehat{\mathrm{Spt}_G^\Delta}$.*

Proof Let $h\big((X^\bullet)_{fG}\big)$ and $h\big((Y^\bullet)_{fG}\big)$ be arbitrary objects in the image of the functor $\widehat{(\text{-})_{fG}}$. Given a weak equivalence $f^\bullet \colon (X^\bullet)_{fG} \to (Y^\bullet)_{fG}$ in $c(\mathrm{Spt}_G)$ (we are not assuming that $f^\bullet$ is obtained by applying $(-)_{fG}$ to some map $X^\bullet \to Y^\bullet$ in $c(\mathrm{Spt}_G)$), the map

$$\big((-)^G\big)\big(\operatorname{holim}_{\Delta} f^\bullet\big) \colon \operatorname{holim}_{\Delta}\big((X^\bullet)_{fG}\big)^G \xrightarrow{\simeq} \operatorname{holim}_{\Delta}\big((Y^\bullet)_{fG}\big)^G$$

is a weak equivalence of spectra. Therefore, the functor $(-)^G$ is homotopical on the full subcategory spanned by the image of $\widehat{(\text{-})_{fG}}$ (it is automatic that this full subcategory is a homotopical category), giving the desired conclusion. □

The above result immediately implies that the functor $(-)^G \colon \widehat{\mathrm{Spt}_G^\Delta} \to \mathrm{Spt}$ is right deformable. Notice that there is the natural transformation

$$\big(\eta_{(\text{-})}\big)^G \colon (-)^G \to (-)^{h_\delta G}, \quad h\big(X^\bullet\big) \mapsto \Big[\big(\eta_{h(X^\bullet)}\big)^G \colon \operatorname{holim}_{\Delta} (X^\bullet)^G \to \operatorname{holim}_{\Delta}\big((X^\bullet)_{fG}\big)^G\Big].$$

Corollary 4.2 *Let G be any profinite group. Then the pair $\big((-)^{h_\delta G}, \big(\eta_{(\text{-})}\big)^G\big)$ is a right approximation of the functor $(-)^G \colon \widehat{\mathrm{Spt}_G^\Delta} \to \mathrm{Spt}$.*

Proof By [12, 41.2, (ii)], Lemma 4.1 implies that the pair $\big((-)^G \circ \widehat{(\text{-})}_{fG}, \big(\eta_{(\text{-})}\big)^G\big)$ is a right approximation of $(-)^G$, and this pair is identical to the desired pair. $\square$

Let $\widehat{\gamma_G} \colon \widehat{\mathrm{Spt}_G^\Delta} \to \mathrm{Ho}\big(\widehat{\mathrm{Spt}_G^\Delta}\big)$ and $\gamma \colon \mathrm{Spt} \to \mathrm{Ho}(\mathrm{Spt})$ be the usual localization functors to the corresponding homotopy categories. Also, let

$$\mathrm{Ho}\big((-)^{h_\delta G}\big) \colon \mathrm{Ho}\big(\widehat{\mathrm{Spt}_G^\Delta}\big) \to \mathrm{Ho}(\mathrm{Spt})$$

be the usual functor on the respective homotopy categories that is induced by the homotopical functor $(-)^{h_\delta G}$ and let

$$\gamma\big(\big(\eta_{(\text{-})}\big)^G\big) \colon \gamma \circ (-)^G \to \gamma \circ (-)^{h_\delta G}$$

be the natural transformation induced by γ between the stated composite functors from $\widehat{\mathrm{Spt}_G^\Delta}$ to $\mathrm{Ho}(\mathrm{Spt})$. Then the above results and [12, 41.5] immediately give the following result.

Theorem 4.3 *For any profinite group G, the pair $\big(\mathrm{Ho}\big((-)^{h_\delta G}\big), \gamma\big(\big(\eta_{(\text{-})}\big)^G\big)\big)$ is a total right derived functor of $(-)^G \colon \widehat{\mathrm{Spt}_G^\Delta} \to \mathrm{Spt}$.*

The above result gives the desired conclusion: in the context of homotopical categories, delta-discrete *homotopy* fixed points (more precisely, the functor $\mathrm{Ho}\big((-)^{h_\delta G}\big)$) are the total right derived functor $\mathbf{R}\big((-)^G\big) \colon \mathrm{Ho}\big(\widehat{\mathrm{Spt}_G^\Delta}\big) \to \mathrm{Ho}(\mathrm{Spt})$.

5 Several properties of the homotopy fixed points of delta-discrete G–spectra

Suppose that P is a finite group and let Z be a P–spectrum. Recall (for example, from [5, Section 5]) that if Z' is a P–spectrum and a fibrant object in Spt, with a map $Z \xrightarrow{\simeq} Z'$ that is P–equivariant and a weak equivalence in Spt, then $Z^{h'P}$, the usual homotopy fixed point spectrum $\mathrm{Map}_P(EP_+, Z')$ in the case when P is a finite discrete group, can also be defined as

$$Z^{h'P} = \operatorname*{holim}_P Z'. \qquad \textbf{(5.1)}$$

Then the following result shows that the homotopy fixed points $(-)^{h_\delta G}$ of Definition 1.7 agree with those of (**5.1**), when the profinite group G is finite and discrete.

Theorem 5.2 *Let G be a finite discrete group and let $\operatorname{holim}_\Delta X^\bullet$ be a delta-discrete G–spectrum (that is, $X^\bullet$ is a cosimplicial G–spectrum, with each X^n a fibrant spectrum). Then there is a weak equivalence*

$$(\operatorname*{holim}_\Delta X^\bullet)^{h_\delta G} \xrightarrow{\simeq} (\operatorname*{holim}_\Delta X^\bullet)^{h'G}.$$

Proof Given $[n] \in \Delta$, by Jardine [22, Proposition 6.39], the canonical map

$$(X^n)^{hG} = ((X^n)_{fG})^G \cong \lim_G (X^n)_{fG} \xrightarrow{\simeq} \operatorname*{holim}_G (X^n)_{fG}$$

is a weak equivalence. Also, notice that the target (since $(X^n)_{fG}$ is a fibrant spectrum, by [5, Lemma 3.10]) and the source of this weak equivalence are fibrant spectra. Thus, there is a weak equivalence

$$(\operatorname*{holim}_\Delta X^\bullet)^{h_\delta G} = \operatorname*{holim}_{[n]\in\Delta} (X^n)^{hG} \xrightarrow{\simeq} \operatorname*{holim}_{[n]\in\Delta} \operatorname*{holim}_G (X^n)_{fG} \cong \operatorname*{holim}_G \operatorname*{holim}_{[n]\in\Delta} (X^n)_{fG}.$$

The proof is finished by noting that

$$\operatorname*{holim}_G \operatorname*{holim}_{[n]\in\Delta} (X^n)_{fG} = (\operatorname*{holim}_\Delta X^\bullet)^{h'G};$$

this equality, which is an application of (**5.1**), is due to the fact that the map

$$\operatorname*{holim}_\Delta X^\bullet \xrightarrow{\simeq} \operatorname*{holim}_{[n]\in\Delta} (X^n)_{fG}$$

is G–equivariant and a weak equivalence (since each map $X^n \to (X^n)_{fG}$ is a weak equivalence between fibrant objects in Spt), with target a fibrant spectrum. □

Now let G be any profinite group and let X be a discrete G–spectrum. We will show that there is a G–equivariant map $X \to \mathrm{cc}_G(X)$ that is a weak equivalence of spectra, along with a weak equivalence $X^{hG} \to (\mathrm{cc}_G(X))^{h_\delta G}$. Since these weak equivalences exist for any G and all $X \in \mathrm{Spt}_G$, we can think of the category of delta-discrete G–spectra as being a generalization of the category Spt_G.

