THE HOMOTOPY FIXED POINT SPECTRA OF PROFINITE GALOIS EXTENSIONS

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Abstract. Let E be a k-local profinite G-Galois extension of an E_∞-ring spectrum A (in the sense of Rognes). We show that E may be regarded as producing a discrete G-spectrum. We show that if E is a faithful k-local profinite extension which satisfies certain extra conditions, then the forward direction of Rognes's Galois correspondence extends to the profinite setting. We show the function spectrum $F_A((E^{hH})_k, (E^{hK})_k)$ is equivalent to the homotopy fixed point spectrum $((E[[G/H]])^{hK})_k$ where H and K are closed subgroups of G. Applications to Morava E-theory are given.

Contents

1. Introduction

In [30], John Rognes develops a Galois theory of commutative S-algebras which mimics Galois theory for rings. Let k be a spectrum, and let $(-)_k$ denote Bousfield localization with respect to k. Given a cofibrant commutative k-local S-algebra A, and a cofibrant commutative k-local A-algebra E, Rognes gives the following definition of a finite k-local Galois extension.

Definition 1.0.1 (Finite Galois extension)**.** The spectrum E is a *k-local G-Galois extension* of A for a finite group G if it satisfies the following conditions:

(1) G acts on E through commutative A-algebra maps.
(2) The canonical map $A \to E^{hG}$ is an equivalence.
(3) The canonical map $(E \wedge_A E)_k \to \mathrm{Map}(G, E)$ is an equivalence.

E is said to be *k-locally faithful* over A if $(M \wedge_A E)_k \simeq *$ implies that $M_k \simeq *$ for every A-module M. In the context of k-local Galois extensions we shall simply to such extentions as being *faithful*.

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Rognes does not require that A and E be k-local, but for convenience, we do.

Remark 1.0.2. Rognes (see also Baker-Richter [1]) shows that a G-Galois extension is faithful if and only if the additive form of Hilbert's Theorem 90 holds:

$$(E^{tG})_k \simeq *.$$

We will always consider faithful Galois extensions, because these are the Galois extensions for which the fundamental theorem of Galois theory holds. It is not known whether there exist non-faithful Galois extensions.

Let G be a profinite group. Rognes defines a (faithful) k-local profinite G-Galois extension E of A to be a colimit (in the category of commutative S-algebras) of (faithful) k-local G/U_α-Galois extensions E_α of A, for a cofinal system of open normal subgroups U_α of G (see Definition 6.2.1). Since a colimit of k-local spectra need not be k-local, the spectrum E is not necessarily k-local.

In [5], the second author develops a category of discrete G-spectra and defines their homotopy fixed points (see also [35], [19], [26], [10], [21]). In this paper we examine profinite k-local G-Galois extensions E of A as objects in the category of discrete G-spectra, and study the spectra of A-module maps between the various homotopy fixed point spectra of E. Unfortunately, to say meaningful things it seems that we must impose more hypotheses on our profinite Galois extensions.

Assumption 1.0.3. In this paper we shall only concern ourselves with localizations $(-)_k$ which are given as a composite of two localization functors $((-)_T)_M$ where $(-)_T$ is a smashing localization and $(-)_M$ is a localization with respect to a finite spectrum M. The spectra S, $H\mathbb{F}_p$, $E(n)$, and $K(n)$ are all examples of such localizations [3],[15].

For a cofibrant commutative S-algebra B and a cofibrant commutative B-algebra C, the k-local Amitsur derived completion $B^\wedge_{k,C}$ is the homotopy limit of the cosimplicial spectrum

$$C_k \Rightarrow (C \wedge_B C)_k \Rrightarrow (C \wedge_B C \wedge_B C)_k \cdots$$

(see, for example, [30]).

Definition 1.0.4. Let E be a profinite k-local G-Galois extension of A.

(1) The extension E is *consistent* if the k-local Amitsur derived completion

$$A \to A^\wedge_{k,E}$$

is an equivalence.

(2) The extension E is of *finite virtual cohomological dimension* (finite vcd) if the profinite group G has an open subgroup U of finite cohomological dimension (there exists a d so that $H^s_c(U, M) = 0$ for each $s > d$ and each discrete U-module M).

Assumption 1.0.3 ensures that if E has finite vcd, then the condition of E being consistent is equivalent to requiring that the map

$$A \to (E^{hG})_k$$

is an equivalence. This is proven as Corollary 6.3.2.

It will then follow that the maps

$$E_\alpha \to (E^{hU_\alpha})_k$$

are equivalences (Lemma 6.3.4). The consistency hypothesis may be unnecessary, since we do not know of any profinite Galois extensions which are not consistent.

The main concern of this paper is the study of the intermediate homotopy fixed point spectra E^{hH} with respect to *closed* subgroups H of G. We prove the "forward" direction of the Galois correspondence.

Theorem (7.2.1)**.** Suppose that E is a consistent faithful k-local G-Galois extension of A of finite vcd, and that H is a closed subgroup of G.

(1) The spectrum E is k-locally H-equivariantly equivalent to a consistent faithful k-local H-Galois extension of $(E^{hH})_k$ of finite vcd.
(2) If H is a normal subgroup of G, then the spectrum E^{hH} is k-locally equivalent to a faithful k-local G/H-Galois extension of A. If the quotient G/H has finite vcd, then E^{hH} is consistent (and of finite vcd) over A.

We also identify the function spectra of A-module maps between any two such homotopy fixed point spectra.

Theorem (7.3.1)**.** Let E be a k-local consistent faithful profinite G-Galois extension of finite vcd, and let H and K be closed subgroups of G. Then there is an equivalence

$$F_A((E^{hH})_k, (E^{hK})_k) \simeq ((E[[G/H]])^{hK})_k. \tag{1.1}$$

The spectrum $E[[G/H]]$ that appears on the right-hand side of (1.1) is the *continuous* G-spectrum with the diagonal action. The case where $K = H = \{e\}$ is the trivial subgroup of G was handled by Rognes [30, Sec. 8].

Example 1.0.5. One important example of a profinite Galois extension is given by Morava E-theory. Let $k = K(n)$ be the nth Morava K-theory spectrum and let $A = S_{K(n)}$ be the $K(n)$-local sphere spectrum. Let $G = \mathbb{G}_n$ be the nth extended Morava stabilizer group $\mathbb{S}_n \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$. Since the group $\mathbb{G}_n$ is a p-adic analytic group, it has finite virtual cohomological dimension, but for general n the group $\mathbb{G}_n$ does not have finite cohomological dimension. Let E_n be the nth Morava E-theory spectrum, where $(E_n)_* = W(\mathbb{F}_{p^n})[[u_1, ..., u_{n-1}]][u^{\pm 1}]$, such that the degree of u is -2 and the complete power series ring is in degree zero. Goerss and Hopkins [11], building on work of Hopkins and Miller [29], have shown that $\mathbb{G}_n$ acts on E_n by maps of commutative S-algebras. Devinatz and Hopkins [7] have given constructions of homotopy fixed point spectra E_n^{hH} for closed subgroups H of $\mathbb{G}_n$. In particular, they show that there is an equivalence

$$E_n^{h\mathbb{G}_n} \simeq S_{K(n)}.$$

Thus, the homotopy fixed point spectra of E_n are intimately related to the nth chromatic layer of the sphere spectrum. Also, these homotopy fixed point spectra coincide with those given for continuous G-spectra [4]. Rognes [30] observed for U an open normal subgroup of $\mathbb{G}_n$, that the work of Devinatz and Hopkins proves that E_n^{hU} is a $K(n)$-local $\mathbb{G}_n/U$-Galois extension of $S_{K(n)}$. Therefore, the discrete $\mathbb{G}_n$-spectrum

$$F_n = \operatorname*{colim}_{U \trianglelefteq_o \mathbb{G}_n} E_n^{hU}$$

is a profinite $K(n)$-local $\mathbb{G}_n$-Galois extension of $S_{K(n)}$. The spectrum E_n is recovered by the equivalence [7]

$$E_n \simeq (F_n)_{K(n)}.$$

Consistency and faithfulness follow immediately from the fact that we are working $K(n)$-locally. In this context, (1.1) was proven in [9] for H and K finite. It was also suggested by the authors of [9] that (1.1) should be true for arbitrary closed subgroups of $\mathbb{G}_n$. Another source of motivation for this work arises from the fact that a special case of (1.1) (Corollary 7.3.2) was needed in an essential way by the first author in [2].

The paper is organized as follows. Our notion of homotopy fixed point spectra uses the framework of equivariant spectra (with respect to a profinite group) as developed by second author [5]. The foundations in [5] use Bousfield-Friedlander spectra. Since we need to work with structured ring spectra to do Galois theory, it is essential for this paper that we reformulate the second author's work in the context of symmetric spectra. A terse summary of these foundations appears in Section 2. In Section 3 we describe properties of the homotopy fixed point functor. In Section 4, we describe continuous G-spectra, generalizing somewhat the setting of [5]. In Section 5, we explain how to extend our constructions to categories of modules and commutative algebras of spectra. In Section 6 we explain how profinite Galois extensions give rise to discrete G-spectra, and show that the homotopy fixed points with respect to open subgroups of the Galois group give rise to intermediate finite Galois extensions. In Section 7 we prove our results concerning the homotopy fixed point spectra with respect to closed subgroups of the Galois group.

Acknowledgments. This paper is an extension of ideas contained in [7], [9], and [30]. The first author benefited from the input of Halvard Fausk and Daniel Isaksen. The authors also thank Paul Goerss for helpful discussions related to this work.

2. Discrete symmetric G-spectra

Let G be a profinite group. We begin this section by describing the basic categories of discrete G-objects that will be used in this paper. We then describe and compare the model structures on the categories of discrete G-objects of Bousfield-Friedlander and symmetric spectra. We end this section with descriptions of some basic constructions in the category of discrete G-spectra. More detailed accounts of some of these model categories and constructions can be found in [5].

2.1. **Simplicial discrete G-sets.** A G-set Z is said to be discrete if for every element $z \in Z$, the stabilizer $\mathrm{Sta}_G(z)$ is open in G. We may express this condition as saying that Z is the colimit of its fixed points:

$$Z = \operatorname*{colim}_{U \leq_o G} Z^U,$$

where the colimit is taken over all open subgroups. These conditions are equivalent to the condition that the action map $G \times Z \to Z$ is continuous, when Z is given the discrete topology. A simplicial discrete G-set is a simplicial object in the category of discrete G-sets.

2.2. **Discrete G-spectra.** Define the category of discrete G-spectra Sp_G to be the category of Bousfield-Friedlander spectra of simplicial discrete G-sets. An object $X \in \mathrm{Sp}_G$ consists of a sequence $\{X_i\}_{i \geq 0}$, where each X_i is a pointed simplicial discrete G-set, together with G-equivariant maps

$$\sigma_i : S^1 \wedge X_i \to X_{i+1}.$$

Here, S^1 is given the trivial G-action.

A map $f : X \to Y$ of discrete G-spectra is a sequence of G-equivariant maps of simplicial sets $f_i : X_i \to Y_i$ which are compatible with the spectrum structure maps.

2.3. **Discrete symmetric G-spectra.** Define the category of discrete symmetric G-spectra $\Sigma\mathrm{Sp}_G$ to be the category of symmetric spectra of simplicial discrete G-sets (see [16], [28], [32] for accounts of symmetric spectra). An object $X \in \Sigma\mathrm{Sp}_G$ consists of a sequence $\{X_i\}_{i\geq 0}$, where each X_i is a pointed simplicial discrete $G\times\Sigma_i$-set, together with suitably compatible $G \times \Sigma_i \times \Sigma_j$-equivariant maps

$$\sigma_{i,j} : S^i \wedge X_j \to X_{i+j}.$$

Here, $S^i = (S^1)^{\wedge i}$ is given the trivial G-action, and Σ^i permutes the smash factors. When G is finite, a discrete symmetric G-spectrum is a simply a naïve symmetric G-spectrum, and not a genuine equivariant symmetric G-spectrum in the sense of [25].

