July 17, 2005

Dear Mike,

This note consists of some questions about your work on the problem of realizing E_n as a G_n-spectrum with a continuous structured action. The note also contains a discussion of my work on this problem.

1. Trying to understand your approach to this problem

Mark Behrens sent me a sketch of a proof, based on your argument, that the pro-object $\{L_n M_I\}_I$ is an H_∞ object in the category of pro-spectra, where $I = (p^{i_0}, v_1^{i_1}, ..., v_{n-1}^{i_{n-1}}) \subset BP_*$, M_I is the corresponding generalized Moore spectrum, and $\{I\}$ is a cofinal collection of ideals such that $L_{K(n)}S^0 = \mathrm{holim}_I\, L_n M_I$. My understanding is that your actual argument was that E_n is a continuous G_n-E_∞ ring spectrum, in some sense, by using pro-spectra.

I have been thinking about how to realize E_n as a continuous G_n-A_∞ ring spectrum since 2000. Thus, I am very interested in learning more about what your ideas are regarding this problem. Also, because this problem is important to me, I am eager to learn what progress you have made on this problem.

Based on the above result that Mark told me, one might hope to argue in the following way. (Thus, if a step has not been rigorously verified, then it seems plausible.) The speculative argument begins with some definitions.

Definition 1.1. Let G be a profinite group. If S is a G-set, then S is a *discrete G-set* if the action map $G \times S \to S$ is continuous, where S is given the discrete topology. If X is a (naive) G-symmetric spectrum of simplicial sets, such that, for each $k \geq 0$, X_k is a simplicial discrete G-set, then X is a *discrete G-spectrum.*

We need the following categories.

Definition 1.2. Let Spt_G denote the category of discrete G-spectra. Let Spt_G^a be the category of discrete G-spectra that are symmetric ring spectra (that is, A_∞ ring spectra in symmetric spectra) such that the G-action is through maps of symmetric ring spectra. Also, let $(\mathrm{pro}\!-\!\mathrm{Spt}_G)_{\mathcal{C}_\infty}$ be the category of objects in $\mathrm{pro}\!-\!\mathrm{Spt}_G$, the category of pro-objects in Spt_G, that are $\mathcal{C}_\infty$ objects, where $\mathcal{C}$ is H or E.

Let $\{U_i\}_{i\geq 0}$ be a descending chain of open normal subgroups of G_n, such that $\bigcap_i U_i = \{e\}$. Let E_∞ be the category of commutative S-algebras. By [7], the G_n/U_i-action on $E_n^{hU_i}$ is given by a functor

$$\text{the groupoid } \{G_n/U_i\} \to E_\infty, \quad G_n/U_i \mapsto E_n^{hU_i}.$$

Composing this functor with the functor $\{(-) \wedge M_I\}$ yields a functor

$$\text{the groupoid } \{G_n/U_i\} \to (\mathrm{pro}\!-\!\mathrm{Spt}_{G_n/U_i})_{H_\infty}, \quad G_n/U_i \mapsto \{E_n^{hU_i} \wedge M_I\}.$$

By the projection $G_n \to G_n/U_i$, for each i,

$$\{E_n^{hU_i} \wedge M_I\} \in (\mathrm{pro}\!-\!\mathrm{Spt}_{G_n})_{H_\infty}.$$

Without loss of generality, one can assume that certain cofibrancy conditions are satisfied so that

$$\{\operatorname*{colim}_i (E_n^{hU_i} \wedge M_I)\} \in (\mathrm{pro}\!-\!\mathrm{Spt}_{G_n})_{H_\infty},$$

where the colimit is formed in Spt_{G_n} (which is formed in symmetric spectra).

By [7], $E_n \wedge M_I \simeq \mathrm{colim}_i(E_n^{hU_i} \wedge M_I)$. Since $\mathrm{holim}_I(E_n \wedge M_I)$ is the G_n-spectrum E_n, $\mathrm{holim}_I \, \mathrm{colim}_i(E_n^{hU_i} \wedge M_I) \simeq E_n$ is a weak equivalence that respects the G_n-actions. Thus, one can say that E_n is *a continuous G_n-H_∞ spectrum*, where this terminology indicates precisely that

$$\{\operatorname*{colim}_i (E_n^{hU_i} \wedge M_I)\} \in (\mathrm{pro-Spt}_{G_n})_{H_\infty}.$$