Given any spectrum Z, it is not hard to see that there is an isomorphism

$$\mathrm{Tot}(\mathrm{cc}^\bullet(Z)) \cong Z;$$

this was noted, for example, in the setting of simplicial sets, in Dwyer, Miller and Neisendorfer [13, Section 1] and is verified for an arbitrary simplicial model category in Hess [19, Remark B.16]. Since the Reedy category Δ has fibrant constants (see Hirschhorn [20, Corollary 15.10.5]), the canonical map $\mathrm{Tot}(\mathrm{cc}^\bullet(Z)) \to \operatorname{holim}_\Delta \mathrm{cc}^\bullet(Z)$ is a weak equivalence, whenever Z is a fibrant spectrum, by [20, Theorem 18.7.4, (2)]. Thus, if Z is a fibrant spectrum, there is a weak equivalence

$$\phi_Z\colon Z \cong \mathrm{Tot}(\mathrm{cc}^\bullet(Z)) \xrightarrow{\simeq} \operatorname*{holim}_\Delta \mathrm{cc}^\bullet(Z)$$

in Spt. In particular, since the discrete G–spectrum $\widehat{X}$ is a fibrant spectrum, the map $\phi_{\widehat{X}}$ is a weak equivalence. Therefore, since the map $\psi\colon X \to \widehat{X}$ is a weak equivalence in Spt_G (by Lemma 2.1), the G–equivariant map

$$\phi_{\widehat{X}} \circ \psi\colon X \xrightarrow{\simeq} \widehat{X} \xrightarrow{\simeq} \operatorname*{holim}_{\Delta} \mathrm{cc}^\bullet(\widehat{X}) = \mathrm{cc}_G(X)$$

and the map

$$\phi_{(\widehat{X})^{hG}} \circ (\psi)^{hG}\colon X^{hG} \xrightarrow{\simeq} (\widehat{X})^{hG} \xrightarrow{\simeq} \operatorname*{holim}_{\Delta} \mathrm{cc}^\bullet\big((\widehat{X})^{hG}\big) = (\mathrm{cc}_G(X))^{h_\delta G}$$

are weak equivalences (the map $\phi_{(\widehat{X})^{hG}}$ is a weak equivalence because $(\widehat{X})^{hG}$ is a fibrant spectrum).

Remark 5.3 The weak equivalences $\phi_{(X^N)_f}$, for $N \lhd_o G$, combined with the argument for Corollary 2.4, show that there is a weak equivalence

$$X \xrightarrow{\simeq} \operatorname*{colim}_{N \lhd_o G} \operatorname*{holim}_{\Delta} \mathrm{cc}^\bullet\big((X^N)_f\big)$$

in Spt_G, and, as in Corollary 2.4, this "colimit presentation" of X comes from a filtered diagram (the diagram $\{\operatorname*{holim}_{\Delta} \mathrm{cc}^\bullet\big((X^N)_f\big)\}_{N \lhd_o G}$) of smaller delta-discrete G–spectra. However, this colimit presentation does not have one of the nice properties that is possessed by the presentation of Corollary 2.4 (see Remark 2.5): in general, $(X^N)_f$ is not a fibrant discrete G/N–spectrum, so that, in general,

$$\operatorname*{holim}_{\Delta} \mathrm{cc}^\bullet\big((X^N)_f\big) \cong \operatorname*{holim}_{\Delta}{}^{G/N} \mathrm{cc}^\bullet\big((X^N)_f\big)$$

need not be a fibrant discrete G/N–spectrum.

Remark 5.4 It is easy to see that the weak equivalence $\Psi\colon X \xrightarrow{\simeq} X_\delta$ (from Definition 2.2) has the canonical factorization

$$X \xrightarrow{\simeq} \lim_{\Delta} \mathrm{cc}^\bullet(\widehat{X}) \xrightarrow{\cong} \mathrm{Tot}(\mathrm{cc}^\bullet(\widehat{X})) \xrightarrow{\simeq} \operatorname*{holim}_{\Delta} \mathrm{cc}^\bullet(\widehat{X}) \xrightarrow{\widetilde{i}_{\widehat{X}}} X_\delta,$$

and hence, the map $\widetilde{i}_{\widehat{X}}$, a morphism in $\widehat{\mathrm{Spt}_G^\Delta}$, is a weak equivalence in Spt, but, interestingly, the map $\widetilde{i}_{\widehat{X}}$, in general, is not a weak equivalence in $\widehat{\mathrm{Spt}_G^\Delta}$, since (for example), in codegree 0, $\pi_t(-)$ applied to the map of diagrams "behind" $\widetilde{i}_{\widehat{X}}$ can be identified with the monomorphism, $i\colon \pi_t(X) \to \mathrm{Map}_c(G, \pi_t(X))$, so that the map behind $\widetilde{i}_{\widehat{X}}$ is not, in general, an objectwise weak equivalence in $c(\mathrm{Spt}_G)$.

6 Iterated homotopy fixed points for delta-discrete G–spectra

Throughout this section (except in Convention 6.1), we assume that the profinite group G has finite vcd. We will show that G–homotopy fixed points for delta-discrete G–spectra have the transitivity property. To do this, we make use of the convention stated below.

Convention 6.1 Let P be a discrete group and let $\mathcal{X}$ be a space. By Dwyer and Wilkerson [14, Remark 10.3], a *proxy action* of P on $\mathcal{X}$ is a space $\mathcal{Y}$ that is homotopy equivalent to $\mathcal{X}$ and has an action of P. Then [14, Remark 10.3] establishes the convention that $\mathcal{X}^{hP}$ is *equal* to $\mathcal{Y}^{hP}$ and a proxy action is sometimes referred to as an action. This convention is an important one: for example, in [14, Section 10], this convention plays a role in Lemmas 10.4 and 10.6 and in the proof that P–homotopy fixed points have the transitivity property (their Lemma 10.5). Thus, in this section, we will make use of the related convention described below.