Maps $f : X \to Y$ of discrete symmetric G-spectra are sequences of $G \times \Sigma_i$ equivariant maps of simplicial sets $f_i : X_i \to Y_i$ which are compatible with the spectrum structure maps.

2.4. **Model structures.** Let $G-\mathrm{Sets}_{df}$ be the site of finite G-sets. Consider the following categories.

$$\begin{aligned}
\mathrm{PreSp}(G-\mathrm{Sets}_{df}) &= \text{category of presheaves of spectra on } G-\mathrm{Sets}_{df}.\\
\mathrm{ShvSp}(G-\mathrm{Sets}_{df}) &= \text{category of sheaves of spectra on } G-\mathrm{Sets}_{df}.\\
\mathrm{Pre}\Sigma\mathrm{Sp}(G-\mathrm{Sets}_{df}) &= \text{category of presheaves of symmetric spectra on } G-\mathrm{Sets}_{df}.\\
\mathrm{Shv}\Sigma\mathrm{Sp}(G-\mathrm{Sets}_{df}) &= \text{category of sheaves of symmetric spectra on } G-\mathrm{Sets}_{df}.
\end{aligned}$$

Then we have the following diagram of categories and Quillen adjoint pairs of functors.

$$\begin{array}{ccccc}
\mathrm{PreSp}(G-\mathrm{Sets}_{df}) & \underset{\mathbb{U}}{\overset{\mathcal{L}^2}{\rightleftarrows}} & \mathrm{ShvSp}(G-\mathrm{Sets}_{df}) & \underset{R}{\overset{L}{\rightleftarrows}} & \mathrm{Sp}_G \\
\mathbb{U}\uparrow\downarrow\mathbb{P}_{Pre} & & \mathbb{U}\uparrow\downarrow\mathbb{P}_{Shv} & & \mathbb{U}\uparrow\downarrow\mathbb{P}_{\Sigma} \\
\mathrm{Pre}\Sigma\mathrm{Sp}(G-\mathrm{Sets}_{df}) & \underset{\mathbb{U}}{\overset{\mathcal{L}^2}{\rightleftarrows}} & \mathrm{Shv}\Sigma\mathrm{Sp}(G-\mathrm{Sets}_{df}) & \underset{R}{\overset{L}{\rightleftarrows}} & \Sigma\mathrm{Sp}_G
\end{array} \tag{2.1}$$

The functors $\mathbb{U}$ are forgetful functors, the functors $\mathcal{L}^2$ are sheafification functors, and the functors $\mathbb{P}_{(-)}$ are prolongation functors (left Kan extensions). The functor L applied to a sheaf $\mathcal{F}$ gives the discrete G-spectrum

$$\operatorname*{colim}_{U\leq_o G} \mathcal{F}(G/U).$$

It has a right adjoint R which, given a discrete G-spectrum X, returns the sheaf $\mathcal{F}$ whose values on orbits are given by

$$\mathcal{F}(G/U) = \mathrm{Map}_G(G/U, E).$$

The top row of Diagram 2.1 is studied in detail in [5]. The pairs (L, R) are equivalences of categories. Therefore, the model structures on $\mathrm{ShvSp}(G-\mathrm{Sets}_{df})$ [12] and $\mathrm{Shv}\Sigma\mathrm{Sp}(G-\mathrm{Sets}_{df})$ [22] give model structures on Sp_G and $\Sigma\mathrm{Sp}_G$.

The model category structures on the categories of presheaves are defined so that the pairs $(\mathcal{L}^2, \mathcal{U})$ are Quillen equivalences [13], [22]. Jardine [22] also shows that the pair $(\mathbb{P}_{Pre}, \mathbb{U})$ is a Quillen equivalence. We therefore have the following proposition, which explains how to translate the results of [4], which take place in the category Sp_G, to the category $\Sigma\mathrm{Sp}_G$.

Proposition 2.4.1. The pair of functors $(\mathbb{P}_\Sigma, \mathbb{U})$ gives a Quillen equivalence between Sp_G and $\Sigma\mathrm{Sp}_G$.

The model structure on Sp_G is shown in [5] to have a concrete description: the cofibrations and weak equivalences are detected by forgetting the G-action, and, as usual, this determines the fibrations. The same is also true of the model structure on $\Sigma\mathrm{Sp}_G$.

From this point on, we shall be working in the world of symmetric spectra, and shall refer to a symmetric spectrum as simply a spectrum.

2.5. **Mapping spectra.** Let K and L be discrete G-sets. Then the set of (non-equivariant) functions $\mathrm{Map}(K, L)$ is a G-set with G acting by conjugation. For $g \in G$ and $f \in \mathrm{Map}(K, L)$, $g \cdot f$ is the map

$$(g \cdot f)(z) = gf(g^{-1}z).$$

Observe that $\mathrm{Map}(K, L)$ is not in general a discrete G-set, but it is if K is finite.

For a finite set K and a spectrum X, we define the mapping spectrum $\mathrm{Map}(K, X)$ to be the discrete G-spectrum whose mth space is given by

$$\mathrm{Map}(K, X)_m = \mathrm{Map}(K, X_m),$$

where the n-simplices of $\mathrm{Map}(K, X_m)$ is the set $\mathrm{Map}(K, (X_m)_n)$. If X is a discrete G-spectrum, then $\mathrm{Map}(K, X)$ is as well.

If $K = \lim_i K_i$ is a profinite set and X is a spectrum, then the spectrum of continuous maps is the spectrum

$$\mathrm{Map}^c(K, X) = \operatorname*{colim}_i \mathrm{Map}(K_i, X).$$

If K is a continuous G-set and X is a discrete G-spectrum, then $\mathrm{Map}^c(K, X)$ is a discrete G-spectrum.

Lemma 2.5.1. Let $K = \lim_i K_i$ be a Mittag-Leffler profinite set. The functor

$$\mathrm{Map}^c(K, -) : \Sigma\mathrm{Sp} \to \Sigma\mathrm{Sp}$$

preserves all stable equivalences.

Proof. A map of symmetric spectra is a stable equivalence if and only if it induces isomorphisms on all generalizes cohomology theories. Let $f : X \to Y$ be a stable equivalence of symmetric spectra, and let E be a generalized cohomology theory. The Milnor sequences give the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E^*(X)[[K]] & \longrightarrow & E^*(\mathrm{Map}^c(K, X)) & \longrightarrow & {\lim_i}^1 E^*(X)[K_i] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & E^*(Y)[[K]] & \longrightarrow & E^*(\mathrm{Map}^c(K, Y)) & \longrightarrow & {\lim_i}^1 E^*(Y)[K_i] & \longrightarrow & 0 \end{array}$$

where the top and bottom rows are exact. The five lemma implies that the map

$$f_* : \mathrm{Map}^c(K, X) \to \mathrm{Map}^c(K, Y)$$

is an E^*-isomorphism. Since E was arbitrary, we conclude that f is a stable equivalence. □

2.6. **Permutation spectra.** Let K be a discrete G-set. Then for X a discrete G-spectrum, we may define the permutation spectrum $X[K]$ to be the spectrum whose nth space is given by

$$X[K]_n = X_n \wedge K_+.$$

This is a discrete G-spectrum with G acting diagonally.

2.7. **Smash products.** Given discrete G-spectra X and Y, we define their smash product

$$X \wedge Y$$

to be the smash product of the underlying symmetric spectra with G acting diagonally. It is immediate from the definition of the smash product that this action is discrete. There is an isomorphism

$$X \wedge S[K] \cong X[K],$$

where S is the sphere spectrum.

3. Homotopy fixed points of discrete G-spectra

Much of the material in this section is assembled from [35], [20], [10], [21], [27], [23], and [5]. Let G be a profinite group. We begin this section with an account of the model category theoretic definition of G-homotopy fixed points. We then describe the comparison with hypercohomology spectra. Finite index restriction and induction functors, as well as iterated homotopy fixed points for finite index subgroups are then discussed. We explain how continuous homomorphisms of groups induce various "change of group functors," of which induction, coinduction, fixed points, and restriction functors are all special cases. We then describe the various technical difficulties related to the fixed point construction for closed subgroups of G. The technical difficulties are observed to vanish if G has finite cohomological dimension.

3.1. **The homotopy fixed point spectrum.** For a discrete G-spectrum X, we define the fixed point spectrum by taking the fixed points level-wise

$$(X^G)_i = (X_i)^G.$$

The G-fixed point functor is right adjoint to the functor *triv* which associates to a spectrum the associated discrete G-spectrum with trivial G-action.

$$\mathit{triv} : \Sigma\mathrm{Sp} \leftrightarrows \Sigma\mathrm{Sp}_G : (-)^G.$$

Lemma 3.1.1. The adjoint functors $(\mathit{triv}, (-)^G)$ form a Quillen pair.

Proof. The functor *triv* preserves cofibrations and weak equivalences. □

Let $\alpha_{G,X} : X \to X_{fG}$ denote a functorial fibrant replacement functor for the model category $\Sigma\mathrm{Sp}_G$, where $\alpha_{G,X}$ is a trivial cofibration of discrete G-spectra. The homotopy fixed point functor $(-)^{hG}$ is the Quillen right derived functor of $(-)^G$, and is thus given by

$$X^{hG} = (X_{fG})^G.$$

3.2. **Hypercohomology spectra.** The functor $\Gamma_G = \mathrm{Map}^c(G, -)$ is a coaugmented comonad on the category of spectra, with coproduct

$$\psi : \Gamma_G = \mathrm{Map}^c(G, -) \to \mathrm{Map}^c(G \times G, -) = \Gamma_G \circ \Gamma_G$$

induced from the product on G, counit

$$\Gamma_G = \mathrm{Map}^c(G, -) \to \mathrm{Map}^c(pt, -) = \mathrm{Id}$$

induced from the unit on G, and coaugmentation

$$\mathrm{Id} \to \mathrm{Map}^c(G, -)$$

given by the inclusion of the constant maps.

Discrete G-spectra are coalgebras over the comonad Γ_G. In [5], the homotopy fixed point spectrum was shown to have the following alternate description, provided G is sufficiently nice (see also [26], [10], [20]).

Theorem 3.2.1. Suppose that G has finite vcd. Then there are equivalences

$$\begin{aligned} X^{hG} &\simeq \operatorname*{holim}_{\Delta} \Gamma^{\bullet}_G X_{fG} \\ &\simeq \operatorname*{holim}_{\Delta} \Gamma^{\bullet}_G X \\ &= \mathbb{H}_c(G; X) \end{aligned}$$

Here, $\mathbb{H}_c(G; X)$ is the hypercohomology spectrum.

Proof. A proof of the first equivalence may be found in [5] (the cosimplicial object defining the hypercohomology spectrum is different, but isomorphic to that appearing in [5]). The second equivalence follows since the fibrant replacement map $X \to X_{fG}$ is an equivalence. The functor $\Gamma_G(-)$ preserves equivalences by Lemma 2.5.1, and the homotopy limit construction sends level-wise equivalences to equivalences since it is a Quillen derived functor. □

3.3. **Iterated homotopy fixed points.** Let U be an open normal subgroup of G, so that G/U is finite.

Proposition 3.3.1. Let X be a discrete G-spectrum.

(1) The U fixed point spectrum $(X_{fG})^U$ is fibrant as a G/U-spectrum.
(2) The fibrant G-spectrum X_{fG} is fibrant as a discrete U-spectrum.
(3) The homotopy fixed point spectrum X^{hU} is naturally equivalent to a G/U-spectrum.
(4) There is an equivalence $X^{hG} \simeq (X^{hU})^{hG/U}$.