Thus, the diagram $\{\mathrm{colim}_i(E_n^{hU_i} \wedge M_I)\}$ has enough structure so that

$$\operatorname*{holim}_I \operatorname*{colim}_i (E_n^{hU_i} \wedge M_I)$$

is an H_∞ spectrum such that the G_n-action is through H_∞ maps of spectra. (If there's not enough structure, for this to be true, then perhaps one can show that

$$\operatorname*{holim}_I \operatorname*{colim}_i (E_n^{hU_i} \wedge M_I)$$

is an A_∞ spectrum with a G_n-action through A_∞ maps.) Recall from [4] that $E_n \simeq \mathrm{holim}_I \, \mathrm{colim}_i(E_n^{hU_i} \wedge M_I)$ makes E_n a continuous G_n-spectrum. Similarly, the fact that $\mathrm{holim}_I \, \mathrm{colim}_i(E_n^{hU_i} \wedge M_I)$ is H_∞ makes E_n a continuous G_n-H_∞ spectrum (or hopefully, at least a continuous G_n-A_∞ spectrum).

The above argument is what I've guessed might be your strategy for producing a continuous structured action. In addition to what Mark told me about your argument, I've seen Rognes's comment that "Hopkins has suggested that a weaker form of structured commutativity, in terms of pro-spectra, may instead be available" [9, pg. 25].

Given the above, I'm wondering: is the above argument the kind of argument that you have in mind? Is your actual argument that it should all go through in the E_∞ setting, and not just in the H_∞ setting; that is, are you able to show that

$$\{\operatorname*{colim}_i (E_n^{hU_i} \wedge M_I)\} \in (\mathrm{pro-Spt}_{G_n})_{E_\infty}?$$

I really would like to learn about your ideas for this problem, because this kind of realization problem is difficult and I don't want to pursue a strategy that is erroneous or inefficient - in §3, I explain where I am at in my own approach to this problem, which was, for the most part, done while I was doing my Ph.D. with Paul.

2. Useful observations regarding this problem

The following definition is useful.

Definition 2.1. Let

$$F_n = \operatorname*{colim}_i E_n^{hU_i}.$$

After trying various realization strategies, I decided that the best route was to try to realize $(E_n)_*/I$ by a discrete G_n-symmetric ring spectrum E_n/I, that is, by a spectrum $E_n/I \in \mathrm{Spt}^a_{G_n}$.

Before describing my work on this problem in §3, below I make a series of remarks that help to frame the problem.

(2.2) By taking cofibrant replacements and working with cofibrations, as needed, the discrete G_n-spectrum F_n is a commutative symmetric ring spectrum such that the discrete G_n-action is by maps of commutative symmetric ring spectra. (Jeff Smith explained to me that there is a model category structure on commutative symmetric ring spectra that makes this work.) Thus, the interesting spectrum F_n is a discrete G_n-commutative symmetric ring spectrum; the continuous action on F_n is E_∞.

(2.3) By [6], the diagram $\{M_I\}$ can be assumed to be a diagram of homotopy ring spectra. Then, by [8, Lemma 2.2], the pro-object $\{F_n \wedge M_I)\}$ is a diagram of discrete G_n-spectra that are homotopy ring spectra.

Therefore, presumably after running through the above arguments (including those in Remark (2.2)) with a little care, one obtains that the discrete G_n-action on $F_n \wedge M_I$ is through maps of homotopy ring spectra, so that $E_n \wedge M_I \simeq F_n \wedge M_I$ is a discrete G_n-homotopy ring spectrum. However, though this would give a continuous weakly structured action, we really only care about structure that occurs on the point-set level, so that this is not really interesting.

(2.4) It has been known for a long time that $E_n \wedge M_I$ almost definitely is an A_∞ ring spectrum, since its close relative $E(n)/I_n^k$ is A_∞, by [2]. Now, by [1], it is a theorem that $E_n \wedge M_I$ is an A_∞ ring spectrum. However, this is far from knowing that G_n acts discretely on $F_n \wedge M_I$ by maps of A_∞ ring spectra.

(2.5) I am excited about the idea that there is a spectrum $E_n/I \in \mathrm{Spt}^a_{G_n}$, because this result should have a surprising consequence (first observed by Charles Rezk). Suppose that $E_n/I \in \mathrm{Spt}_{G_n}$. Then the forgetful functor

$$U \colon \mathrm{Spt}^a_{G_n} \to \text{symmetric ring spectra}$$

should be well-behaved enough, so that the construction

$$\operatorname*{holim}_{\Delta} \mathrm{Map}_c(G_n^{\bullet+1}, (E_n/I)_{f,G_n})^{G_n} \simeq (E_n/I)^{hG_n}$$

can be done entirely in $\mathrm{Spt}^a_{G_n}$, so that $(E_n/I)^{hG_n} \simeq E_n^{hG_n} \wedge M_I$ is an A_∞ ring spectrum. This implies that $E_n^{hG_n} \wedge M_I \simeq L_{K(n)}M_I$ is A_∞.