Let G be any profinite group and let $X^{\bullet,\bullet}$ be a bicosimplicial discrete G–spectrum (that is, $X^{\bullet,\bullet}$ is a cosimplicial object in $c(\mathrm{Spt}_G)$), such that, for all $m,n \geq 0$, $X^{m,n}$ is a fibrant spectrum. Let $\{X^{n,n}\}_{[n]\in\Delta}$ be the cosimplicial discrete G–spectrum that is the diagonal of $X^{\bullet,\bullet}$: $\{X^{n,n}\}_{[n]\in\Delta}$ is defined to be the composition

$$\Delta \to \Delta \times \Delta \to \mathrm{Spt}_G, \quad [n] \mapsto ([n],[n]) \mapsto X^{n,n}.$$

Then there is a natural G–equivariant map

$$\operatorname*{holim}_{\Delta\times\Delta} X^{\bullet,\bullet} \xrightarrow{\simeq} \operatorname*{holim}_{[n]\in\Delta} X^{n,n} \tag{6.2}$$

that is a weak equivalence (see, for example, Thomason [29, Lemma 5.33] and Hirschhorn [20, Remark 19.1.6; Theorem 19.6.7, (2)]). Notice that the target of (6.2), $\operatorname*{holim}_{[n]\in\Delta} X^{n,n}$, is a delta-discrete G–spectrum. Thus, we identify the source of (6.2), the G–spectrum $\operatorname*{holim}_{\Delta\times\Delta} X^{\bullet,\bullet}$, with the delta-discrete G–spectrum $\operatorname*{holim}_{[n]\in\Delta} X^{n,n}$, so that

$$(\operatorname*{holim}_{\Delta\times\Delta} X^{\bullet,\bullet})^{h_\delta G} := (\operatorname*{holim}_{[n]\in\Delta} X^{n,n})^{h_\delta G}. \tag{6.3}$$

Let $\operatorname*{holim}_{\Delta\times\Delta} (X^{\bullet,\bullet})^{hG}$ denote $\operatorname*{holim}_{([m],[n])\in\Delta\times\Delta} (X^{m,n})^{hG}$ and notice that, by Theorem 3.2 and Lemma 3.3, there is a zigzag of weak equivalences

$$\operatorname*{holim}_{\Delta\times\Delta} (X^{\bullet,\bullet})^{hG} \xrightarrow{\simeq} \operatorname*{holim}_{\Delta\times\Delta} ((X^{\bullet,\bullet})_{\mathcal{Fib}})^G \xleftarrow{\simeq} \big(\mathbf{R}\big(\lim_{\Delta\times\Delta}(-)^G\big)\big)\big(X^{\bullet,\bullet}\big).$$

Hence, it is natural to define the homotopy fixed points of the "$(\Delta\times\Delta)$–discrete G–spectrum" $\operatorname*{holim}_{\Delta\times\Delta} X^{\bullet,\bullet}$ as

$$(\operatorname*{holim}_{\Delta\times\Delta} X^{\bullet,\bullet})^{hG} = \operatorname*{holim}_{\Delta\times\Delta} (X^{\bullet,\bullet})^{hG}.$$

Since each $(X^{m,n})^{hG}$ is a fibrant spectrum, then, as in (6.2), there is a weak equivalence

$$(\operatorname*{holim}_{\Delta\times\Delta} X^{\bullet,\bullet})^{hG} \xrightarrow{\simeq} \operatorname*{holim}_{[n]\in\Delta} (X^{n,n})^{hG} = (\operatorname*{holim}_{\Delta\times\Delta} X^{\bullet,\bullet})^{h_\delta G},$$

which further justifies the convention given in (6.3). □

As in the Introduction, let H and K be closed subgroups of G, with H normal in K. Recall from (2.8) that, given $X \in \mathrm{Spt}_G$, there is an identification

$$X^{hH} = \operatorname*{holim}_{\Delta} \mathrm{Map}_c(G^\bullet, \widehat{X})^H.$$

Since the map $\psi\colon X \to \widehat{X}$ is natural, it is clear from [6, page 145; the proof of Theorem 5.2] that the above identification is natural in X.

Now let $\operatorname{holim}_\Delta X^\bullet$ be any delta-discrete G–spectrum. Using the naturality of the above identification, we have

$$(\operatorname*{holim}_{\Delta} X^\bullet)^{h_\delta H} = \operatorname*{holim}_{[n]\in\Delta} (X^n)^{hH} = \operatorname*{holim}_{[n]\in\Delta} \operatorname*{holim}_{[m]\in\Delta} (\mathrm{Map}_c(G^\bullet, \widehat{X^n})^m)^H.$$

Because of the isomorphism

$$\operatorname*{holim}_{[n]\in\Delta} \operatorname*{holim}_{[m]\in\Delta} (\mathrm{Map}_c(G^\bullet, \widehat{X^n})^m)^H \cong \operatorname*{holim}_{\Delta\times\Delta} \mathrm{Map}_c(G^\bullet, \widehat{X^\bullet})^H$$

and because homotopy limits are ends, which are only unique up to isomorphism, we can set

$$(\operatorname*{holim}_{\Delta} X^\bullet)^{h_\delta H} = \operatorname*{holim}_{\Delta\times\Delta} \mathrm{Map}_c(G^\bullet, \widehat{X^\bullet})^H.$$

By Lemma 2.7, for each $m, n \geq 0$, $\mathrm{Map}_c(G^\bullet, \widehat{X^n})^m$ is a fibrant discrete H–spectrum, so that $(\mathrm{Map}_c(G^\bullet, \widehat{X^n})^m)^H$ is a fibrant spectrum. Also, since the diagram $\mathrm{Map}_c(G^\bullet, \widehat{X^\bullet})$ is a bicosimplicial discrete K–spectrum, $\mathrm{Map}_c(G^\bullet, \widehat{X^\bullet})^H$ is a bicosimplicial discrete K/H–spectrum. Thus, the discussion above in Convention 6.1 implies that there is a K/H–equivariant map

$$(\operatorname*{holim}_{\Delta} X^\bullet)^{h_\delta H} = \operatorname*{holim}_{\Delta\times\Delta} \mathrm{Map}_c(G^\bullet, \widehat{X^\bullet})^H \xrightarrow{\simeq} \operatorname*{holim}_{[n]\in\Delta} (\mathrm{Map}_c(G^\bullet, \widehat{X^n})^n)^H$$

that is a weak equivalence, and the target of this weak equivalence is a delta-discrete K/H–spectrum. Therefore, by Convention 6.1, we can identify $(\operatorname{holim}_\Delta X^\bullet)^{h_\delta H}$ with the delta-discrete K/H–spectrum $\operatorname{holim}_{[n]\in\Delta}(\mathrm{Map}_c(G^\bullet, \widehat{X^n})^n)^H$, and hence, by (6.3) and as in the proof of Theorem 2.11, we have

$$\begin{aligned}
\big((\operatorname*{holim}_{\Delta} X^\bullet)^{h_\delta H}\big)^{h_\delta K/H} &= \big(\operatorname*{holim}_{[n]\in\Delta} (\mathrm{Map}_c(G^\bullet, \widehat{X^n})^n)^H\big)^{h_\delta K/H} \\
&= \operatorname*{holim}_{[n]\in\Delta} \big(\big((\mathrm{Map}_c(G^\bullet, \widehat{X^n})^n)^H\big)_{fK/H}\big)^{K/H} \xleftarrow{\simeq} \operatorname*{holim}_{[n]\in\Delta} \big((\mathrm{Map}_c(G^\bullet, \widehat{X^n})^n)^H\big)^{K/H} \\
&= \operatorname*{holim}_{[n]\in\Delta} (\mathrm{Map}_c(G^\bullet, \widehat{X^n})^n)^K \xleftarrow{\simeq} \operatorname*{holim}_{[n]\in\Delta} \operatorname*{holim}_{[m]\in\Delta} (\mathrm{Map}_c(G^\bullet, \widehat{X^n})^m)^K \\
&= \operatorname*{holim}_{[n]\in\Delta} (X^n)^{hK} = (\operatorname*{holim}_{\Delta} X^\bullet)^{h_\delta K}.
\end{aligned}$$

We summarize our work above in the following theorem.