Proof. To prove (1), observe that since U is normal, for any discrete G-spectrum Y, the U-fixed point spectrum Y^U is naturally a G/U spectrum. There is an adjoint pair of functors $(\mathrm{Res}^G_{G/U}, (-)^U)$

$$\mathrm{Res}^G_{G/U} : \Sigma\mathrm{Sp}_G \leftrightarrows \Sigma\mathrm{Sp}_{G/U} : (-)^U,$$

where the $\mathrm{Res}^G_{G/U}$ is restriction along the quotient homomorphism $G \to G/U$. Since $\mathrm{Res}^G_{G/U}$ preserves cofibrations and weak equivalences, the functor $(-)^U$ preserves fibrant objects.

(2) is similar (compare with [21, Rmk. 6.26]). Define the induction functor on a discrete U-spectrum Y to be the Borel construction

$$\mathrm{Ind}^G_U Y = G_+ \wedge_U Y.$$

Here, the Borel construction is taken regarding G and U as being discrete groups, but this is easily seen to produce a discrete G-spectrum since U is a normal subgroup of finite index. The induction functor is the left adjoint of an adjunction

$$\mathrm{Ind}_U^G : \Sigma\mathrm{Sp}_U \leftrightarrows \Sigma\mathrm{Sp}_G : \mathrm{Res}_G^U,$$

where Res_G^U is restriction along the inclusion $U \hookrightarrow G$. Since non-equivariantly we have an isomorphism

$$\mathrm{Ind}_U^G Y \cong G/U_+ \wedge Y,$$

we see that Ind_U^G preserves cofibrations and weak equivalences, from which it follows that Res_G^U preserves fibrant objects.

The spectrum $(X_{fG})^U$ is a G/U-spectrum. By (2), the spectrum X_{fG} is fibrant as a discrete U-spectrum, so there is a weak equivalence of U spectra

$$X_{fG} \to X_{fU}.$$

This gives an equivalence

$$(X_{fG})^U \to (X_{fU})^U = X^{hU}.$$

This proves (3).

(4) is proven using our fibrancy results. There are equivalences:

$$X^{hG} \simeq X_{fG}^G = (X_{fG}^U)^{G/U} \simeq (X^{hU})^{hG/U}.$$

□

3.4. **Homomorphisms of groups.** If $f : H \to G$ is a continuous homomorphism of profinite groups, we may regard discrete G-sets as discrete H-sets. For a discrete H-set Z, we define the coninduced discrete G-set by

$$f_* Z = \mathrm{CoInd}_H^G Z = \mathrm{Map}_H^c(G, Z) = \operatorname*{colim}_{U \trianglelefteq_o G} \mathrm{Map}_H(G/U, Z).$$

An element $g \in G$ acts on an element $\alpha \in \mathrm{Map}_H(G/U, Z)$ by the formula

$$(g \cdot \alpha)(g'U) = \alpha(g'gU).$$

This construction extends to simplicial discrete G-sets and discrete G-spectra in the obvious manner to give a functor

$$f_* : \Sigma\mathrm{Sp}_H \to \Sigma\mathrm{Sp}_G.$$

The functor f_* is the right adjoint of an adjoint pair (f^*, f_*), where

$$f^* = \mathrm{Res}_G^H : \Sigma\mathrm{Sp}_G \to \Sigma\mathrm{Sp}_H$$

is the restriction functor along the homomorphism f. Since f^* clearly preserves cofibrations and weak equivalences, we have the following lemma.

Lemma 3.4.1. The adjoint functors (f^*, f_*) form a Quillen pair. In particular, f_* preserves fibrations and weak equivalences between fibrant objects.

We make the following observations.

1. The Quillen pair (f^*, f_*) gives rise to a derived adjoint pair (Lf^*, Rf_*).
2. Since the functor f^* preserves all weak equivalences, there are equivalences $Lf^* X \simeq f^* X$ for all discrete G-spectra X.

(3) If $j : H \hookrightarrow G$ is the inclusion of a closed subgroup, then for a discrete H-spectrum X, we have a *non-equivariant* isomorphism
$$j_* X = \mathrm{Map}^c_H(G, X) \cong \mathrm{Map}^c(G/H, X).$$
By Lemma 2.5.1, we see that j_* preserves weak equivalences, and therefore there is an equivalence $j_* X \simeq Rj_* X$.

(4) The adjoint pair $(\mathit{triv}, (-)^G)$ of Section 3.1 agrees with the adjoint pair (r^*, r_*) where $r : G \to e$ is the homomorphism to the trivial group. Therefore, the homotopy fixed point functor is given by $(-)^{hG} = Rr_*$.

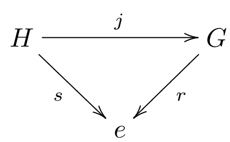
(5) If $H \xrightarrow{f} G \xrightarrow{g} K$ is a composible pair of continuous homomorphisms, there are natural isomorphisms $(g \circ f)_* \cong g_* \circ f_*$ and $(g \circ f)^* \cong f^* \circ g^*$. We get similar formulas on the level of derived functors.

(6) If $i : U \hookrightarrow G$ is the inclusion of an open subgroup, then the induction functor $i_! = \mathrm{Ind}_U^G$ (Proposition 3.3.1) is the left adjoint of a Quillen pair $(i_!, i^*)$.

We use these derived functors to prove a version of Shapiro's Lemma.

Lemma 3.4.2. Let X be a discrete G-spectrum, and suppose that H is a closed subgroup of G. Then there is an equivalence
$$\mathrm{Map}^c(G/H, X)^{hG} \xrightarrow{\simeq} X^{hH}.$$

Proof. Consider the following diagram of groups.



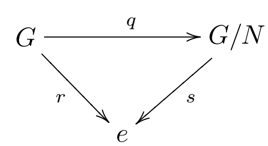
If Z is a discrete G-set, there is a G-equivariant bijection
$$\delta : j_* j^* Z = \mathrm{Map}^c_H(G, Z) \xrightarrow{\cong} \mathrm{Map}^c(G/H, X).$$
The map δ sends a map α in $\mathrm{Map}^c_H(G, Z)$ to the map
$$\delta(\alpha) : gH \mapsto g\alpha(g^{-1}).$$
The inverse map δ^{-1} sends a map β in $\mathrm{Map}^c(G/H, Z)$ to the map
$$\delta^{-1}(\beta) : g \mapsto g\beta(g^{-1}H).$$
The isomorphism δ induces for a discrete G-spectrum Y an isomorphism
$$\delta : j_* j^* Y \xrightarrow{\cong} \mathrm{Map}^c(G/H, Y).$$
By Lemma 3.4.1, the functor j_* sends H-fibrant objects to G-fibrant objects. Therefore we have equivalences:
$$\begin{aligned} \mathrm{Map}(G/U, X)^{hG} &= Rr_* j_* j^* X \\ &\simeq Rr_* Rj_* j^* X \\ &\simeq Rs_* j^* X \\ &= X^{hH}. \end{aligned}$$
□

3.5. **Iterated fixed points for closed subgroups.** Let $j : H \hookrightarrow G$ be the inclusion of a closed subgroup. We wish to extend the results of Section 3.3 to the closed subgroup H. The following proposition may be compared to [21, Lem. 6.35]

Proposition 3.5.1. Let N be a closed normal subgroup of G, and let X be a discrete G-spectrum. Then there is an equivalence

$$((X_{fG})^N)^{hG/N} \simeq X^{hG}.$$

Proof. Consider the following diagram.



There is an equivalence

$$((X_{fG})^N)^{hG/N} = Rs_*Rq_*X \simeq Rr_*X = X^{hG}.$$

□

The reader might hope that X_{fG} was fibrant as a discrete H-spectrum, but this does not appear to hold for closed subgroups H. We discuss these difficulties in Section 3.6. The mysterious spectrum $(X_{fG})^H$ may be identified as the following colimit.

Corollary 3.5.2. Let G be of finite cohomological dimension. Then there is an equivalence

$$(X_{fG})^H \simeq \operatorname*{colim}_{H \leq U \leq_o G} X^{hU}.$$

Proof. Since H acts discretely on X_{fG}, we have an equality

$$(X_{fG})^H = \operatorname*{colim}_{H \leq U \leq_o G} (X_{fG})^U.$$

By Proposition 3.3.1, the spectrum X_{fG} is fibrant as a discrete U-spectrum, so there are equivalences $(X_{fG})^U \simeq X^{hU}$. □

If we confine ourselves to the case where G has finite cohomological dimension, then the iterated homotopy fixed point theorem takes on a more satisfactory form.

Proposition 3.5.3. Suppose that G has finite cohomological dimension, and suppose that X is a discrete G-spectrum. Then the natural map

$$\operatorname*{colim}_{H \leq U \leq_o G} X^{hU} \to X^{hH}$$

is an equivalence.

Corollary 3.5.4. Let G be of finite cohomological dimension, and let X be a discrete G-spectrum.

(1) There is an equivalence $(X_{fG})^H \simeq X^{hH}$.
(2) Suppose H is normal in G. Then, using the above model for the H-homotopy fixed point spectrum, there is an equivalence $(X^{hH})^{hG/H} \simeq X^{hG}$.

In Section 7.1 we will see that we may extend Proposition 3.5.3 to groups of finite *virtual* cohomological dimension provided that the thing we are taking homotopy fixed points of is a consistent faithful profinite Galois extension.

3.6. **The difficulties concerning arbitrary closed fixed points.** Let H be a closed subgroup of an arbitrary profinite group G. We would be able to remove the finite cohomological dimension hypothesis in Section 3.5 if we knew that the restriction functor

$$\mathrm{Res}^H_G : \Sigma\mathrm{Sp}_G \to \Sigma\mathrm{Sp}_H$$

sent G-fibrant objects to H-fibrant objects. While we know of no counterexamples, we also doubt that this is true in general.

We saw in Proposition 3.3.1 that for U an open subgroup of G, the presence of an induction functor Ind^G_U which was a left Quillen adjoint to Res^U_G allowed us to prove that Res^U_G preserved fibrant objects.

However, as pointed out to the second author by Jeff Smith, Res^H_G cannot possess a left adjoint, since it does not preserve limits. This can be seen as follows.

For a profinite group K, and a diagram $\{X_\alpha\}$ in the category $\Sigma\mathrm{Sp}_K$, let

$$\lim_\alpha{}^K X_\alpha$$

denote the limit computed in the category of discrete K-spectra. This limit is give by the following formula:

$$\lim_\alpha{}^K X_\alpha = \operatorname*{colim}_{U \trianglelefteq_o K} (\lim_\alpha{}^{\mathrm{Sp}} X_\alpha)^U.$$

Here, the limit $\lim^{\mathrm{Sp}}$ is the limit computed in the underlying category of symmetric spectra.

Thus, given a diagram $\{X_\alpha\}$, the restriction of the limit is given by

$$\mathrm{Res}^H_G \lim_\alpha{}^G X_\alpha = \operatorname*{colim}_{U \trianglelefteq_o G} (\lim_\alpha{}^{\mathrm{Sp}} X_\alpha)^U.$$

However, the limit of the restriction is computed to be

$$\lim_\alpha{}^H \mathrm{Res}^H_G X_\alpha = \operatorname*{colim}_{V \trianglelefteq_o H} (\lim_\alpha{}^{\mathrm{Sp}} X_\alpha)^V.$$

These subspectra of ${\lim_\alpha}^{\mathrm{Sp}} X_\alpha$ in general do not agree.

One might suspect that one could still prove that the map

$$\operatorname*{colim}_{H \le U \le_o G} X^{hU} \to X^{hG}$$

is an equivalence if G was a group of finite *virtual* cohomological dimension by a comparison of descent spectral sequences. This approach, however, also presents difficulties. For K a profinite group of finite vcd, and Y a discrete K-spectrum, let $\{E_r(K;Y)\}$ denote the conditionally convergent descent spectral sequence

$$E_2(K;Y) = H^*_c(K;\pi_*(Y)) \Rightarrow \pi_*(Y^{hK}).$$

There is a map of spectral sequences

$$E'_r(H;X) = \operatorname*{colim}_{H \le U \le_o G} E_r(U;X) \to E_r(H;X)$$

which can be shown to be an isomorphism on the level of E_2-terms. The problem is that the colimit of the spectral sequences does not converge to the colimit of the abutments in general.

