FUNCTION SPECTRA AND CONTINUOUS G-SPECTRA

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Abstract. Let G be a profinite group, $\{X_\alpha\}_\alpha$ a cofiltered diagram of discrete G-spectra, and Z a spectrum with trivial G-action. We show how to define the homotopy fixed point spectrum $F(Z, \operatorname{holim}_\alpha X_\alpha)^{hG}$ and that when G has finite virtual cohomological dimension (*vcd*), it is equivalent to $F(Z, \operatorname{holim}_\alpha (X_\alpha)^{hG})$. With these tools, we show that the $K(n)$-local Spanier-Whitehead dual is always a homotopy fixed point spectrum, a well-known Adams-type spectral sequence is actually a descent spectral sequence, and, for a sufficiently nice k-local profinite G-Galois extension E, with $K \lhd G$ and closed, the equivalence $(E^{h_kK})^{h_kG/K} \simeq E^{h_kG}$ (due to Behrens and the author), where $(-)^{h_k(-)}$ denotes k-local homotopy fixed points, can be upgraded to an equivalence that just uses ordinary (*non-local*) homotopy fixed points, when G/K has finite vcd.

1. Introduction

In this paper, all of our spectra are symmetric spectra of simplicial sets and we use G to denote a profinite group. Also, as in [1, Section 2.3], we let $\Sigma\mathrm{Sp}_G$ be the category of discrete G-spectra. Thus, if $X \in \Sigma\mathrm{Sp}_G$, then, in particular, X is a symmetric spectrum with a G-action and the symmetric sequence $\{X_i\}_{i\geq 0}$ of simplicial G-sets has the property that, for each $j \geq 0$, the action map on j-simplices,

$$G \times (X_i)_j \to (X_i)_j,$$

is continuous, when the set $(X_i)_j$ is regarded as a discrete space.

As in [1, Section 4], let

$$\{X_\alpha\}_\alpha$$

be a cofiltered diagram in $\Sigma\mathrm{Sp}_G$; thus, $\{X_\alpha\}_\alpha$ is a pro-discrete G-spectrum. Following [1, Section 4], we refer to the diagram $\{X_\alpha\}_\alpha$ as a *continuous G-spectrum* and the *G-homotopy fixed point spectrum* of the spectrum $\operatorname{holim}_\alpha X_\alpha$ is defined by

$$\big(\operatorname*{holim}_\alpha X_\alpha\big)^{hG} = \operatorname*{holim}_\alpha (X_\alpha)^{hG}. \tag{1.1}$$

Now let Z be any spectrum with trivial G-action and let $F(Z, \operatorname{holim}_\alpha X_\alpha)$ be the function spectrum. We can write the spectrum Z as

$$Z \simeq \operatorname*{hocolim}_\beta Z_\beta, \tag{1.2}$$

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a homotopy colimit of a directed system of finite spectra Z_β, and hence,

$$\begin{aligned} F(Z, \operatorname*{holim}_\alpha X_\alpha) &\simeq \operatorname*{holim}_\beta F(Z_\beta, \operatorname*{holim}_\alpha X_\alpha) \\ &\simeq \operatorname*{holim}_{\alpha,\beta} (X_\alpha \wedge DZ_\beta), \end{aligned}$$

where DZ_β is the Spanier-Whitehead dual of Z_β. We regard each DZ_β as having trivial G-action, so that each $X_\alpha \wedge DZ_\beta$, with the diagonal G-action, is a discrete G-spectrum. Thus, the diagram $\{X_\alpha \wedge DZ_\beta\}_{\alpha,\beta}$ is a continuous G-spectrum and hence, it is natural to make the following definition.

Definition 1.3. If $\{X_\alpha\}_\alpha$ is a continuous G-spectrum and Z is a spectrum with trivial G-action, then we define

$$\begin{aligned} F(Z, \operatorname*{holim}_\alpha X_\alpha)^{hG} &= \big(\operatorname*{holim}_{\alpha,\beta} (X_\alpha \wedge DZ_\beta)\big)^{hG} \\ &= \operatorname*{holim}_{\alpha,\beta} (X_\alpha \wedge DZ_\beta)^{hG}, \end{aligned}$$

where the second equality follows immediately from Definition (1.1).