Theorem 6.4 *If G has finite vcd and $\operatorname*{holim}_{\Delta} X^\bullet$ is a delta-discrete G–spectrum, then there is a weak equivalence*

$$\big((\operatorname*{holim}_{\Delta} X^\bullet)^{h_\delta H}\big)^{h_\delta K/H} \xleftarrow{\simeq} (\operatorname*{holim}_{\Delta} X^\bullet)^{h_\delta K}.$$

7 The relationship between X^{hH} and $(X_\delta)^{h_\delta H}$, in general

Let G be an arbitrary profinite group and let X be any discrete G–spectrum. Also, as usual, let H and K be closed subgroups of G, with H normal in K. As mentioned in Section 1.1, it is not known, in general, that the "discrete homotopy fixed points" X^{hH} have a K/H–action. In this section, we consider this issue by using the framework of delta-discrete G–spectra.

Given the initial data above, there is a commutative diagram

$$\begin{tikzcd}
X_{fH} \arrow[dr, "\Psi_H"'] & X \arrow[l, "f_H"', "\simeq"] \arrow[r, "\Psi", "\simeq"'] \arrow[d, "\operatorname*{colim}_{U \triangleleft_o H} \Psi^U"] & X_\delta \\
& \operatorname*{colim}_{U \triangleleft_o H} (X_\delta)^U, \arrow[ur] &
\end{tikzcd}$$

where $\operatorname{colim}_{U \triangleleft_o H} \Psi^U$, a morphism in Spt_H whose label is a slight abuse of notation, is defined to be the composition

$$X \cong \operatorname*{colim}_{U \triangleleft_o H} X^U \to \operatorname*{colim}_{U \triangleleft_o H} (X_\delta)^U,$$

and Ψ_H, a morphism between fibrant objects in Spt_H, exists because, in Spt_H, f_H is a trivial cofibration and $\operatorname*{colim}_{U \triangleleft_o H} (X_\delta)^U$ is fibrant (by [6, Theorem 3.5]).

Definition 7.1 The map Ψ_H induces the map

$$X^{hH} = (X_{fH})^H \xrightarrow{(\Psi_H)^H} \big(\operatorname*{colim}_{U \lhd_o H} (X_\delta)^U\big)^H \cong \operatorname*{holim}_{\Delta} \operatorname{Map}_c(G^\bullet, \widehat{X})^H$$

and (as in the proof of Lemma 2.9) there is a weak equivalence

$$\operatorname*{holim}_{\Delta} \operatorname{Map}_c(G^\bullet, \widehat{X})^H \xrightarrow{\simeq} \operatorname*{holim}_{\Delta} \operatorname{Map}_c(G^\bullet, \widehat{X})^{hH} = (X_\delta)^{h_\delta H}.$$

The composition of these two maps defines the map

$$\rho(X)_H \colon X^{hH} \to (X_\delta)^{h_\delta H}.$$

(The map $\rho(X)_H$ is not the same as the map of Lemma 2.9 (when $L = H$), since $\rho(X)_H$ does not use the identification of (**2.8**).)

Notice that if the map $\operatorname{colim}_{U \lhd_o H} \Psi^U$ is a weak equivalence in Spt, then the map Ψ_H is a weak equivalence in Spt_H, and hence, $(\Psi_H)^H$ is a weak equivalence. Thus, if $\operatorname{colim}_{U \lhd_o H} \Psi^U$ is a weak equivalence in Spt, then $\rho(X)_H$ is a weak equivalence. This observation, together with [6, proof of Theorem 4.2] and Mitchell [25, Proposition 3.3], immediately yields the following result.

Theorem 7.2 *If G is any profinite group and $X \in \mathrm{Spt}_G$, then the map*

$$\rho(X)_H \colon X^{hH} \xrightarrow{\simeq} (X_\delta)^{h_\delta H}$$

is a weak equivalence, whenever any one of the following conditions holds:

- *(i) H has finite vcd;*
- *(ii) G has finite vcd;*
- *(iii) there exists a fixed integer p such that $H^s_c(U; \pi_t(X)) = 0$, for all $s > p$, all $t \in \mathbb{Z}$ and all $U \lhd_o H$;*
- *(iv) there exists a fixed integer q such that $H^s_c(U; \pi_t(X)) = 0$, for all $t > q$, all $s \geq 0$ and all $U \lhd_o H$; or*
- *(v) there exists a fixed integer r such that $\pi_t(X) = 0$, for all $t > r$.*

In the statement of Theorem 7.2, note that (ii) implies (i) and (v) implies (iv). Also, it is not known, in general, that $\operatorname{colim}_{U \lhd_o H} \Psi^U$ is a weak equivalence, so that we do not know, in general, that X^{hH} and $(X_\delta)^{h_\delta H}$ are equivalent.

As noted in Definition 7.1, $(X_\delta)^{h_\delta H}$ is equivalent to the delta-discrete K/H–spectrum $\operatorname*{holim}_{\Delta} \operatorname{Map}_c(G^\bullet, \widehat{X})^H$, and hence, it is natural to identify them and to set

$$\big((X_\delta)^{h_\delta H}\big)^{h_\delta K/H} = \big(\operatorname*{holim}_{\Delta} \operatorname{Map}_c(G^\bullet, \widehat{X})^H\big)^{h_\delta K/H}.$$

Theorem 7.3 *If G is any profinite group and $X \in \mathrm{Spt}_G$, then*

$$\big((X_\delta)^{h_\delta H}\big)^{h_\delta K/H} \simeq (X_\delta)^{h_\delta K}.$$

Proof As in the proof of Theorem 2.11, it is easy to see that there is a zigzag of weak equivalences

$$\big((X_\delta)^{h_\delta H}\big)^{h_\delta K/H} \xleftarrow{\simeq} \operatorname*{holim}_{\Delta} \mathrm{Map}_c(G^\bullet, \widehat{X})^K \xrightarrow{\simeq} (X_\delta)^{h_\delta K}. \qquad \square$$

Given any profinite group G, Theorem 7.3 shows that by using $(-)_\delta$ and $(-)^{h_\delta H}$, delta-discrete homotopy fixed points for discrete G–spectra are transitive.

8 Comparing two different models for the iterated homotopy fixed point spectrum

Let G be any profinite group and let X be a discrete G–spectrum. As defined by the author in [7, Definition 4.1], X is a hyperfibrant discrete G–spectrum if the map

$$\psi(X)^G_L \colon (X_{fG})^L \to ((X_{fG})_{fL})^L$$

is a weak equivalence for every closed subgroup L of G. (We use "ψ" in the notation "$\psi(X)^G_L$," because we follow the notation of [7]; this use of "ψ" is not related to the map ψ of Lemma 2.1.)