4. Continuous G-spectra

For us, a *continuous G-spectrum* is a pro-object in the category of discrete G-spectra. In this section we extend some of our constructions for $\Sigma\mathrm{Sp}_G$ to the category of continuous G-spectra. For continuous G-spectra that are indexed over $\{0 \leftarrow 1 \leftarrow 2 \leftarrow \cdots\}$, part of this material appears in greater depth in [5].

4.1. **Pro-objects in discrete G-spectra.** Following standard usage, a pro-object in a category $\mathcal{C}$ is a cofiltered diagram in $\mathcal{C}$. We define the category of continuous G-spectra $\Sigma\mathrm{Sp}_G^c$ to be the category of pro-objects in $\Sigma\mathrm{Sp}_G$. Thus, a continuous G-spectrum is a cofiltered diagram $\mathbf{X} = \{X_i\}_{i\in I}$ of discrete G-spectra. Maps in the category of continuous G-spectra are given by

$$\Sigma\mathrm{Sp}_G^c(\mathbf{X}, \mathbf{Y}) = \lim_j \operatorname*{colim}_i \Sigma\mathrm{Sp}_G(X_i, Y_j).$$

Any pro-spectrum $\mathbf{X} = \{X_i\}$ gives rise to a spectrum X via the homotopy limit functor:

$$X = \operatorname*{holim}_i X_i.$$

We shall always denote our pro-spectra by boldface type and their homotopy limits with non-boldface type.

Remark 4.1.1. A more general theory of pro-spectra, including a model category structure, has been developed by Isaksen [18]. Fausk and Isaksen are developing a category of continuous *genuine* G-spectra where G is a pro-Lie group. The notion of continuous G-spectrum in this paper should then correspond to a naïve G-spectrum.

4.2. **Continuous mapping spectra.** Let $K = \lim_i K_i$ be a profinite G-set. Given a continuous G-spectrum $\mathbf{X}$, the continuous mapping spectrum $\mathbf{Map}^c(K, \mathbf{X})$ is defined to be the continuous G-spectrum

$$\{\mathrm{Map}^c(K, X_j)\}_j.$$

We denote the homotopy limit of $\mathbf{Map}^c(K, \mathbf{X})$ by $\mathrm{Map}^c(K, X)$. If the derived functors $\lim_j^s \mathrm{Map}^c(K, \pi_t(X_j)) = 0$, for all $s > 0$ and all $t \in \mathbb{Z}$, then the Bousfield-Kan spectral sequence

$$\lim\nolimits_j^s \mathrm{Map}^c(K, \pi_t(X_j)) \Rightarrow \pi_*(\mathrm{Map}^c(K, X))$$

collapses, and thus,

$$\pi_*(\mathrm{Map}^c(K, X)) \cong \mathrm{Map}^c(K, \lim_j \pi_*(X_j)).$$

4.3. **Continuous permutation spectra.** Let K be as above, and let each finite set K_j be a discrete G-set. Also, let $\mathbf{X}$ be a continuous G-spectrum. Define the permutation spectrum $\mathbf{X}[[K]]$ to be the continuous G-spectrum given by the pro-spectrum

$$\{X_i[K_j]\}_{i,j}.$$

We denote the homotopy limit of $\mathbf{X}[[K]]$ by $X[[K]]$.

Note that if E is a discrete G-spectrum, then $\mathbf{E}[[K]]$ is the continuous G-spectrum $\{E[K_j]\}_j$. If $\lim_{i,j}^s \pi_t(X_i)[K_j] = 0$, for all $s > 0$ and all $t \in \mathbb{Z}$ (where $\pi_t(X_i)[K_j]$ is an abelian group), then

$$\pi_*(X[[K]]) \cong \lim_{i,j} \pi_*(X_i)[K_j] \equiv \lim_i \pi_*(X_i)[[K]].$$

4.4. **Continuous homotopy fixed points.** For a continuous G-spectrum $\mathbf{X}$, define the homotopy fixed point spectrum $\mathbf{X}^{hG}$ to be the pro-spectrum given by

$$\{X_i^{hG}\}_i.$$

We denote the homotopy inverse limit of $\mathbf{X}^{hG}$ by X^{hG}.

Example 4.4.1. Consider the $\mathbb{G}_n$-Galois extension F_n of $S_{K(n)}$ (Example 1.0.5). Let $\{M_I\}_I$ be a cofinal collection of generalized Moore spectra [17]. Let H be a closed subgroup of G. Then the pro-spectrum $\mathbf{E}_n = \{F_n \wedge M_I\}_I$ is a continuous H-spectrum. Using the fact that F_n is $E(n)$-local, the homotopy fixed points are identified by

$$\begin{aligned} E_n^{hH} &= \lim_I F_n^{hH} \wedge M_I \\ &= (F_n^{hH})_{K(n)}. \end{aligned}$$

Thus the homotopy fixed points of the continuous H-spectrum $\mathbf{E}_n$ coincide with the $K(n)$-localization of the homotopy fixed points of the discrete H-spectrum F_n.

4.5. **Continuous hypercohomology spectra.** Define a coaugmented comonad $\mathbf{\Gamma}_G : (\text{pro}-\text{spectra}) \to (\text{pro}-\text{spectra})$ by

$$\mathbf{\Gamma}_G(\mathbf{X}) = \mathbf{Map}^c(G, \mathbf{X}).$$

Let $\Gamma_G(X)$ be the homotopy inverse limit of $\mathbf{\Gamma}_G(\mathbf{X})$.

If $\mathbf{X}$ is a continuous G-spectrum, then it is a coalgebra over $\mathbf{\Gamma}_G$. The following theorem is proven in [5].

Theorem 4.5.1 ([5])**.** Suppose that $\mathbf{X} = \{X_i\}$ is a continuous G-spectrum, and suppose that G has finite vcd. Let $\mathbf{X}_{fG}$ be the continuous G-spectrum given by the pro-spectrum $\{(X_i)_{fG}\}$. Then there is are equivalences:

$$X^{hG} \simeq \operatorname*{holim}_{\Delta} \operatorname{Map}^c(G^\bullet, X_{fG}) = \mathbb{H}_c(G; X_{fG}) \simeq \mathbb{H}_c(G; X).$$

4.6. **Homotopy fixed point spectral sequence.** Assume that G has finite vcd. The homotopy fixed point spectral sequence is the Bousfield-Kan spectral sequence that computes the homotopy groups of the spectrum

$$X^{hG} \simeq \operatorname*{holim}_{\Delta} \operatorname*{holim}_{i} \operatorname{Map}^c(G^\bullet, X_i).$$

Thus, the homotopy fixed point spectral sequence has the form

$$E_2^{s,t} = \pi^s \pi_t(\operatorname*{holim}_i \operatorname{Map}^c(G^\bullet, X_i)) \Rightarrow \pi_{t-s}(X^{hG}).$$

If $\lim_i^s \operatorname{Map}^c(G^k, \pi_t(X_i)) = 0$, for all $s > 0$, all $k \geq 0$, and all $t \in \mathbb{Z}$, then the Bousfield-Kan spectral sequence

$$\lim\nolimits_i^s \operatorname{Map}^c(G^k, \pi_t(X_i)) \Rightarrow \pi_*(\operatorname{holim}_i \operatorname{Map}^c(G^k, X_i))$$

collapses, and thus,

$$\begin{aligned} E_2^{s,t} &\cong \pi^s(\lim_i \operatorname{Map}^c(G^\bullet, \pi_t(X_i))) \\ &\cong H^s(\operatorname{Map}^c(G^\bullet, \lim_i \pi_t(X_i)) \\ &\cong H^s(\operatorname{Map}^c(G^\bullet, \pi_t(X)) \\ &\cong H_c^s(G, \pi_*(X)). \end{aligned}$$

4.7. **Completed smash products.** Although we do not use it in this paper, we remark that there is a well behaved smash product on the category of continuous G-spectra. If $\mathbf{X}$ and $\mathbf{Y}$ are continuous G-spectra, we define the completed smash product $\mathbf{X} \wedge_c \mathbf{Y}$ to be the continuous G-spectrum given by the pro-spectrum

$$\{X_i \wedge Y_j\}.$$

Remark 4.7.1. The reader may notice that we have not defined for continuous G-spectra $\mathbf{X}$ and $\mathbf{Y}$ a continuous function spectrum $\mathbf{F}_c(\mathbf{X}, \mathbf{Y})$. The difficulty is that some sort of finiteness condition must be placed on $\mathbf{X}$ similar to the case of $\mathbf{Map}^c(K, \mathbf{X})$ to ensure that the terms in the pro-system are discrete. We do not pursue these subtleties here.

5. Structured modules and algebras of discrete G-spectra

Let A be a commutative symmetric ring spectrum, and let G be a profinite group. In this section we describe the model categories of discrete G-A-modules and commutative discrete G-A-algebras, and form the homotopy fixed point spectra within these categories. These new homotopy fixed point constructions are observed to agree in the stable homotopy category with the old ones. We then spend some time commenting on the comparisons between filtered homotopy colimits and filtered colimits of modules and commutative algebras, and conclude that, when properly interpreted, they all coincide in the stable homotopy category. We close the section by describing how to make the hypercohomology spectra of commutative discrete G-A-algebras take values in the category of commutative A-algebras.

5.1. **Modules of G-spectra.** Let A be a commutative symmetric ring spectrum. By a discrete G-A-module, we shall mean a discrete G-spectrum X which also possesses the structure of an A-module. We require these structures to be compatible in the following sense: for every element $g \in G$, the following diagram must commute.

$$\begin{array}{ccc} A \wedge X & \xrightarrow{\xi} & X \\ \downarrow{\scriptstyle 1\wedge g} & & \downarrow{\scriptstyle g} \\ A \wedge X & \xrightarrow{\xi} & X \end{array}$$

Here ξ is the A-module structure map. Let $\mathrm{Mod}_{G,A}$ denote the category of discrete G-A-modules. Following [28], the general framework of [31] may be applied to our situation to provide the following model structure on $\mathrm{Mod}_{G,A}$.

Proposition 5.1.1. The category $\mathrm{Mod}_{G,A}$ is a model category, where the fibrations and weak equivalences are the fibrations and weak equivalences of the underlying discrete G-spectra, and the cofibrations are the cofibrations of the underlying A-modules (as defined in [28]).

Given a G-A-module X, let X_{fGA} denote a functorial fibrant replacement. We define the homotopy fixed point A-module to be the fixed point spectrum

$$X^{h_A G} = (X_{fGA})^G.$$

The nature of our model category structures immediately gives the following lemma.

Lemma 5.1.2. The spectrum X_{fGA} is fibrant as a discrete G-spectrum, and there exists a weak equivalence

$$X_{fG} \xrightarrow{\simeq} X_{fGA}.$$

Since $(-)^G$ is preserves weak equivalences between G-fibrant spectra, there is an equivalence

$$X^{hG} \xrightarrow{\simeq} X^{h_A G}.$$

The monad $\Gamma_G(-) = \mathrm{Map}^c(G, -)$ restricts to the category of discrete G-A-spectra, with the A-module structure given by the composition

$$A \wedge \mathrm{Map}^c(G, X) \to \mathrm{Map}^c(G, A \wedge X) \xrightarrow{\xi_*} \mathrm{Map}^c(G, X).$$

Given discrete G-A-modules X and Y, their smash product $X \wedge_A Y$ is easily seen to be a discrete G-A-module with the diagonal action.

The category $\mathrm{Mod}^c_{G,A}$ of continuous G-A-modules is the category of pro-objects in $\mathrm{Mod}_{G,A}$.

Let X be a discrete or continuous A-module. Since X_{fAG} is G-fibrant, the following proposition is an immediate consequence of Theorem 3.2.1.

Proposition 5.1.3. Suppose G is of finite vcd. Then there is an equivalence of A-modules

$$X^{h_A G} \simeq \operatorname*{holim}_{\Delta} \mathrm{Map}^c(G, X_{fGA}) \simeq \mathbb{H}_c(G; X).$$

We shall henceforth drop the distinction between $(-)^{h_A G}$ and $(-)^{hG}$. All homotopy fixed points of A-modules will implicitly be taken in the category of A-modules.