Now suppose that G has *finite vcd* (that is, "finite virtual cohomological dimension"): this assumption means exactly that there is a natural number m and an open subgroup U of G such that $H^s_c(U; M) = 0$, whenever $s > m$, for all discrete U-modules M. Here, $H^s_c(U; M)$ is the continuous cohomology of the profinite group U, with coefficients in M.

The above assumption of finite vcd, combined with [3, Remark 7.16] and the fact that each DZ_β is a finite spectrum, justifies the first equivalence in the following:

$$\begin{aligned} \operatorname*{holim}_{\alpha,\beta} (X_\alpha \wedge DZ_\beta)^{hG} &\simeq \operatorname*{holim}_{\alpha,\beta} \big((X_\alpha)^{hG} \wedge DZ_\beta\big) \\ &\simeq \operatorname*{holim}_\alpha F\big(Z, (X_\alpha)^{hG}\big) \\ &\simeq F\big(Z, (\operatorname*{holim}_\alpha X_\alpha)^{hG}\big). \end{aligned}$$

The above string of equivalences and the discussion that precedes it prove the following result.

Theorem 1.4. *If the profinite group G has finite vcd, $\{X_\alpha\}_\alpha$ is a continuous G-spectrum, and Z is any spectrum with trivial G-action, then*

$$F(Z, \operatorname*{holim}_\alpha X_\alpha)^{hG} \simeq F\big(Z, (\operatorname*{holim}_\alpha X_\alpha)^{hG}\big).$$

In the case when G is finite, Theorem 1.4 is well-known. Also, there is a quick and interesting application of this result to chromatic homotopy theory. To see this, we need to pause to introduce the main actors, along with some notation.

Let $n \geq 1$, let p be a fixed prime, and let E_n be the Lubin-Tate spectrum with

$$\pi_*(E_n) = W(\mathbb{F}_{p^n})[\![u_1, ..., u_{n-1}]\!][u^{\pm 1}],$$

where $W(\mathbb{F}_{p^n})$ is the ring of Witt vectors of the field $\mathbb{F}_{p^n}$, each u_i has degree zero, and the degree of u is -2. Also, set

$$G_n = S_n \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p),$$

the extended Morava stabilizer group; G_n is a profinite group of finite vcd. Finally, let $K(n)$ be the nth Morava K-theory spectrum, $L_{K(n)}(S^0)$ the $K(n)$-local sphere, and

$$M_0 \leftarrow M_1 \leftarrow M_2 \leftarrow \cdots \leftarrow M_i \leftarrow \cdots$$

a tower of generalized Moore spectra, each of which is finite, such that there is an equivalence $L_{K(n)}(S^0) \simeq \operatorname{holim}_i L_{E(n)}(M_i)$, where $E(n)$ is the Johnson-Wilson spectrum (see [10, Section 2]).

We recall that in [7], for any closed subgroup K of G_n, Devinatz and Hopkins construct a commutative S-algebra E_n^{dhK} that behaves like a K-homotopy fixed point spectrum. (We note that instead of the notation E_n^{dhK}, [7] uses "E_n^{hK}." However, we reserve the notation E_n^{hK} for the homotopy fixed point spectrum of E_n that is formed with respect to the continuous action of K on E_n (as in [3, Definition 9.2]).) Following [3], we use this construction of Devinatz and Hopkins to define

$$F_n = \operatorname*{colim}_{N \triangleleft_o G_n} E_n^{dhN},$$

where the colimit is over all open normal subgroups of G_n; since each E_n^{dhN} is a (G_n/N)-spectrum, F_n is a discrete G_n-spectrum and $\{F_n \wedge M_i\}_i$ is a continuous G_n-spectrum. Then there is the equivalence

$$E_n^{hG_n} = \big(\operatorname*{holim}_i (F_n \wedge M_i)\big)^{hG_n} \simeq L_{K(n)}(S^0),$$

thanks to [7, Theorem 1, (iii)] and [1, Theorem 8.2.1].