Now suppose that X is a hyperfibrant discrete G–spectrum and, as usual, let H and K be closed subgroups of G, with H normal in K. These hypotheses imply that

(a) the map $\psi(X)^G_H \colon (X_{fG})^H \xrightarrow{\simeq} ((X_{fG})_{fH})^H$ is a weak equivalence;

(b) the source of the map $\psi(X)^G_H$, the spectrum $(X_{fG})^H$, is a discrete K/H–spectrum; and

(c) since the composition $X \to X_{fG} \to (X_{fG})_{fH}$ is a trivial cofibration and the target of the weak equivalence $X \to X_{fH}$ is fibrant, in Spt_H, there is a weak equivalence $\upsilon \colon (X_{fG})_{fH} \to X_{fH}$ between fibrant objects, and hence, there is a weak equivalence

$$(X_{fG})^H \xrightarrow{\simeq} X^{hH}$$

that is defined by the composition

$$\upsilon^H \circ \psi(X)^G_H \colon (X_{fG})^H \xrightarrow{\simeq} ((X_{fG})_{fH})^H \xrightarrow{\simeq} (X_{fH})^H = X^{hH}.$$

Thus, following [7, Definition 4.5], it is natural to define

$$(X^{hH})^{hK/H} := ((X_{fG})^H)^{hK/H}.$$

Let G have finite vcd, so that, by Lemma 2.10, $X^{hH} = \operatorname{holim}_{\Delta} \operatorname{Map}_c(G^\bullet, \widehat{X})^H$ is a delta-discrete K/H–spectrum. Thus,

$$(X^{hH})^{h_\delta K/H} = \operatorname*{holim}_{[n]\in\Delta} \big((\operatorname{Map}_c(G^\bullet, \widehat{X})^n)^H\big)^{hK/H},$$

and, by Theorem 2.11, there is a weak equivalence

$$\textbf{(8.1)} \qquad \operatorname*{holim}_{\Delta} \operatorname{Map}_c(G^\bullet, \widehat{X})^K \xrightarrow{\simeq} (X^{hH})^{h_\delta K/H}.$$

The above discussion shows that when G has finite vcd and X is a hyperfibrant discrete G–spectrum, there are two different models for the iterated homotopy fixed point spectrum, $(X^{hH})^{hK/H}$ and $(X^{hH})^{h_\delta K/H}$, and we would like to know when they agree with each other. The following result gives a criterion for when $(X^{hH})^{hK/H}$ and $(X^{hH})^{h_\delta K/H}$ are equivalent.

Theorem 8.2 *Let G have finite vcd and suppose that X is a hyperfibrant discrete G–spectrum. If the map $\psi(X)^K_H\colon (X_{fK})^H \to ((X_{fK})_{fH})^H$ is a weak equivalence, then there is a weak equivalence*

$$\textbf{(8.3)} \qquad (X^{hH})^{hK/H} \xrightarrow{\simeq} (X^{hH})^{h_\delta K/H}.$$

Remark 8.4 Let G and X be as in Theorem 8.2. By definition, if X is a hyperfibrant discrete K–spectrum, then each map $\psi(X)^K_H$ is a weak equivalence (here, as usual, H is any closed subgroup of G that is normal in K), giving the weak equivalence of (**8.3**). We refer the reader to [7, Sections 3, 4] for further discussion about hyperfibrancy. The spectral sequence considerations of [7, pages 2887–2888] are helpful for understanding when X is a hyperfibrant discrete K–spectrum.

Proof of Theorem 8.2. It follows from Theorem 7.2, (ii) and Definition 7.1 that $(\Psi_K)^K$ is a weak equivalence, so that, by composing with the weak equivalence of (**8.1**), there is a weak equivalence

$$X^{hK} \xrightarrow{\simeq} \operatorname*{holim}_{\Delta} \operatorname{Map}_c(G^\bullet, \widehat{X})^K \xrightarrow{\simeq} (X^{hH})^{h_\delta K/H}.$$

Therefore, to obtain (**8.3**), it suffices to show that there is a weak equivalence

$$(X^{hH})^{hK/H} = ((X_{fG})^H)^{hK/H} \to X^{hK}.$$

It is useful for the argument below to recall from the proof of [7, Lemma 4.6] that the functor $(-)^H\colon \mathrm{Spt}_K \to \mathrm{Spt}_{K/H}$ is a right Quillen functor.

Since $X \to X_{fG}$ is a trivial cofibration in Spt_K and the map $X \to X_{fK}$ has a fibrant target, there is a weak equivalence

$$\mu\colon X_{fG} \xrightarrow{\simeq} X_{fK}$$

in Spt_K. Thus, there is a commutative diagram

$$\begin{tikzcd}
((X_{fG})^H)_{fK/H} \arrow[dr, "\widetilde{\mu^H}"'] & (X_{fG})^H \arrow[l, "\eta"', "\simeq"] \arrow[r, "\psi(X)^G_H", "\simeq"'] \arrow[d, "\mu^H"] & ((X_{fG})_{fH})^H \arrow[d, "(\mu_{fH})^H", "\simeq"'] \\
& (X_{fK})^H \arrow[r, "\psi(X)^K_H"] & ((X_{fK})_{fH})^H.
\end{tikzcd}$$

The map $\widetilde{\mu^H}$ exists in $\mathrm{Spt}_{K/H}$, because in $\mathrm{Spt}_{K/H}$ the map η is a trivial cofibration and $(X_{fK})^H$ is fibrant. Also, as noted in the diagram, the map $\psi(X)^G_H$ is a weak equivalence, since X is a hyperfibrant discrete G–spectrum, and the rightmost vertical map, $(\mu_{fH})^H = \mu^{hH}$, is a weak equivalence, since μ is a weak equivalence in Spt_H.

Now suppose that $\psi(X)^K_H$ is a weak equivalence. Then μ^H is a weak equivalence, and hence, $\widetilde{\mu^H}$ is a weak equivalence between fibrant objects in $\mathrm{Spt}_{K/H}$. Therefore, there is a weak equivalence

$$\left(\widetilde{\mu^H}\right)^{K/H}\colon ((X_{fG})^H)^{hK/H} = (((X_{fG})^H)_{fK/H})^{K/H} \xrightarrow{\simeq} ((X_{fK})^H)^{K/H} = X^{hK},$$

completing the proof. □

9 Delta-discrete homotopy fixed points are always discrete homotopy fixed points

Let G be any profinite group. In this final section, we obtain in two different ways the conclusion stated in the section title. Somewhat interestingly, the discrete homotopy fixed points, in both cases, are those of a discrete G–spectrum that, in general, need not be equivalent (in the sense of Definition 9.8) to the delta-discrete G–spectrum (whose delta-discrete homotopy fixed points are under consideration).

Let $\operatorname{holim}_{\Delta} X^\bullet$ be any delta-discrete G–spectrum and define

$$C(X^\bullet) := \operatorname*{colim}_{N \triangleleft_o G} \left(\operatorname*{holim}_{[n]\in\Delta} (X^n)_{fG}\right)^N.$$

Notice that $C(X^\bullet)$ is a discrete G–spectrum and

$$(\operatorname*{holim}_{\Delta} X^\bullet)^{h_\delta G} \cong \big(\operatorname*{holim}_{[n]\in\Delta} (X^n)_{fG}\big)^G \cong (C(X^\bullet))^G \xrightarrow{\simeq} (C(X^\bullet))^{hG},$$

where the justification for the second isomorphism above is as in [6, proof of Theorem 2.3] and the weak equivalence is due to the fact that the spectrum $C(X^\bullet)$ is a fibrant discrete G–spectrum (by [6, Corollary 2.4]). Also, notice that the weak equivalence

$$(\operatorname*{holim}_{\Delta} X^\bullet)^{h_\delta G} \xrightarrow{\simeq} (C(X^\bullet))^{hG}, \tag{9.1}$$

defined above, is partly induced by the canonical map ι_G (defined by a colimit of inclusions of fixed points) in the zigzag

$$C(X^\bullet) \xrightarrow{\iota_G} \operatorname*{holim}_{[n]\in\Delta} (X^n)_{fG} \xleftarrow{\simeq} \operatorname*{holim}_{\Delta} X^\bullet \tag{9.2}$$

of G–equivariant maps. In (9.2), the second map, as indicated, is a weak equivalence of delta-discrete G–spectra.