5.2. **Algebras of discrete G-spectra.** By a discrete commutative G-A-algebra, we shall mean a discrete G-A-module E together with a commutative A-algebra multiplication

$$\mu : E \wedge_A E \to E$$

such that G acts through maps of A-algebras. Let $\mathrm{Alg}_{A,G}$ denote the category of discrete commutative G-A-algebras. The arguments of [28] go through in this context to give the following result.

Proposition 5.2.1.

(1) The category of discrete G-spectra admits a positive model structure, which we denote $\Sigma\mathrm{Sp}^+_G$, where the cofibrations are the positive cofibrations of underlying symmetric spectra, the weak equivalences are the stable equivalences of underlying symmetric spectra, and the positive fibrations are determined.
(2) The identity functor from discrete G-spectra with the positive model structure to discrete G-spectra with the stable model structure is the left adjoint of a Quillen equivalence.
(3) The category of discrete commutative G-A-modules admits a model structure where the cofibrations are the cofibrations of underlying commutative symmetric ring spectra, the weak equivalences are the stable equivalences of underlying symmetric spectra, and the fibrations are the positive fibrations of discrete G-spectra.

We consequently have the following corollary.

Corollary 5.2.2. We have the following Quillen adjoint pairs:

$$triv : \Sigma\mathrm{Sp}^+ \leftrightarrows \Sigma\mathrm{Sp}^+_G : (-)^G,$$
$$triv : \mathrm{Alg}_A \leftrightarrows \mathrm{Alg}_{A,G} : (-)^G,$$

For a discrete G-spectrum X we denote X_{fG^+} to be the functorial fibrant replacement in the positive model structure, and denote the corresponding homotopy fixed point spectrum by

$$X^{h^+G} = (X_{f^+G})^G.$$

We have the following lemma

Lemma 5.2.3. There is an equivalence

$$X^{h^+G} \xrightarrow{\simeq} X^{hG}.$$

Proof. Since fibrant discrete G-spectra are positively fibrant discrete G-spectra, there is an stable equivalence

$$\alpha : X_{f^+G} \to X_{fG}.$$

Since $(-)^G$ preserves stable equivalences between positively fibrant discrete G-spectra, α induces an equivalence

$$X^{h^+G} = (X_{f^+G})^G \xrightarrow{\alpha_*} (X_{fG})^G = X^{hG}.$$

□

For a discrete commutative G-A-algebra E, we shall denote the functorial fibrant replacement by $E_{fGA-\mathrm{Alg}}$. We define the homotopy fixed point A-algebra to be the fixed point spectrum

$$E^{h_{\mathrm{Alg}}G} = (E_{fGA-\mathrm{Alg}})^G.$$

We have the following lemma.

Lemma 5.2.4. If E is a discrete commutative G-A-algebra, there is a zig-zag of equivalences

$$E^{hG} \xleftarrow{\simeq} E^{h^+G} \xrightarrow{\simeq} E^{h_{\mathrm{Alg}}G}.$$

We shall henceforth not distinguish between the various equivalent types of homotopy fixed points discussed here. Homotopy fixed points of discrete G-A-algebras will always implicitly be taken in the category of discrete G-A-algebras.

5.3. **Filtered colimits.** We will make frequent use of filtered colimits. Filtered colimits of spectra of simplicial sets are rather well behaved. We have the following lemma, given in [35], [27] in the context of Bousfield-Friedlander spectra, whose proof in the context of symmetric spectra is identical. We remind the reader that a fibrant spectrum is positively fibrant.

Lemma 5.3.1. In the category of symmetric spectra of simplicial sets, filtered colimits preserve cofibrations and (positive) fibrations. Filtered colimits preserve weak equivalences between (positively) fibrant spectra.

Corollary 5.3.2. Given a filtered diagram $\{X_\alpha\}_{\alpha\in I}$ of (positively) fibrant spectra, there is an equivalence

$$\operatorname*{hocolim}_{\alpha} X_\alpha \xrightarrow{\simeq} \operatorname*{colim}_{\alpha} X_\alpha.$$

Proof. Let $\{\widetilde{X}_\alpha\} \xrightarrow{\phi} \{X_\alpha\}$ be a cofibrant replacement in the category of I-shaped diagrams of spectra, so that ϕ is a level-wise acyclic fibration. Then each spectrum $\widetilde{X}_\alpha$ is (positively) fibrant, and we have

$$\operatorname*{hocolim}_\alpha X_\alpha = \operatorname*{colim}_\alpha \widetilde{X}_\alpha \underset{\simeq}{\xrightarrow{\phi}} \operatorname*{colim}_\alpha X_\alpha.$$

□

Corollary 5.3.3. (Positively) fibrant discrete G-spectra are (positively) fibrant as non-equivariant spectra.

Proof. Let X be a (positively) fibrant discrete G-spectrum. Let U be an open normal subgroup of G. By Proposition 3.3.1, X is (positively) fibrant as a discrete U-spectrum. Therefore, by Lemma 3.1.1, the U-fixed points X^U is (positively) fibrant as a non-equivariant spectrum. The formula

$$X = \operatorname*{colim}_{U \trianglelefteq_o G} X^U$$

shows that X is (positively) fibrant as a non-equivariant spectrum. □

Lemma 5.3.4. Filtered colimits computed in the category of A-modules agree with those computed in the category of spectra.

Proof. Let $\{E_\alpha\}$ be a filtered diagram of commutative A-algebras. Then the colimit is easily seen to have the structure of a commutative A-algebra with multiplication given by

$$\begin{aligned} (\operatorname*{colim}_\alpha E_\alpha) \wedge_A (\operatorname*{colim}_\beta E_\beta) &\cong \operatorname*{colim}_\alpha \operatorname*{colim}_\beta E_\alpha \wedge_A E_\beta \\ &\cong \operatorname*{colim}_\alpha E_\alpha \wedge_A E_\alpha \\ &\to \operatorname*{colim}_\alpha E_\alpha. \end{aligned}$$

This filtered colimit is easily seen to satisfy the universal property. □

We shall henceforth never take homotopy colimits of spectra, but only colimits, with the understanding that we may have to implicitly take functorial (positively) fibrant replacements before computing the filtered colimit if the terms in the colimit are not already (positively) fibrant.

If X_α is a diagram of commutative discrete G-A-algebras, when we are taking a filtered colimit, we shall implicitly be taking the filtered colimit $\operatorname{colim}_\alpha X_{fGA-\mathrm{Alg}}$ of the functorial fibrant replacements. The underlying spectrum of $X_{fGA-\mathrm{Alg}}$ is positively fibrant.

For the sake of comparing our filtered colimits of commutative algebras with those homotopy colimits of A-algebras appearing in [7], we present the following lemma, whose proof is identical to that of Corollary 5.3.2.

Lemma 5.3.5. Suppose that $\{E_\alpha\}$ is a filtered diagram of fibrant commutative A-algebras. Then there is an equivalence

$$\operatorname*{colim}_\alpha E_\alpha \simeq \operatorname*{hocolim}_\alpha{}^{\mathrm{Alg}} E_\alpha,$$

where the homotopy colimit $\operatorname{hocolim}^{\mathrm{Alg}}$ is taken in the category of commutative A-algebras.

5.4. **Hypercohomology algebras.** Let E be a commutative discrete G-A-algebra. For any finite set K, the the mapping spectrum $\mathrm{Map}(K, E)$ is naturally a commutative A-algebra, using the diagonal on K. Therefore, by Lemma 5.3.4, the continuous mapping spectrum

$$\mathrm{Map}^c(G, E) = \operatorname*{colim}_{U \trianglelefteq_o G} \mathrm{Map}(G/U, E)$$

is a commutative A-algebra.

Since the category of spectra with the positive model structure is equivalent to the category of spectra with the stable model structure, there is an equivalence

$$\mathbb{H}^+_c(G; E) = \operatorname*{holim}_{\Delta}{}^+ \mathrm{Map}^c(G^\bullet, E) \simeq \operatorname*{holim}_{\Delta} \mathrm{Map}^c(G^\bullet, E) = \mathbb{H}_c(G; E)$$

between the homotopy limits computed in the positive and stable model structures. Since the homotopy limit of commutative A-algebras is computed in the underlying positive model structure, we have the following lemma.

Lemma 5.4.1. The hypercohomology spectrum $\mathbb{H}_c(G; E)$ is equivalent to a commutative A-algebra $\mathbb{H}^+_c(G; E)$.

When we take hypercohomology spectra of A-algebras, we shall always implicitly be taking the homotopy limit with respect to the positive model structure.

6. Profinite Galois extensions

Although the homotopy limit of k-local objects is k-local, it is not true in general that k-localization commutes with homotopy limits. We begin this section by explaining how, under appropriate hypothesis, Assumption 1.0.3 allows us to commute these two functors. We then explain how a profinite Galois extension naturally gives rise to a discrete G-spectrum, and how our consistency hypothesis allows us to recover the intermediate finite Galois extensions using the homotopy fixed point construction.

6.1. **Properties of k-localization.** Recall that we have assumed that the k-localization functor is given by $((-)_T)_M$, where localization with respect to T is smashing, and M is a finite spectrum (Assumption 1.0.3). These localizations are the functorial fibrant replacements in appropriately localized model categories. In this section we shall establish some lemmas concerning such k-localizations.

Lemma 6.1.1. If X is a k-local spectrum, then it is a T-local spectrum.

Proof. Let $f : A \to B$ be a T-local equivalence. Then it induces an equivalence on T-local spectra

$$f_T : A_T \xrightarrow{\simeq} B_T$$

and hence on the M-localization

$$f_k : A_k \simeq (A_T)_M \xrightarrow{\simeq} (B_T)_M \simeq B_k.$$

Therefore, f is a k-local equivalence, and since X is k-local, the induced map

$$f^* : [B, X] \xrightarrow{\cong} [A, X]$$

is an isomorphism. Since this is true for every T-local equivalence f, we deduce that X is T-local. $\square$

Lemma 6.1.2. Let X_i be a diagram of spectra. Then there is an equivalence

$$(\operatorname*{holim}_i X_i)_M \simeq \operatorname*{holim}_i (X_i)_M.$$

Proof. The homotopy inverse limit $\operatorname{holim}(X_i)_M$ is M-local, so there is a map

$$f : (\operatorname{holim} X_i)_M \to \operatorname{holim}(X_i)_M.$$

Smashing with M, and using the fact that M is a finite complex, we have the following diagram of equivalences

$$\begin{array}{ccc} M \wedge \operatorname{holim}_i X_i & \xrightarrow{\simeq} & \operatorname{holim}_i(M \wedge X_i) \\ \simeq \downarrow & & \downarrow \simeq \\ M \wedge (\operatorname{holim}_i X_i)_M & \xrightarrow[f \wedge M]{} & \operatorname{holim}_i(M \wedge (X_i)_M) \end{array}$$

from which we deduce that f is an M-local equivalence. Since f is a map between M-local spectra, the map f is an equivalence. □

Arbitrary localizations do not commute with homotopy inverse limits. Our reason for making Assumption 1.0.3 on k-localization is that we may deduce the following corollary.

Corollary 6.1.3. Let X_i be a diagram of T-local spectra. Then there is an equivalence

$$(\operatorname*{holim}_i X_i)_k \simeq \operatorname*{holim}_i (X_i)_k.$$

Proof. Since the spectra X_i are T-local, the homotopy inverse limit $\operatorname{holim}_i X_i$ is T-local. Using Lemma 6.1.2, we have equivalences:

$$\begin{aligned} (\operatorname*{holim}_i X_i)_k &\simeq (\operatorname*{holim}_i X_i)_M \\ &\simeq \operatorname*{holim}_i (X_i)_M \\ &\simeq \operatorname*{holim}_i (X_i)_k. \end{aligned}$$

□

Since T-localization is smashing, it possesses the following pleasant properties.