Paul, Charles, and Jeff have told me that it would be quite interesting if $L_{K(n)}M_I$ turns out to be A_∞. More generally, $(E_n/I)^{hG} \simeq E_n^{hG} \wedge M_I$, for any closed G in G_n, would be A_∞.

(2.6) The implications of the conjecture that E_n/I is a discrete G_n-symmetric ring spectrum for Rognes's Galois extensions and associative Galois extensions are explored in my manuscript [5].

(2.7) Jim and Haynes told me that it is very doubtful that $E_n \wedge M_I$ has the homotopy type of an E_∞ ring spectrum. Charles sketched for me an argument that seems to imply that, for all n, p, and I, $E_n \wedge M_I$ fails to have the homotopy type of an E_∞ ring spectrum. (Charles said this argument was yours; Jim said the argument has an important antecedent in work of Mark Steinberger.)

There is no reason to hope that E_n/I is a discrete G_n-commutative symmetric ring spectrum. Thus, the only way to show that $\mathrm{holim}_I(E_n/I)$ makes E_n a continuous G_n-E_∞ ring spectrum is to show that, thanks to enough structure being present in each E_n/I, the homotopy limit $\mathrm{holim}_I(E_n/I)$ yields a G_n-action through E_∞ maps. Over the years, I have tried to set up various obstruction theoretic machines, in the spirit of Hopkins-Miller and Goerss-Hopkins, and I believe that the machinery required for the type of realization problem described in the preceding sentence would be extremely technically formidable.

The fact that $\{E_n/I\}$ cannot be E_∞ has the effect of making our problem more well-defined, and, together with Remark (2.4), it indicates that trying to show that $E_n/I \in \mathrm{Spt}^a_{G_n}$ is a plausible first step in producing a continuous structured action.

(2.8) Jeff has done work leading him to believe that, roughly speaking, something like the following theorem is true: "Let $\mathcal{C}$ be a symmetric monoidal model category of R-modules, where R is a monoid (in a symmetric monoidal model category $\mathcal{D}$). If $[\Sigma^n R, R]_{\mathrm{Ho}(\mathcal{C})} = 0$, whenever n is odd, and if $\alpha \subset \pi_*(R)$ is a regular sequence, then R/α is also a monoid in $\mathcal{C}$."

Jeff and I have talked about how this theorem could be useful for the problem of constructing a continuous structured action. (An easy corollary of this theorem is that E_n/I is A_∞, which, as noted earlier, is already known.) For example, let $\mathcal{C}$ be the category ${}_{F_n}\mathrm{Spt}^a_{G_n}$ of discrete G_n-symmetric ring spectra that are twisted F_n-modules (that is, the module structure map $F_n \wedge F_n \to F_n$ is G_n-equivariant, where the source has the diagonal action). However, $\pi_*(F_n^{hG_n})$ is not known, so that the hypotheses of the theorem cannot be verified to conclude that $F_n/I \simeq E_n/I \in \mathrm{Spt}^a_{G_n}$. Thus, this theorem is not known to be helpful. (Also, it not known that I is a regular sequence in $\pi(F_n)$.)

(2.9) The previous remark points out that it might be useful to know what $\pi_*(F_n)$ is. Paul computed this in the case $n = 1$ and $p = 2$ and obtained that

$$\pi_*(F_1) = \begin{cases} \mathbb{Z}/p^\infty & * = \text{odd}, \neq -1, \\ \mathbb{Z}_p & * = 0, \\ \mathbb{Q}_p & * = -1, \\ 0 & \text{otherwise.} \end{cases}$$

This computation led Paul to suspect that F_n and the spectrum E_n/I_n^∞ (which plays a role in Neil Strickland's proof of Gross-Hopkins duality [10, pp. 1029-1031]) are closely related to each other. (Jeff also thinks that F_n and E_n/I_n^∞ seem to be close to each other.) Note that F_n and E_n/I_n^∞ cannot be identical to each other, because the identify $M_n E_n = \Sigma^{-n} E/I_n^\infty$ (by [10, pg. 1030]) shows that E/I_n^∞ is $K(n)$-local, whereas F_n is not $K(n)$-local (see [4]).