Now we are ready to return to Theorem 1.4: if $G = G_n$, $\{X_\alpha\}_\alpha$ is set equal to $\{F_n \wedge M_i\}_i$, and $Z = E_n \simeq \operatorname{holim}_i (F_n \wedge M_i)$, then this result – together with Definition 1.3 – makes precise and justifies the assertion

$$F(E_n, L_{K(n)}(S^0)) \simeq F(E_n, E_n)^{hG_n},$$

which occurs in [11, end of Section 8.1]. More generally, if Z is any spectrum with trivial G_n-action, there is the equivalence

$$F(Z, L_{K(n)}(S^0)) \simeq F(Z, E_n)^{hG_n},$$

where the left-hand side in this equivalence,

$$F(Z, L_{K(n)}(S^0)) \simeq F\big(L_{K(n)}(Z), L_{K(n)}(S^0)\big),$$

is equivalent to the $K(n)$-local Spanier-Whitehead dual of the $K(n)$-local spectrum $L_{K(n)}(Z)$. Thus, the functional dual in the $K(n)$-local category is given by G_n-homotopy fixed points.

Remark 1.5. As shown in [5, Section 1; Corollary 5.3], there is an equivalence $(F_n)^{hG_n} \simeq L_{K(n)}(S^0)$, and hence, for Z an arbitrary spectrum with trivial G_n-action,

$$F\big(L_{K(n)}(Z), L_{K(n)}(S^0)\big) \simeq F\big(Z, (F_n)^{hG_n}\big) \simeq F(Z, F_n)^{hG_n}.$$

Thus, we can conclude that the functional dual of $L_{K(n)}(Z)$ in the $K(n)$-local category is also given by the G_n-homotopy fixed points of the function spectrum $F(Z, F_n)$, which, curiously, is not necessarily $K(n)$-local (for example, when $Z = S^0$, $F(Z, F_n) \cong F_n$ is not $K(n)$-local, by [3, Lemma 6.7]).

In addition to the above conclusions, Theorem 1.4 is also useful for further developing the theory of homotopy fixed points in at least two other ways: it plays a role in obtaining Theorem 2.5, which is a result about iterated homotopy fixed points for a certain type of profinite Galois extension; and, for K any closed subgroup of G_n and Z any spectrum, we show in Theorem 3.4 that the strongly convergent Adams-type spectral sequence with abutment $(E_n^{dhK})^*(Z)$, constructed by Devinatz and Hopkins in [7], is actually a descent spectral sequence for the homotopy fixed point spectrum $F(Z, E_n)^{hK}$. To keep this Introduction from being unnecessarily redundant, we defer a fuller exposition of these two applications to Sections 2 and 3.

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2. Iterated homotopy fixed points for profinite Galois extensions

Our first extended application of the tools developed in the Introduction is to the theory of profinite Galois extensions; for background on these extensions, we refer the reader to [1] and [11].

We begin by establishing some notation. As in [1], suppose that $L_k(-)$, Bousfield localization with respect to the spectrum k, satisfies the equivalence

$$L_k(-) \simeq L_M(L_T(-)),$$

where M is a finite spectrum and T is smashing. Also, suppose that A is a cofibrant commutative symmetric ring spectrum that is k-local. Finally, given a profinite group H, let $(-)^{h_k H}$ denote the right derived functor of the fixed points $(-)^H$, with respect to the k-local model structure on discrete H-spectra (see [1, Section 6.1]): given a discrete H-spectrum Y,

$$Y^{h_k H} = (Y_{f_k H})^H,$$

where $Y \to Y_{f_k H}$ is a natural trivial cofibration, with $Y_{f_k H}$ fibrant, in the k-local model structure on discrete H-spectra (henceforth, we will say that $Y_{f_k H}$ is a "k-locally fibrant discrete H-spectrum").