Lemma 6.1.4.

(1) Colimits of T-local spectra are T-local.
(2) If X is a T-local spectrum, and Y is any spectrum, $X \wedge Y$ is T-local.
(3) If X is T-local, then $\operatorname{Map}^c(G, X)$ is T-local.

We end this section with the following lemma.

Lemma 6.1.5. Suppose that $f : X \to Y$ is a k-local equivalence of T-local discrete G-spectra. Then the induced map

$$f_* : \mathbb{H}_c(G; X)_k \to \mathbb{H}_c(G; Y)_k$$

is an equivalence.

Proof. Using Lemma 6.1.4, we see that the hypercohomology functor

$$\mathbb{H}_c(G; -) = \operatorname*{holim}_{\Delta} \mathrm{Map}^c(G^\bullet, -)$$

sends T-local spectra to T-local spectra. Therefore, we just need to check that the map

$$f_* : \mathbb{H}_c(G; X) \to \mathbb{H}_c(G; Y)$$

is an M-local equivalence. Since M is finite and f is an M-local equivalence we see that $M \wedge f_*$ is an equivalence. Indeed, it is equivalent to the composite

$$\begin{aligned} M \wedge \mathbb{H}_c(G; X) &\simeq \mathbb{H}_c(G; M \wedge X) \\ &\xrightarrow{(M\wedge f)_*} \mathbb{H}_c(G; M \wedge Y) \\ &\simeq M \wedge \mathbb{H}_c(G; Y). \end{aligned}$$

□

6.2. **Profinite Galois extensions as discrete G-spectra.** We first recall some terminology from [30]. Let A be a cofibrant k-local commutative symmetric ring spectrum, let E be a commutative A-algebra, and let G be a profinite group.

Definition 6.2.1 (Profinite Galois extension)**.** The spectrum E is a *(faithful) k-local G-Galois extension* of A if

(1) There is a directed system of (faithful) finite k-local G/U_α-Galois extensions E_α of A for U_α a cofinal system of open normal subgroups of G.
(2) All of the maps $E_\alpha \to E_\beta$ are G-equivariant and are cofibrations of underlying commutative A-algebras.
(3) For $\alpha \leq \beta$, letting $K_{\alpha,\beta}$ denote the quotient U_α/U_β, the natural maps $E_\alpha \to E_\beta^{hK_{\alpha,\beta}}$ are equivalences.
(4) The spectrum E is the filtered colimit $\mathrm{colim}_\alpha E_\alpha$.

Remark 6.2.2. The spectra E_α are k-local, but the spectrum E need not be k-local. Our assumptions on k-localization do imply that E is T-local.

For the remainder of this section, we shall assume that E is a faithful k-local G-Galois extension of A. We shall make the following additional assumptions on E.

- E is *consistent* over A: the map $A \to A^\wedge_{k,E}$ is an equivalence.
- E has *finite vcd* over A: the profinite group G has finite virtual cohomological dimension.

Proposition 6.2.3 (Rognes [30])**.** Then there are natural equivalences:

$$(E \wedge_A E)_k \xrightarrow{\simeq} (\mathrm{Map}^c(G, E))_k,$$
$$(E[[G]])_k \xrightarrow{\simeq} F_A(E_k, E_k).$$

Proposition 6.2.4. The spectrum E is a discrete commutative G-A-algebra.

Proof. Clearly, E_α is a commutative discrete G-A-algebra. Discrete G-A-algebras are closed under filtered colimits. □

6.3. **The consistent hypothesis.**

Proposition 6.3.1. Assume that the Galois extension E of A has finite vcd. Then there is a natural equivalence

$$A^{\wedge}_{k,E} \to (E^{hG})_k$$

between the k-local Amitsur derived completion and the k-localization of the homotopy fixed point spectrum.

Proof. By iterating Proposition 6.2.3, the natural map

$$(\underbrace{E \wedge_A E \wedge_A \cdots \wedge_A E}_{n+1})_k \to (\mathrm{Map}^c(G^n, E))_k$$

is an equivalence. Totalizing the associated cosimplicial spectra and using Corollary 6.1.3 and Theorem 3.2.1, we have:

$$\begin{aligned} A^{\wedge}_{k,E} &= \operatorname*{holim}_{\Delta}(E^{\wedge_A \bullet+1})_k \\ &\xrightarrow{\simeq} \operatorname*{holim}_{\Delta}(\mathrm{Map}^c(G^\bullet, E))_k \\ &\simeq (\operatorname*{holim}_{\Delta} \mathrm{Map}^c(G^\bullet, E))_k \\ &= (E^{hG})_k. \end{aligned}$$

□

Corollary 6.3.2. Let E is be a k-local profinite G-Galois extension of A of finite vcd. Then the extension is consistent if and only if the natural map

$$A \to (E^{hG})_k$$

is an equivalence.

We shall say that a k-local A-module X is *k-locally dualizable* if the map

$$(D_A(X) \wedge_A X)_k \to F_A(X, X)$$

is an equivalence. Here, $D_A(-) = F_A(-, A)$ is the Spanier-Whitehead dual in the category of A-modules. We shall repeatedly use the following dualizability result [30, Sec. 6], [1].

Proposition 6.3.3. If E is a *finite* k-local Galois extension of A, then E is a k-locally dualizable A-module. There is a natural map (the *discriminant map*)

$$E \to D_A(E)$$

which is an equivalence.

Being consistent implies the following consistency result.

Lemma 6.3.4. Suppose that $E = \mathrm{colim}_\alpha E_\alpha$ is a consistent k-local profinite G-Galois extension of finite vcd. Then for each α, the natural map

$$E_\alpha \to (E^{hU_\alpha})_k$$

is an equivalence.

Proof. Since E_α is G/U_α-Galois over A, we have a chain of equivalences:

$$\begin{aligned}(E_\alpha \wedge_A E)_k &\simeq (E_\alpha \wedge_A \operatorname*{colim}_{\beta\geq\alpha} E_\beta)_k \\ &\simeq (\operatorname*{colim}_{\beta\geq\alpha}(E_\alpha \wedge_A E_\alpha) \wedge_{E_\alpha} E_\beta)_k \\ &\simeq (\operatorname*{colim}_{\beta\geq\alpha} \mathrm{Map}(G/U_\alpha, E_\alpha) \wedge_{E_\alpha} E_\beta)_k \\ &\simeq (\operatorname*{colim}_{\beta\geq\alpha} \mathrm{Map}(G/U_\alpha, E_\beta))_k \\ &\simeq (\mathrm{Map}(G/U_\alpha, E))_k.\end{aligned}$$

By Corollary 6.3.2, the natural map

$$A \to (E^{hG})_k \tag{6.1}$$

is an equivalence. Smashing (6.1) over A with E_α, using Theorem 3.2.1, employing the fact that E_α is k-local, and k-locally dualizable, and applying Corollary 6.1.3 and Shapiro's Lemma (3.4.2), we have equivalences:

$$\begin{aligned}E_\alpha &\simeq (E_\alpha \wedge_A A)_k \\ &\simeq (E_\alpha \wedge_A E^{hG})_k \\ &\simeq (E_\alpha \wedge_A \operatorname*{holim}_{\Delta} \mathrm{Map}^c(G^\bullet, E))_k \\ &\simeq (\operatorname*{holim}_{\Delta} \mathrm{Map}^c(G^\bullet, E_\alpha \wedge_A E))_k \\ &\simeq \operatorname*{holim}_{\Delta} \mathrm{Map}^c(G^\bullet, E_\alpha \wedge_A E)_k \\ &\simeq \operatorname*{holim}_{\Delta} \mathrm{Map}^c(G^\bullet, \mathrm{Map}(G/U_\alpha, E))_k \\ &\simeq (\operatorname*{holim}_{\Delta} \mathrm{Map}^c(G^\bullet, \mathrm{Map}(G/U_\alpha, E)))_k \\ &\simeq (\mathrm{Map}^c(G/U_\alpha, E)^{hG})_k \\ &\simeq (E^{hU_\alpha})_k.\end{aligned}$$

□

Adding the faithful hypothesis allows us to forget the system of finite Galois extensions which gave E.

Proposition 6.3.5. Let E be a consistent faithful k-local profinite G-Galois extension of A of finite vcd.

(1) For each open normal subgroup U of G, $(E^{hU})_k$ is a faithful k-local G/U-Galois extension of A.
(2) If $U \leq V$ are a pair of open subgroups of G with U normal in V, then $(E^{hU})_k$ is a faithful k-local V/U-Galois extension of $(E^{hV})_k$.

Proof. (1) We repeatedly use the fundamental theorem of Galois theory. Choose α so that U_α is contained in U. Then by Lemma 6.3.4, there is an equivalence $E_\alpha \simeq (E^{hU_\alpha})_k$, so $(E^{hU_\alpha})_k$ is a faithful k-local G/U_α-Galois extension of A. Proposition 3.3.1 implies that there is an equivalence

$$(E^{hU})_k \simeq ((E^{hU_\alpha})^{hU/U_\alpha})_k \simeq (E_\alpha)^{hU/U_\alpha}.$$

Galois theory therefore implies that $(E^{hU})_k$ is a faithful k-local G/U-Galois extension of A.

(2) Let N be an open normal subgroup of G contained in U. By (1) we know that $(E^{hN})_k$ is a faithful k-local G/N-Galois extension of A. By Proposition 3.3.1, we have

$$(E^{hV})_k \simeq ((E^{hN})^{hV/N})_k \simeq ((E^{hN})_k)^{hV/N}.$$

The fundamental theorem of Galois theory [30] implies that $(E^{hN})_k$ is a faithful k-local V/N-Galois extension of $(E^{hV})_k$. Again, by Proposition 3.3.1, we have

$$(E^{hU})_k \simeq ((E^{hN})^{hU/N})_k \simeq ((E^{hN})_k)^{hU/N}.$$

Since U/N is normal in V/N, with quotient V/U, the fundamental theorem of Galois theory implies that $(E^{hU})_k$ is a faithful k-local V/U-Galois extension of $(E^{hV})_k$. □

7. Closed fixed points of profinite Galois extensions

Let E be a consistent faithful k-local profinite G-Galois extension of A of finite vcd. We begin by showing that under these hypotheses, the H-fixed points functor is extremely well behaved for H an arbitrary closed subgroup of G. We then prove the forward part of the profinite Galois correspondence, and compute the homotopy type of the function spectrum between arbitrary k-local closed homotopy fixed point spectra.

7.1. **Iterated Galois fixed points.** In this section we will extend the results of Section 3.5 to all closed subgroups H of G. Let $j : H \hookrightarrow G$ be the inclusion. In this section we shall prove the following theorem, and derive consequences from it.

Theorem 7.1.1. Suppose that G has finite vcd. Then the natural map

$$(\operatorname*{colim}_{H\leq U\leq_o G} E^{hU})_k \to (E^{hH})_k$$

is an equivalence.

We shall first need the following fundamental lemma.

Lemma 7.1.2. Suppose that V is an open normal subgroup of G, and that H is a closed subgroup of G. Then the natural map

$$((E^{hV})_k \wedge_{(E^{hHV})_k} E^{hH})_k \to (E^{h(H\cap V)})_k$$

is an equivalence.