In this context, it is natural to compute $\pi_*(F_n^{hG_n})$ using its descent spectral sequence. Also, I have wondered if $F_n^{hG_n}$ is closely related to $L_n S^0$. I think that a close relationship between $F_n^{hG_n}$ and $L_n S^0$ would be the discrete analogue of the equivalence that comes from this relationship's I_n-adic completion

$$L_{K(n)}(L_n S^0) = L_{K(n)} S^0 = E_n^{hG_n} = L_{K(n)}(F_n^{hG_n}),$$

where the last identity is shown in [4]. Once, Paul quickly did a descent spectral sequence computation and concluded that, for $n = 1$ and $p > 2$, $F_n^{hG_n}$ and $L_n S^0$ are probably not the same.

(2.10) In trying to realize $E_n/I \in \mathrm{Spt}^a_{G_n}$, most of the time when one wants to (a) prove something about the G_n-action on $E_n \wedge M_I$, one can just as well obtain the desired result by instead (b) proving the analogous thing about the G_n/U_i-action on $E_n^{hU_i} \wedge M_I$, for each i.

An advantage of (a) is that G_n/U_i is a finite discrete group, so that its topology is simpler. For example, in (b) one simply asks for a finite group to act by A_∞-maps, but in (a), the A_∞-action must also respect the discrete topology on all the $(E_n/I)_k$ and the profinite topology of G_n. Also, conceptually, it is easier to work with the category of modules over $\pi_*(E_n^{hU_i})[G_n/U_i]$ than with $(E_n)_*[[G_n]]$, because with the former twisted group ring, one can ignore the topology. This makes the homological algebra involved in (b) simpler than the homological algebra for (a). When working with the homological algebra for (a), I often was forced to work with an Ext, whose topological algebra was so complicated that I never succeeded in getting it to behave as needed.

However, a disadvantage of (b) is that, in general, $\pi_*(E_n^{hU_i} \wedge M_I)$ is not explicitly known, whereas, in (a), $\pi_*(E_n \wedge M_I)$ is complete known. In trying (b), I almost always find myself in a situation where I cannot proceed beyond the first step, due to the lack of computational knowledge about $\pi_*(E_n^{hU_i} \wedge M_I)$. This disadvantage of (b) has always meant that I have to proceed with (a), and abandon (b).

3. My approach to this problem

Now I briefly describe my work in trying to realize E_n/I as a discrete G_n-symmetric ring spectrum. For simplicity, let's view the Hopkins-Miller and Goerss/Hopkins machinery for constructing A_∞ and E_∞-actions of G_n on E_n as consisting of a universal coefficient spectral sequence and an obstruction theory for realizing F over an Adams spectrum E. For example, in the Hopkins-Miller theorem, $E = F = E_n$.

I have made two major efforts towards obtaining $E_n/I \in \mathrm{Spt}^a_{G_n}$. In both efforts, $F = E_n/I$. In the first attempt, I let $E = E_n$ and tried for a long time to build a universal coefficient sequence for Spt_{G_n} of the form

$$\mathrm{Ext}^{*,*}_{(E_n)_*[[G_n]]}(\mathrm{Map}_c(G_n, (E_n)_*/I, (E_n)_*) \Rightarrow \pi_* \mathrm{Map}_{G_n}(E_n/I, E_n/I).$$

But I never succeeded in constructing this spectral sequence because I could not get the requisite homological algebra of topological modules to work out.

Then I tried a different strategy: build the obstruction theory for $E = F = E_n/I$. To begin with a (partly) developed a version of André-Quillen cohomology for twisted discrete associative R-G-algebras, where R is a twisted discrete commutative G-ring. The Hopkins/Miller and Goerss/Hopkins machinery and [3] imply that the obstructions to the existence of $E_n/I \in \mathrm{Spt}^a_{G_n}$ are in

$$R^{t+2}\mathrm{Der}^{G_n}_{(E_n)_*/I}((E_n/I)_*(E_n/I), E_{n,*+t}/I), \quad t \geq 0.$$

I tried to compute these obstructions for the case $n = 1$ and $p > 2$, but I never succeeded in this.

Since finishing my Ph.D. two years ago, I have not tried to take the two efforts above any further. Regarding the second effort, I concluded that $(E_n/I)_*(E_n/I)$ is probably too nasty to work with successfully. Regarding the first effort, at one point, I thought that I had succeeded in getting the requisite homological algebra to work out, but I did not have time to work through it carefully. The constraints of time during my Ph.D. forced me to stop the first effort completely and work on the second one.