Though (9.1) shows that the delta-discrete homotopy fixed points of any delta-discrete G–spectrum can be realized as the discrete homotopy fixed points of a discrete G–spectrum, there is a slight incongruity here: the map ι_G in (9.2) does not have to be a weak equivalence. For example, if ι_G is a weak equivalence, then $\pi_0(\operatorname{holim}_\Delta X^\bullet)$ is a discrete G–module (since $\pi_0(C(X^\bullet))$ is a discrete G–module), but this is not the case, for example, when $\operatorname{holim}_\Delta X^\bullet$ is the delta-discrete $\mathbb{Z}_q$–spectrum $Y^{h\mathbb{Z}/p}$ (this characterization of $Y^{h\mathbb{Z}/p}$ is obtained by applying Lemma 2.10) that was referred to in Section 1.1, since $\pi_0(Y^{h\mathbb{Z}/p})$ is not a discrete $\mathbb{Z}_q$–module (as shown by Wieland and the author in [7, Appendix A]).

Now we consider a second and more interesting way to realize $(\operatorname{holim}_\Delta X^\bullet)^{h_\delta G}$ as a discrete homotopy fixed point spectrum. As in Section 3, let $(X^\bullet)_{\mathcal{Fib}}$ denote the fibrant replacement of $X^\bullet$ in the model category $c(\mathrm{Spt}_G)$. Then there is again a zigzag

$$\lim_{\Delta} (X^\bullet)_{\mathcal{Fib}} \xrightarrow{\lambda} \operatorname*{holim}_{\Delta} (X^\bullet)_{\mathcal{Fib}} \xleftarrow[\gamma]{\simeq} \operatorname*{holim}_{\Delta} X^\bullet \tag{9.3}$$

of canonical G–equivariant maps, where λ is the usual map in Spt from the limit to the homotopy limit and the map γ is a weak equivalence of delta-discrete G–spectra. We will show that, as in the case of zigzag (9.2), $\lim_\Delta (X^\bullet)_{\mathcal{Fib}}$ is a discrete G–spectrum and its discrete homotopy fixed points are equivalent to $(\operatorname{holim}_\Delta X^\bullet)^{h_\delta G}$, but, as before, the map λ need not be a weak equivalence.

Lemma 9.4 *Let $\operatorname*{holim}_{\Delta} X^\bullet$ be a delta-discrete G–spectrum. Then $\lim_{\Delta} (X^\bullet)_{\mathcal{Fib}}$ is a discrete G–spectrum.*

Proof Given two morphisms $Y_0 \rightrightarrows Y_1$ in Spt_G, let $\mathrm{equal}[\, Y_0 \rightrightarrows Y_1\,]$ denote the equalizer in Spt. Also, let $\mathrm{equal}_G[\, Y_0 \rightrightarrows Y_1\,]$ denote the equalizer in Spt_G. Due to the fact that $X^\bullet$ and $(X^\bullet)_{\mathcal{Fib}}$ are cosimplicial discrete G–spectra, to prove the lemma it suffices to show that $\lim_\Delta X^\bullet$ is a discrete G–spectrum.

It is a standard fact about limits of cosimplicial diagrams that the canonical G–equivariant map $\lim_\Delta X^\bullet \xrightarrow{\cong} \mathrm{equal}\Big[X^0 \overset{d^0}{\underset{d^1}{\rightrightarrows}} X^1\Big]$ is an isomorphism in Spt. Then the proof is completed by noting that for the diagram $\big[X^0 \rightrightarrows X^1\big] \equiv \Big[X^0 \overset{d^0}{\underset{d^1}{\rightrightarrows}} X^1\Big]$, there are isomorphisms

$$\begin{aligned}
\mathrm{equal}_G\big[X^0 \rightrightarrows X^1\big] &\cong \operatorname*{colim}_{N \lhd_o G} \big(\mathrm{equal}\big[X^0 \rightrightarrows X^1\big]\big)^N \\
&\cong \mathrm{equal}\Big[\operatorname*{colim}_{N \lhd_o G} (X^0)^N \rightrightarrows \operatorname*{colim}_{N \lhd_o G} (X^1)^N\Big] \\
&\cong \mathrm{equal}\big[X^0 \rightrightarrows X^1\big] \\
&\cong \lim_\Delta X^\bullet,
\end{aligned}$$

with each isomorphism G–equivariant. In this string of isomorphisms, the first one is because of [5, Remark 4.2] and the third one follows from the fact that X^0 and X^1 are discrete G–spectra. $\square$

If $\mathcal{C}$ is a small category and $Y^\bullet$ is an object in $(\mathrm{Spt}_G)^{\mathcal{C}}$, we use $\lim_{\mathcal{C}}{}^G Y^\bullet$ to denote the corresponding limit in Spt_G. By the opening lines in the proof of Theorem 3.5, the functor $\lim_{\mathcal{C}}{}^G(-)\colon (\mathrm{Spt}_G)^{\mathcal{C}} \to \mathrm{Spt}_G$ is a right Quillen functor, where, as usual, $(\mathrm{Spt}_G)^{\mathcal{C}}$ is equipped with the injective model structure. Now we give an application of Lemma 9.4.

Lemma 9.5 *Let $\operatorname{holim}_\Delta X^\bullet$ be a delta-discrete G–spectrum. Then there is an equivalence*

$$(\lim_\Delta (X^\bullet)_{\mathcal{Fib}})^{hG} \simeq (\operatorname{holim}_\Delta X^\bullet)^{h_\delta G}.$$

Proof Since the spectrum $\lim_\Delta (X^\bullet)_{\mathcal{Fib}}$ is in Spt_G, there are isomorphisms

$$\textbf{(9.6)}\qquad \lim_\Delta (X^\bullet)_{\mathcal{Fib}} \cong \operatorname*{colim}_{N \lhd_o G} \big(\lim_\Delta (X^\bullet)_{\mathcal{Fib}}\big)^N \cong \lim_\Delta{}^G (X^\bullet)_{\mathcal{Fib}}$$

in Spt_G and $\lim_\Delta{}^G (X^\bullet)_{\mathcal{Fib}}$ is a fibrant discrete G–spectrum. Thus, the canonical map

$$(\lim_\Delta{}^G (X^\bullet)_{\mathcal{Fib}})^G \xrightarrow{\simeq} (\lim_\Delta{}^G (X^\bullet)_{\mathcal{Fib}})^{hG} \cong (\lim_\Delta (X^\bullet)_{\mathcal{Fib}})^{hG}$$

is a weak equivalence.

The proof is completed by the zigzag of equivalences

$$(\lim_{\Delta}{}^{G}(X^\bullet)_{\mathcal{Fib}})^G \cong \lim_{\Delta}((X^\bullet)_{\mathcal{Fib}})^G \xrightarrow{\simeq} \operatorname{holim}_{\Delta}((X^\bullet)_{\mathcal{Fib}})^G \xleftarrow{\simeq} (\operatorname{holim}_{\Delta} X^\bullet)^{h_\delta G},$$

where the isomorphism uses (**9.6**), the first weak equivalence is by Theorem 3.2 and the second weak equivalence applies Theorem 3.4. □

The argument that was used earlier to show that ι_G in zigzag (**9.2**) does not have to be a weak equivalence also applies to show that the map λ in zigzag (**9.3**) does not have to be a weak equivalence.