Proof. Let Q be the finite group $HV/V \cong H/(H\cap V)$. By Proposition 6.3.5, the extension

$$(E^{hHV})_k \to (E^{hH})_k$$

is a k-local Q-Galois extension. Under the isomorphism

$$E \cong (E^{hV})_k \wedge_{(E^{hV})_k} E$$

the G action on E is transformed to the diagonal action on $(E^{hV})_k \wedge_{(E^{hV})_k} E$. Therefore, under the equivalences

$$\begin{aligned}((E^{hV})_k \wedge_{(E^{hHV})_k} E)_k &\cong ((E^{hV})_k \wedge_{(E^{hHV})_k} (E^{hV})_k \wedge_{(E^{hV})_k} \wedge E)_k \\ &\simeq (\mathrm{Map}(Q, E^{hV})_k \wedge_{(E^{hV})_k} \wedge E)_k \\ &\simeq \mathrm{Map}(Q, E)_k\end{aligned}$$

the G-action on E is transformed to the conjugation action on $\mathrm{Map}(Q, E)$. Using Theorem 3.2.1, the fact that $(E^{hV})_k$ is a k-locally dualizable $(E^{hHV})_k$-module (Proposition 6.3.3), and Shapiro's Lemma (Lemma 3.4.2) we have:

$$\begin{aligned}
((E^{hV})_k \wedge_{(E^{hHV})_k} E^{hH})_k &\simeq ((E^{hV})_k \wedge_{(E^{hHV})_k} \mathbb{H}_c(H; E))_k \\
&\simeq (\mathbb{H}_c(H; (E^{hV})_k \wedge_{(E^{hHV})_k} E))_k \\
&\simeq (((E^{hV})_k \wedge_{(E^{hHV})_k} E)^{hH})_k \\
&\simeq (\mathrm{Map}(Q, E)^{hH})_k \\
&\cong (\mathrm{Map}(H/(H \cap V), E)^{hH})_k \\
&\cong (E^{h(H \cap V)})_k.
\end{aligned}$$

□

Proof of Theorem 7.1.1. Choose an open normal subgroup V of G of finite cohomological dimension. By Proposition 3.5.3 we see that the map

$$\operatorname*{colim}_{H \leq U \leq_o HV} E^{h(U \cap V)} \to E^{h(H \cap V)} \tag{7.1}$$

is an equivalence. Let $Q = HV/V \cong H/(H \cap V)$ be the finite quotient group. For each open subgroup U of HV containing H, we have $UV = HV$. Therefore there is an isomorphism $Q = UV/V \cong U/(U \cap V)$. By Proposition 6.3.5, the extensions

$$(E^{hU})_k \to (E^{h(U \cap V)})_k,$$
$$(E^{hUV})_k \to (E^{hV})_k$$

are faithful k-local Q-Galois extensions. Therefore, by Remark 1.0.2, the norm maps

$$((E^{h(U \cap V)})_{hQ})_k \to ((E^{h(U \cap V)})^{hQ})_k,$$
$$((E^{h(V)})_{hQ})_k \to ((E^{hV})^{hQ})_k$$

are equivalences. We therefore have, using Proposition 3.3.1 and Lemma 7.1.2, the following sequence of equivalences:

$$\begin{aligned}
(\operatorname*{colim}_{H \leq U \leq_o G} E^{hU})_k &\simeq (\operatorname*{colim}_{H \leq U \leq_o HV} E^{hU})_k \\
&\simeq (\operatorname*{colim}_{H \leq U \leq_o HV} (E^{h(U \cap V)})^{hQ})_k \\
&\simeq (\operatorname*{colim}_{H \leq U \leq_o HV} (E^{h(U \cap V)})_{hQ})_k \\
&\simeq ((\operatorname*{colim}_{H \leq U \leq_o HV} E^{h(U \cap V)})_{hQ})_k \\
&\simeq ((E^{hH \cap V})_{hQ})_k \\
&\simeq (((E^{hV})_k \wedge_{(E^{hHV})_k} E^{hH})_{hQ})_k \\
&\simeq (((E^{hV})_{hQ})_k \wedge_{(E^{hHV})_k} E^{hH})_k \\
&\simeq (((E^{hV})^{hQ})_k \wedge_{(E^{hHV})_k} E^{hH})_k \\
&\cong (((E^{hV})^{h(HV/V)})_k \wedge_{(E^{hHV})_k} E^{hH})_k \\
&\simeq ((E^{hHV})_k \wedge_{(E^{hHV})_k} E^{hH})_k \\
&\simeq (E^{hH})_k.
\end{aligned}$$

□

Using the methods of Section 3.5, Theorem 7.1.1 has the following corollary.

Corollary 7.1.3.

(1) There is an equivalence $((E_{fG})^H)_k \simeq (E^{hH})_k$.
(2) If H is a closed normal subgroup of G, then, using the model for H-homotopy fixed points given by part (1), there is an equivalence

$$((E^{hH})^{hG/H})_k \simeq (E^{hG})_k.$$

7.2. **Intermediate Galois extensions.** In this section we will prove the forward direction of the profinite Galois correspondence.

Theorem 7.2.1. Suppose that H is a closed subgroup of G.

(1) The spectrum E is k-locally H-equivariantly equivalent to a consistent faithful k-local H-Galois extension of $(E^{hH})_k$ of finite vcd.
(2) If H is a normal subgroup of G, then the spectrum E^{hH} is k-locally equivalent to a faithful k-local G/H-Galois extension of A. If the quotient G/H has finite vcd, then E^{hH} is consistent (and of finite vcd) over A.

Remark 7.2.2. It is useful to note that if G is a compact p-adic analytic group, then for any closed normal subgroup H of G, then the quotient group G/H is a compact p-adic analytic group, and therefore must also have finite vcd [8, Theorem 9.6] [34].

Proof of part (1). Since H is a closed subgroup of G, a group of finite vcd, we may conclude that H has finite vcd. The system $\{H \cap V\}_{V \trianglelefteq_o G}$ is cofinal in the system of open normal subgroups of H. Let $U = H \cap V$ be one of these open normal subgroups.

$(E^{hU})_k$ *is k-locally H/U-Galois over* $(E^{hH})_k$. We must check that the extension satisfies the two conditions of Definition 1.0.1. By Proposition 3.3.1 and Corollary 6.1.3, we have

$$\begin{aligned} ((E^{hU})_k)^{hH/U} &\simeq ((E^{hU})^{hH/U})_k \\ &\simeq (E^{hH})_k, \end{aligned}$$

which verifies the first condition. The second condition is verified through the use of Lemma 7.1.2, and the fact that $(E^{hV})_k$ is a faithful k-local HV/V-Galois extension of $(E^{HV})_k$ (Proposition 6.3.5):

$$\begin{aligned} &((E^{hU})_k \wedge_{(E^{hH})_k} (E^{hU})_k)_k \\ &\quad \simeq (((E^{hH})_k \wedge_{(E^{hHV})_k} E^{hV})_k \wedge_{(E^{hH})_k} ((E^{hH})_k \wedge_{(E^{hHV})_k} E^{hV})_k)_k \\ &\quad \simeq ((E^{hH})_k \wedge_{(E^{hHV})_k} (E^{hV})_k \wedge_{(E^{hHV})_k} E^{hV})_k \\ &\quad \simeq ((E^{hH})_k \wedge_{(E^{hHV})_k} \mathrm{Map}(HV/V, (E^{hV})_k))_k \\ &\quad \cong ((E^{hH})_k \wedge_{(E^{hHV})_k} \mathrm{Map}(H/U, (E^{hV})_k))_k \\ &\quad \simeq \mathrm{Map}(H/U, ((E^{hH})_k \wedge_{(E^{hHV})_k} E^{hV})_k) \\ &\quad \simeq \mathrm{Map}(H/U, (E^{hU})_k). \end{aligned}$$

$(E^{hU})_k$ *is k-locally faithful over* $(E^{hH})_k$. Suppose M is an $(E^{hH})_k$-module and that we have

$$((E^{hU})_k \wedge_{(E^{hH})_k} M)_k \simeq *.$$

We must show M_k is null. We use Lemma 7.1.2 to deduce

$$\begin{aligned} * &\simeq ((E^{hV} \wedge_{(E^{hHV})_k} (E^{hH})_k)_k \wedge_{(E^{hH})_k} M)_k \\ &\simeq ((E^{hV})_k \wedge_{(E^{hHV})_k} \wedge M)_k. \end{aligned}$$

By Proposition 6.3.5, we deduce that $(E^{hV})_k$ is k-locally faithful over $(E^{hHV})_k$, so we may conclude that M_k is null.

Let L be the colimit $\mathrm{colim}_{V \trianglelefteq_o G}(E^{hH\cap V})_k$. The spectrum L is a faithful k-local H-Galois extension of $(E^{hH})_k$.

E is k-locally H-equivariantly equivalent to L. By Corollary 7.1.3, for each V we have

$$((E_{fG})^{H\cap V})_k \simeq (E^{hH\cap V})_k.$$

Since E_{fG} is a discrete H-spectrum, we have:

$$\begin{aligned} (E_{fG})_k &= (\operatorname*{colim}_{V \trianglelefteq_o G}(E_{fG})^{H\cap V})_k \\ &\simeq (\operatorname*{colim}_{V \trianglelefteq_o G}(E^{hH\cap V})_k)_k \\ &= L_k. \end{aligned}$$

The fibrant replacement map $E \to E_{fG}$ is an H-equivariant equivalence.

L is consistent over $(E^{hH})_k$. By Corollary 6.3.2, we just need to check that the map

$$(E^{hH})_k \to (L^{hH})_k \tag{7.2}$$

is an equivalence. We have already seen that the map $E \to L$ is a k-local equivalence. By Theorem 3.2.1 and Lemma 6.1.5, we see that the map of Equation (7.2) is an equivalence. □

Proof of part (2). Let K be the colimit $\mathrm{colim}_{U \trianglelefteq_o G}(E^{hHU})_k$. Since H is normal in G, the groups HU is are open normal subgroups of G. By Proposition 6.3.5, the spectra $(E^{hHU})_k$ are k-local faithful G/HU-Galois extensions of A. Therefore, K is a k-local faithful G/H-Galois extension of A. The spectrum K is k-locally equivalent to E^{hH} by Theorem 7.1.1.

Suppose that G/H is of finite vcd. We are left with showing that K is consistent over A. By Corollary 6.3.2, it suffices to check that the map

$$A \to (K^{hG/H})_k$$

is an equivalence. Using Corollary 6.1.3, Theorem 7.1.1, Corollary 7.1.3, and the fact that E is consistent over A, we have:

$$\begin{aligned}(K^{hG/H})_k &= ((\operatorname*{colim}_{U\trianglelefteq_o G}(E^{hHU})_k)^{hG/H})_k \\ &\simeq ((\operatorname*{colim}_{U\trianglelefteq_o G} E^{hHU})^{hG/H})_k \\ &\simeq (((E_{fG})^{H})^{hG/H})_k \\ &\simeq (E^{hG})_k \\ &\simeq A.\end{aligned}$$

□

7.3. **Function spectra.** In this section, we prove the following theorem.

Theorem 7.3.1. Let H and K be closed subgroups of G. Then there is an equivalence

$$F_A((E^{hH})_k, (E^{hK})_k) \simeq ((E[[G/H]])^{hK})_k.$$

Corollary 7.3.2. If H and K are closed subgroups of G, and the left action of K on G/H is trivial, then there is an equivalence

$$F_A((E^{hH})_k, (E^{hK})_k) \simeq (E^{hK}[[G/H]])_k.$$

Proof. We have the following sequence of equivalences:

$$\begin{aligned}F_A((E^{hH})_k, (E^{hK})_k) &\simeq ((E[[G/H]])^{hK})_k \\ &\simeq (\operatorname*{holim}_{H\le U\le_o G}(E[G/U])^{hK})_k \\ &\simeq (\operatorname*{holim}_{H\le U\le_o G} E^{hK}[G/U])_k \\ &\simeq (E^{hK}[[G/H]])_k.\end{aligned}$$

□

Remark 7.3.3. The conclusion of Corollary 7.3.2 is typically far from true for arbitrary H and K. For instance, let n be odd. It is shown in [33] that in the case of $k = K(n)$, $A = S_{K(n)}$ $E = F_n$, $G = \mathbb{G}_n$, $H = e$, and $K = \mathbb{G}_n$, the $K(n)$-local Spanier-Whitehead dual of E_n is given by:

$$F(E_n, E_n^{h\mathbb{G}_n}) \simeq F(E_n, S_{K(n)}) \simeq \Sigma^{-n^2} E_n \not\simeq E_n.$$

We give a criterion that guarantees that the hypothesis of Corollary 7.3.2 is met. First, we recall the definition of a standard concept in group theory.