In the past two years, I have primarily taken a step back and tried to reflect on what is the best strategy for tackling this problem. I have decided that the best way to proceed is to go back to effort one, where $E = E_n$ and $F = E_n/I$, and see if I was correct about finally getting the homological algebra to work out. This way seems best because it seems that it would be closest to the Hopkins/Miller and Goerss/Hopkins approaches. Unfortunately, the level of difficulty of this approach seems such that it would be unwise to take a major chunk of time and pursue it, at this stage of my career. Instead, I believe that I should instead focus on my other projects, which are easier and which allow me to have publications.

Sometime after making this (disappointing) decision, Mark Behrens told me about your alternative strategy of producing a continuous structured action. As far as I can tell your strategy does not help with the problem of realizing $E_n/I \in \mathrm{Spt}^a_{G_n}$, but it does seem to be a doable approach for obtaining an interesting result about E_n.

Because I love this problem and your strategy offers a way of making progress in the immediate future, I am very interested in learning about it.

Sometimes I wonder if the following approach to $E_n/I \in \mathrm{Spt}^a_{G_n}$ offers a better way than that given by $E = E_n$, $F = E_n/I$. Since $E_n \wedge M_I$ is an $L_{K(n)}S^0$-module and $L_{K(n)}S^0$ is a trivial discrete G_n-spectrum, let $E = L_{K(n)}S^0$ and $F = E_n/I$. Ignoring the fact that I don't know if $L_{K(n)}S^0$ is an Adams spectrum (it needs to be one), the obstructions to the existence of $E_n/I \in \mathrm{Spt}^a_{G_n}$ are in

$$R^{t+2}\mathrm{Der}^{G_n}_{\pi_*(L_{K(n)}S^0)}((E_n)_*/I, (E_n)_{*+t}/I), \quad t \geq 0.$$

If we can let $E = L_{K(n)}S^0$ in the Hopkins/Miller and Goerss/Hopkins machinery, and if one is in the range where $\pi_*(L_{K(n)}S^0)$ is computationally accessible, then I wonder if these obstruction groups might be computable; because the action on $\pi_*(L_{K(n)}S^0)$ is trivial, its profinite topology should not present as many problems when one tries to work out the homological algebra. I wonder if you have any thoughts about whether or not this approach might be a good one.

Note that $E_n^{hU_{i+j}}$ is an $E_n^{hU_i}$-module, for any $j \geq 0$. Combining Remark (2.10) with the above strategy, leads one to note that another strategy, which depends on whether or not $E_n^{hU_i}$ is an Adams spectrum to just get off the ground, is to let $E = E_n^{hU_i}$ and $F = E_n^{hU_{i+j}}/I$, where $j \geq 0$. In this case, the obstructions are

$$R^{t+2}\mathrm{Der}^{U_i/U_{i+j}}_{\pi_*(E_n^{hU_i})}(\pi_*(E_n^{hU_i} \wedge E_n^{hU_{i+j}} \wedge M_I), \pi_{*+t}(E_n^{hU_{i+j}}/I)),$$

where U_i/U_{i+j} is a finite group.

Another approach, which might computationally be the best, but which technically would be the most difficult to construct, is to apply the "look at the associated Morava module" technique. If $X \to Y$ is a map between $K(n)$-local spectra, often it is easiest to prove that this is a weak equivalence by showing that the associated map of Morava modules $L_{K(n)}(E_n \wedge X) \to L_{K(n)}(E_n \wedge Y)$ is a weak equivalence. Thus, one might redo the Hopkins/Miller machinery for realizing F by working over E, so that the machinery realizes F by looking at $L_{K(n)}(E_n \wedge F)$, while working over $L_{K(n)}(E_n \wedge E_n)$. Then the obstruction groups might look like

$$R^{t+2}\mathrm{Der}^{G_n}_{\mathrm{Map}_c(G_n,(E_n)_*)}(\mathrm{Map}_c(G_n^2, (E_n)_*/I), \mathrm{Map}_c(G_n, (E_n)_{*+t}/I) \quad \text{and}$$

$$R^{t+2}\mathrm{Der}^{U_i/U_{i+j}}_{\prod_{G_n/U_i}(E_n)_*}(\mathrm{Map}_c(G_n/U_i \times G_n/U_{i+j}, (E_n)_*/I), \textstyle\prod_{G_n/U_{i+j}}(E_n)_{*+t}/I),$$

depending on which strategy is chosen.

Sincerely,
Daniel

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