Remark 9.7 Notice that if $(X^\bullet)_{\mathcal{Fib}}$ is fibrant in $c(\mathrm{Spt})$ (equipped with the injective model structure), then, as in the proof of Theorem 3.2, the map λ is a weak equivalence. Thus, interestingly, fibrations in $c(\mathrm{Spt}_G)$ are not necessarily fibrations in $c(\mathrm{Spt})$ – even though any fibration in Spt_G is a fibration in Spt, by [5, Lemma 3.10]. This observation is closely related to the fact that the forgetful functor $U\colon \mathrm{Spt}_G \to \mathrm{Spt}$ need not be a right adjoint (a discussion of this fact is given by Behrens and the author in [1, Section 3.6]): if one supposes that U is a right adjoint, then U is also a right Quillen functor, since U preserves fibrations and weak equivalences, and hence, by Lurie [24, Remark A.2.8.6], the forgetful functor $U \circ (-)\colon c(\mathrm{Spt}_G) \to c(\mathrm{Spt})$ preserves fibrations.

Let V be an open subgroup of G. By [7, proof of Lemma 3.1], the right adjoint $\mathrm{Res}^V_G\colon \mathrm{Spt}_G \to \mathrm{Spt}_V$ that regards a discrete G–spectrum as a discrete V–spectrum is a right Quillen functor, and hence (again, by [24, Remark A.2.8.6]), the restriction functor $\mathrm{Res}^V_G \circ (-)\colon c(\mathrm{Spt}_G) \to c(\mathrm{Spt}_V)$ preserves fibrations. Thus,

$$(X^\bullet)_{\mathcal{Fib}} \cong \operatorname{colim}_{N \lhd_o G}((X^\bullet)_{\mathcal{Fib}})^N$$

is the filtered colimit of cosimplicial spectra $((X^\bullet)_{\mathcal{Fib}})^N$, each of which is fibrant in $c(\mathrm{Spt})$, since $\mathrm{Res}^N_G \circ (X^\bullet)_{\mathcal{Fib}} = (X^\bullet)_{\mathcal{Fib}}$ is fibrant in $c(\mathrm{Spt}_N)$ and, as in the proof of Theorem 3.2, the functor $(-)^N\colon c(\mathrm{Spt}_N) \to c(\mathrm{Spt})$ preserves fibrant objects. As noted above, when G is $\mathbb{Z}_q$, for any prime q, there are cases where $(X^\bullet)_{\mathcal{Fib}}$ is not fibrant in $c(\mathrm{Spt})$. In these cases, since $\mathbb{Z}_q$ is countably based, the above filtered colimit can be taken to be a sequential colimit indexed over the natural numbers. Therefore, we can conclude, perhaps somewhat surprisingly, that, unlike in Spt, a sequential colimit of fibrant objects in $c(\mathrm{Spt})$ does not have to be fibrant. This also shows that, though the model category $c(\mathrm{Spt})$ is cofibrantly generated (by [24, Proposition A.2.8.2]), it is not almost finitely generated (see Hovey [21, Definition 4.1, Lemma 4.3]).

To place our next result, Theorem 9.9, into context, Definition 9.8 below is helpful. Simple examples of the notion captured by this definition, for which we have not needed a name until now, include the map Ψ in Section 2.1, the map $\phi_{\widehat{X}} \circ \psi$ considered at the end of Section 5, and the map in (**6.2**).

Definition 9.8 Let $\operatorname{holim}_\Delta X^\bullet$ be a delta-discrete G–spectrum and let Y be a G–spectrum (that is, Y is an object in G–Spt; Y can be in Spt_G). We say that Y is *equivalent* to $\operatorname{holim}_\Delta X^\bullet$ if there is a zigzag of weak equivalences between Y and $\operatorname{holim}_\Delta X^\bullet$ in the model category G–Spt. For example, the following map and two zigzags,

$$X \xrightarrow{\simeq_G} Y, \quad X \xleftarrow{\simeq_G} Y_1 \xrightarrow{\simeq_G} Y_2, \quad \text{and} \quad X \xrightarrow{\simeq_G} Y_3 \xleftarrow{\simeq_G} Y_4 \xrightarrow{\simeq_G} Y_5,$$

where $X \equiv \operatorname{holim}_\Delta X^\bullet$, "$\simeq_G$" marks a weak equivalence in G–Spt, and each Y_i is in G–Spt, indicate that Y, Y_2 and Y_5 are each equivalent to $\operatorname{holim}_\Delta X^\bullet$. It is clear from this definition that any object in the essential image or the "weak essential image" (as in Bergner [2, Example 3.3]) of the forgetful functor $\mathbb{U}_G\colon \widehat{\mathrm{Spt}_G^\Delta} \to G$–Spt is equivalent to a delta-discrete G–spectrum.

A priori, we do not expect a delta-discrete G–spectrum, in general, to be equivalent to a discrete G–spectrum, and the fact that zigzags (**9.2**) and (**9.3**), which give natural ways to realize an arbitrary delta-discrete homotopy fixed point spectrum as a discrete homotopy fixed point spectrum, fail, in general, to be equivalences (in the sense of Definition 9.8) is consistent with our expectation.

The next result shows that zigzags (**9.2**) and (**9.3**) are, in fact, directly related to each other (beyond just having the "same general structure"). Given a small category $\mathcal{C}$ and a diagram $Y^\bullet \in (\mathrm{Spt}_G)^{\mathcal{C}}$, we let $\operatorname*{holim}^G_\Delta Y^\bullet$ be the homotopy limit in Spt_G of $Y^\bullet$, as defined in Hirschhorn [20, Definition 18.1.8].

Theorem 9.9 *Let $\operatorname{holim}_\Delta X^\bullet$ be any delta-discrete G–spectrum. Then the map ι_G in (**9.2**) is a weak equivalence if and only if the map λ in (**9.3**) is a weak equivalence.*

Proof Let $Y^\bullet$ be a cosimplicial discrete G–spectrum. Then, by [6, Theorem 2.3] and because homotopy limits are ends and thereby only unique up to isomorphism, there is the identity

$$\operatorname*{holim}^G_\Delta Y^\bullet = \operatorname*{colim}_{N \lhd_o G} (\operatorname*{holim}_\Delta Y^\bullet)^N,$$

which implies that $C(X^\bullet) = \operatorname{holim}^G_{[n]\in\Delta} (X^n)_{fG}$. Similarly, there is the identity

$$\lim_\Delta{}^G (X^\bullet)_{\mathcal{Fib}} = \operatorname*{colim}_{N \lhd_o G} (\lim_\Delta (X^\bullet)_{\mathcal{Fib}})^N.$$

Also, since the map $\{X^n\}_{[n]\in\Delta} \to \{(X^n)_{fG}\}_{[n]\in\Delta}$ is a trivial cofibration in $c(\mathrm{Spt}_G)$, there is a weak equivalence

$$\widetilde{\gamma}\colon \{(X^n)_{fG}\}_{[n]\in\Delta} \to (X^\bullet)_{\mathcal{F}ib}$$

in $c(\mathrm{Spt}_G)$.