Definition 7.3.4. The *core* H_G of H in G is the normal subgroup

$$H_G = \bigcap\nolimits_{g\in G} gHg^{-1}.$$

The core H_G is the largest normal subgroup of G that is contained in H.

Lemma 7.3.5. Let H and K be closed subgroups of G. Then K acts trivially on the left G-set G/UH, for all open normal subgroups U of G, if and only if K is contained in H_G.

Proof. Suppose that $k \cdot gUH = gUH$, for all U, all $k \in K$, and all $g \in G$. Since U is normal in G, this condition is equivalent to the condition that k is contained in $UgHg^{-1}$. Since H is closed, gHg^{-1} is a closed subset of G, so that

$$\bigcap_U UgHg^{-1} = \big(\bigcap_U U\big)gHg^{-1} = \{e\}gHg^{-1} = gHg^{-1}.$$

Thus, G acts trivially on the left G-set G/UH, for all U, if and only if k is an element of gHg^{-1}, for all $k \in K$ and all $g \in G$. This is equivalent to requiring that $K \subset \bigcap g \in G\, gHg^{-1} = H_G$. □

The remainder of this section will be spent proving Theorem 7.3.1. We first prove some technical lemmas. Recall that an A-module X is said to be k-locally *F-small* if the natural map

$$\operatorname*{colim}_i F_A(X, Y_i) \to F_A(X, \operatorname*{colim}_i Y_i)$$

is a k-local equivalence for every filtered diagram $\{Y_i\}_i$ of A-modules. Observe that if X is a k-locally dualizable A-module, then it is k-locally F-small, since we have:

$$\begin{aligned}(\operatorname*{colim}_i F_A(X, Y_i))_k &\simeq (\operatorname*{colim}_i (D_A(X) \wedge_A Y_i))_k \\ &\simeq (D_A(X) \wedge_A \operatorname*{colim}_i Y_i)_k \\ &\simeq F_A(X, \operatorname*{colim}_i Y_i)_k.\end{aligned}$$

Lemma 7.3.6. Suppose that X is a spectrum which is k-locally F-small. Let $T = \lim_i T_i$ be a profinite set. Then the natural map

$$\mathrm{Map}^c(T, F_A(X, Y))_k \to F_A(X, \mathrm{Map}^c(T, Y))_k$$

is a weak equivalence.

Proof. We have:

$$\begin{aligned}\mathrm{Map}^c(T, F_A(X, Y))_k &= (\operatorname*{colim}_i \mathrm{Map}(T_i, F_A(X, Y)))_k \\ &\cong (\operatorname*{colim}_i F_A(X, \mathrm{Map}(T_i, Y)))_k \\ &\simeq F_A(X, \operatorname*{colim}_i \mathrm{Map}(T_i, Y))_k \\ &= F_A(X, \mathrm{Map}^c(T, Y))_k.\end{aligned}$$

□

Lemma 7.3.7. Let $\{Y_j\}$ be any filtered diagram of spectra, and let $T = \lim_i T_i$ be a profinite set. Then the natural map

$$\operatorname*{colim}_j \mathrm{Map}^c(T, Y_j) \to \mathrm{Map}^c(T, \operatorname*{colim}_j Y_j)$$

is an equivalence.

Proof. We have:

$$\begin{aligned}\operatorname*{colim}_j \mathrm{Map}^c(T, Y_j) &= \operatorname*{colim}_j \operatorname*{colim}_i \mathrm{Map}(T_i, Y_j) \\ &\cong \operatorname*{colim}_i \mathrm{Map}(T_i, \operatorname*{colim}_j Y_j) \\ &= \mathrm{Map}^c(T, \operatorname*{colim}_j Y_j).\end{aligned}$$

□

Lemma 7.3.8. Let U be an open subgroup of G, and suppose that V is an open normal subgroup of G contained in U. Then there is an equivalence

$$\psi_V : (E^{hV}[G/U])_k \xrightarrow{\simeq} F_A((E^{hU})_k, (E^{hV})_k).$$

Proof. By Proposition 6.3.5, the spectrum $(E^{hV})_k$ is a k-local G/V-Galois extension of A, and by Proposition 3.3.1, there is an equivalence

$$E^{hU} \simeq (E^{hV})^{hU/V}.$$

By Proposition 6.3.3, the A-module $(E^{hV})_k$ is k-locally A-dualizable. Using Corollary 6.1.3, we have:

$$\begin{aligned}
((E^{hV})_k \wedge_A (E^{hU})_k)_k &\simeq ((E^{hV})_k \wedge_A ((E^{hV})^{hU/V})_k)_k \\
&\simeq ((E^{hV})_k \wedge_A ((E^{hV})_k)^{hU/V})_k \\
&\simeq (((E^{hV})_k \wedge_A (E^{hV})_k)^{hU/V})_k \\
&\simeq \mathrm{Map}(G/V, (E^{hV})_k)^{hU/V} \\
&\simeq \mathrm{Map}(G/U, (E^{hV})_k).
\end{aligned}$$

Applying $F_{(E^{hV})_k}(-, (E^{hV})_k)$ to both sides, we have:

$$\begin{aligned}
F_A((E^{hU})_k, (E^{hV})_k) &\simeq F_{(E^{hV})_k}((E^{hV})_k \wedge_A (E^{hU})_k, (E^{hV})_k) \\
&\simeq F_{(E^{hV})_k}(((E^{hV})_k \wedge_A (E^{hU})_k)_k, (E^{hV})_k) \\
&\simeq F_{(E^{hV})_k}(\mathrm{Map}(G/U, (E^{hV})_k), (E^{hV})_k) \\
&\simeq F_{(E^{hV})_k}((E^{hV})_k, (E^{hV}[G/U])_k) \\
&\simeq (E^{hV}[G/U])_k.
\end{aligned}$$

□

Lemma 7.3.9. Let U be an open subgroup of G, and let V be an open normal subgroup of G, such that $V \le U$. Then there is a map of discrete G-A-modules

$$\xi : A[G/U] \wedge_A E^{hU} \to E^{hV},$$

where G acts on the source of ξ by acting only on $A[G/U]$.

Proof. To produce the map ξ, it suffices to construct the adjoint map of sets

$$\widetilde{\xi} : G/U \to \mathrm{Mod}_A(E^{hU}, E^{hV}).$$

Observe that the multiplication map

$$g : E \to E$$

descends to a map

$$\widetilde{\xi}(gU) = \overline{g} : E^{hU} \to E^{hV}.$$

It is easy to check that this map is independent of choice of coset representative. To show that ξ is G-equivariant we merely must show that $\widetilde{\xi}$ is G-equivariant, where G acts on the morphism set $\mathrm{Mod}_A(E^{hU}, E^{hV})$ by postcomposition. This is clear from the definition of $\widetilde{\xi}$. □

Lemma 7.3.10. Let U be an open subgroup of G. There is an equivalence of discrete G-spectra

$$\phi : (E[G/U])_k \xrightarrow{\simeq} \operatorname*{colim}_{V \trianglelefteq_o G, V \leq U} F_A((E^{hU})_k, (E^{hV})_k).$$

Here, G is acting diagonally on the left hand side and acting only on each $(E^{hV})_k$ on the right hand side.

Remark 7.3.11. Let V be an open normal subgroup of G. Since E^{hV} is a G/V-spectrum, the function spectrum $F_A((E^{hU})_k, (E^{hV})_k)$ is a discrete G-spectrum, where G is acting only on the spectrum E^{hV}. The colimit

$$\operatorname*{colim}_{V \trianglelefteq_o G, V \leq U} F_A((E^{hU})_k, (E^{hV})_k)$$

is therefore a discrete G-spectrum.

Proof of 7.3.10. The map ϕ is given as the composite

$$\begin{aligned} (E[G/U])_k &\simeq ((\operatorname*{colim}_{V \trianglelefteq_o G}(E^{hV})_k)[G/U])_k \\ &\simeq (\operatorname*{colim}_{V \trianglelefteq_o G}(E^{hV}[G/U])_k)_k \\ &\xrightarrow{\psi} (\operatorname*{colim}_{V \trianglelefteq_o G, V \leq U} F_A((E^{hU})_k, (E^{hV})_k))_k. \end{aligned}$$

The map ψ is the colimit of maps

$$\psi_V : (E^{hV}[G/U])_k \to F_A((E^{hU})_k, (E^{hV})_k)$$

whose adjoints $\widetilde{\psi}_V$ are given by the composites

$$\begin{aligned} \widetilde{\psi}_V : (E^{hV})_k \wedge_A (A[G/U] \wedge_A (E^{hU})_k) &\xrightarrow{1 \wedge \xi} (E^{hV})_k \wedge_A (E^{hV})_k \\ &\xrightarrow{\mu} (E^{hV})_k, \end{aligned}$$

where $\mu : (E^{hV})_k \wedge_A (E^{hV})_k \to (E^{hV})_k$ is the multiplication map of the A-algebra $(E^{hV})_k$.

Non-equivariantly, the maps ψ_V correspond to those of Lemma 7.3.8, and are therefore equivalences. Therefore ψ is an equivalence, so ϕ is an equivalence. □

Proof of Theorem 7.3.1. We have equivalences:

$$\begin{aligned}
(E[[G/H]]^{hK})_k &= \operatorname*{holim}_{H\le U\le_o G} \operatorname*{holim}_{\Delta} \mathrm{Map}^c(K^\bullet, E[G/U])_k \\
&\simeq \operatorname*{holim}_{H\le U\le_o G} \operatorname*{holim}_{\Delta} \mathrm{Map}^c(K^\bullet, (E[G/U])_k)_k \\
&\simeq \operatorname*{holim}_{H\le U\le_o G} \operatorname*{holim}_{\Delta} \mathrm{Map}^c(K^\bullet, \operatorname*{colim}_{V\trianglelefteq_o G, V\le U} F_A((E^{hU})_k, (E^{hV})_k))_k \\
&\simeq \operatorname*{holim}_{H\le U\le_o G} \operatorname*{holim}_{\Delta}(\operatorname*{colim}_{V\trianglelefteq_o G, V\le U} \mathrm{Map}^c(K^\bullet, F_A((E^{hU})_k, (E^{hV})_k)))_k \\
&\simeq \operatorname*{holim}_{H\le U\le_o G} \operatorname*{holim}_{\Delta}(\operatorname*{colim}_{V\trianglelefteq_o G, V\le U} F_A((E^{hU})_k, \mathrm{Map}^c(K^\bullet, (E^{hV})_k)))_k \\
&\simeq \operatorname*{holim}_{H\le U\le_o G} \operatorname*{holim}_{\Delta} F_A((E^{hU})_k, \operatorname*{colim}_{V\trianglelefteq_o G, V\le U} \mathrm{Map}^c(K^\bullet, (E^{hV})_k))_k \\
&\simeq \operatorname*{holim}_{H\le U\le_o G} \operatorname*{holim}_{\Delta} F_A((E^{hU})_k, \mathrm{Map}^c(K^\bullet, \operatorname*{colim}_{V\trianglelefteq_o G, V\le U} (E^{hV})_k))_k \\
&\simeq \operatorname*{holim}_{H\le U\le_o G} \operatorname*{holim}_{\Delta} F_A((E^{hU})_k, \mathrm{Map}^c(K^\bullet, E))_k \\
&\simeq \operatorname*{holim}_{H\le U\le_o G} F_A((E^{hU})_k, \operatorname*{holim}_{\Delta} \mathrm{Map}^c(K^\bullet, E))_k \\
&\simeq \operatorname*{holim}_{H\le U\le_o G} F_A((E^{hU})_k, (E^{hK})_k) \\
&\simeq F_A(\operatorname*{colim}_{H\le U\le_o G} (E^{hU})_k, (E^{hK})_k) \\
&\simeq F_A((E^{hH})_k, (E^{hK})_k).
\end{aligned}$$

□

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