Given the above facts, there is the commutative diagram

$$\begin{tikzcd}
\lim\limits_{\Delta}(X^\bullet)_{\mathcal{F}ib} \arrow[r, "\lambda"] & \operatorname*{holim}\limits_{\Delta}(X^\bullet)_{\mathcal{F}ib} & \operatorname*{holim}\limits_{[n]\in\Delta}(X^n)_{fG} \arrow[l, "\simeq"', "\widehat{\widetilde{\gamma}}"] \\
\lim\limits_{\Delta}{}^{G}(X^\bullet)_{\mathcal{F}ib} \arrow[u, "\cong"] \arrow[r, "\simeq", "\lambda'_G"'] & \operatorname*{holim}\limits_{\Delta}{}^{G}(X^\bullet)_{\mathcal{F}ib} \arrow[u] & \operatorname*{holim}\limits_{[n]\in\Delta}{}^{G}(X^n)_{fG} \arrow[u, "\iota_G"'] \arrow[l, "\simeq"', "\left(\widehat{\widetilde{\gamma}}\right)_G"]
\end{tikzcd}$$

of canonical maps: each vertical map is induced by inclusions of fixed points; the map $\widehat{\widetilde{\gamma}}$ is a weak equivalence (of delta-discrete G–spectra) because, in $c(\mathrm{Spt})$, $\widetilde{\gamma}$ is an objectwise weak equivalence between objectwise fibrant diagrams; the leftmost vertical map is an isomorphism thanks to Lemma 9.4; Theorem 3.5 implies that λ'_G is a weak equivalence; and, in $c(\mathrm{Spt}_G)$, the map $\widetilde{\gamma}$ is an objectwise weak equivalence between objectwise fibrant diagrams, so that, by Hirschhorn [20, Theorem 18.5.3, (2)], the induced map $\left(\widehat{\widetilde{\gamma}}\right)_G$ is a weak equivalence.

The desired conclusion now follows immediately from the above diagram. $\square$

References

[1] **M Behrens**, **D G Davis**, *The homotopy fixed point spectra of profinite Galois extensions*, Trans. Amer. Math. Soc. 362 (2010) 4983–5042

[2] **J E Bergner**, *Homotopy limits of model categories and more general homotopy theories*, (2010) 10 pages, arXiv:1010.0717

[3] **F Borceux**, *Handbook of categorical algebra, 3: Categories of sheaves*, volume 52 of *Encyclopedia of Mathematics and its Applications*, Cambridge University Press, Cambridge (1994)

[4] **C Casacuberta**, **B Chorny**, *The orthogonal subcategory problem in homotopy theory*, from: "An alpine anthology of homotopy theory", Contemp. Math. 399, Amer. Math. Soc., Providence, RI (2006) 41–53

[5] **D G Davis**, *Homotopy fixed points for $L_{K(n)}(E_n \wedge X)$ using the continuous action*, J. Pure Appl. Algebra 206 (2006) 322–354

[6] **D G Davis**, *Explicit fibrant replacement for discrete G–spectra*, Homology, Homotopy Appl. 10 (2008) 137–150

[7] **D G Davis**, *Iterated homotopy fixed points for the Lubin-Tate spectrum* (With an appendix by D G Davis and B Wieland), Topology Appl. 156 (2009) 2881–2898

[8] **E S Devinatz**, *A Lyndon-Hochschild-Serre spectral sequence for certain homotopy fixed point spectra*, Trans. Amer. Math. Soc. 357 (2005) 129–150

[9] **E S Devinatz**, **M J Hopkins**, *Homotopy fixed point spectra for closed subgroups of the Morava stabilizer groups*, Topology 43 (2004) 1–47

[10] **D Dikranjan**, *Topological characterization of p–adic numbers and an application to minimal Galois extensions*, Annuaire Univ. Sofia Fac. Math. Méc. 73 (1979) 103–110

[11] **D Dugger**, *Combinatorial model categories have presentations*, Adv. Math. 164 (2001) 177–201

[12] **W G Dwyer**, **P S Hirschhorn**, **D M Kan**, **J H Smith**, *Homotopy limit functors on model categories and homotopical categories*, Providence, RI: American Mathematical Society (AMS) (2004)

[13] **W Dwyer**, **H Miller**, **J Neisendorfer**, *Fibrewise completion and unstable Adams spectral sequences*, Israel J. Math. 66 (1989) 160–178

[14] **W G Dwyer**, **C W Wilkerson**, *Homotopy fixed-point methods for Lie groups and finite loop spaces*, Ann. of Math. (2) 139 (1994) 395–442

[15] **H Fausk**, *Equivariant homotopy theory for pro-spectra*, Geom. Topol. 12 (2008) 103–176

[16] **P G Goerss**, **M J Hopkins**, *Moduli spaces of commutative ring spectra*, from: "Structured ring spectra", London Math. Soc. Lecture Note Ser. 315, Cambridge Univ. Press, Cambridge (2004) 151–200

[17] **P G Goerss**, **J F Jardine**, *Simplicial homotopy theory*, Birkhäuser Verlag, Basel (1999)

[18] **R I Grigorchuk**, *Just infinite branch groups*, from: "New horizons in pro-p groups", Progr. Math. 184, Birkhäuser Boston, Boston, MA (2000) 121–179

[19] **K Hess**, *A general framework for homotopic descent and codescent*, (2010) 68 pages, arXiv:1001.1556v3

[20] **P S Hirschhorn**, *Model categories and their localizations*, volume 99 of *Mathematical Surveys and Monographs*, American Mathematical Society, Providence, RI (2003)

[21] **M Hovey**, *Spectra and symmetric spectra in general model categories*, J. Pure Appl. Algebra 165 (2001) 63–127

[22] **J F Jardine**, *Generalized étale cohomology theories*, Birkhäuser Verlag, Basel (1997)

[23] **J F Jardine**, *Representability theorems for presheaves of spectra*, J. Pure Appl. Algebra 215 (2011) 77–88

[24] **J Lurie**, *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*, Princeton University Press, Princeton, NJ (2009)

[25] **S A Mitchell**, *Hypercohomology spectra and Thomason's descent theorem*, from: "Algebraic K–theory (Toronto, ON, 1996)", Amer. Math. Soc., Providence, RI (1997) 221–277

[26] **S A Morris**, **S Oates-Williams**, **H B Thompson**, *Locally compact groups with every closed subgroup of finite index*, Bull. London Math. Soc. 22 (1990) 359–361

[27] **J Rognes**, *Galois extensions of structured ring spectra*, from: "Galois extensions of structured ring spectra/Stably dualizable groups, in Mem. Amer. Math. Soc., vol. 192 (898)" 1–97, 2008

[28] **J Rosický**, *Generalized Brown representability in homotopy categories*, Theory Appl. Categ. 14 (2005) no. 19, 451–479 (electronic)

[29] **R W Thomason**, *Algebraic K–theory and étale cohomology*, Ann. Sci. École Norm. Sup. (4) 18 (1985) 437–552

[30] **B Toën**, **G Vezzosi**, *Homotopical algebraic geometry, II: Geometric stacks and applications*, Mem. Amer. Math. Soc. 193 (2008) x+224

[31] **J S Wilson**, *On just infinite abstract and profinite groups*, from: "New horizons in pro-p groups", Progr. Math. 184, Birkhäuser Boston, Boston, MA (2000) 181–203

Department of Mathematics, University of Louisiana at Lafayette, Lafayette, LA 70504, U.S.A.

dgdavis@louisiana.edu

http://www.ucs.louisiana.edu/~dxd0799