As in [1, Section 7], we let E be a consistent profaithful k-local profinite G-Galois extension of A of finite vcd and we recall that by [1, Proposition 6.2.3], E is a discrete G-spectrum. Also, we let K be any closed normal subgroup of G. By [1, Proposition 7.1.4], $(E_{f_k G})^K$ is k-locally fibrant as a discrete (G/K)-spectrum, and hence, since the (G/K)-equivariant map

$$\lambda\colon (E_{f_k G})^K \xrightarrow{\simeq_k} \big((E_{f_k G})^K\big)_{f_k G/K}$$

is a k-local equivalence between k-locally fibrant discrete (G/K)-spectra, the induced map $\lambda^{G/K}$, which has the form

$$E^{h_k G} = (E_{f_k G})^G = \big((E_{f_k G})^K\big)^{G/K} \xrightarrow[\lambda^{G/K}]{\simeq_k} \big(\big((E_{f_k G})^K\big)_{f_k G/K}\big)^{G/K} = ((E_{f_k G})^K)^{h_k G/K},$$

is a k-local equivalence. By [1, Proposition 6.1.7, (1)], the source $E^{h_k G}$ and target $((E_{f_k G})^K)^{h_k G/K}$ of $\lambda^{G/K}$ are k-local spectra. Therefore,

$$\lambda^{G/K}\colon E^{h_k G} \xrightarrow{\simeq} ((E_{f_k G})^K)^{h_k G/K} \tag{2.1}$$

is a weak equivalence of spectra.

Notice that there is a zigzag of k-local equivalences

$$E_{f_k G} \xrightarrow{\simeq_k} (E_{f_k G})_{f_k K} \xleftarrow{\simeq_k} E_{f_k K} \tag{2.2}$$

of discrete K-spectra. Then taking the K-fixed points of zigzag (2.2) gives the zigzag

$$(E_{f_k G})^K \xrightarrow{\simeq_k} \big((E_{f_k G})_{f_k K}\big)^K \xleftarrow{\simeq} (E_{f_k K})^K = E^{h_k K}, \tag{2.3}$$

where the second map is a weak equivalence of spectra, since it is the K-fixed points of a k-local equivalence between k-locally fibrant discrete K-spectra, and the first map is a k-local equivalence, due to the combination of the last conclusion (about the second map) and the end of [1, proof of Theorem 7.1.6].

In the k-local model structure on discrete (G/K)-spectra, the weak equivalences are those morphisms that are k-local equivalences, and hence, though $\big((E_{f_kG})_{f_kK}\big)^K$ and $(E_{f_kK})^K$ do not necessarily carry (pertinent, nontrivial) (G/K)-actions, zigzag (2.3) makes it reasonable – in the k-local setting – to identify the discrete (G/K)-spectrum $(E_{f_kG})^K$ with E^{h_kK}, so that equivalence (2.1) can be interpreted as the equivalence

$$(E^{h_kK})^{h_kG/K} \simeq E^{h_kG}. \tag{2.4}$$

Equivalence (2.4) says that given a sufficiently nice profinite Galois extension E, the iterated *k-local* homotopy fixed point spectrum $(E^{h_kK})^{h_kG/K}$ can be formed, and it behaves in a natural way, in that it is equivalent to E^{h_kG} and thereby mimics the fixed point identity

$$(E^K)^{G/K} = E^G.$$

Though equivalence (2.4) is a step forward in the theory of profinite Galois extensions, we would like to have such a result about iterated homotopy fixed points that *avoids* the k-local setting that is used in (2.4). This is not an easy thing to achieve: as explained in detail in [4, Sections 1, 3, and 4] and [1, Section 3.6], there are subtleties with (non-local) iterated homotopy fixed points that, in general, make even forming the iterated homotopy fixed point spectrum a difficult task. However, with E as above, we are able to show that $L_M(E)$ is the homotopy limit of a continuous G-spectrum, so that one can form $(L_M(E))^{hK}$, and this last spectrum is the homotopy limit of a continuous (G/K)-spectrum, so that one can form the iterated homotopy fixed point spectrum $\big((L_M(E))^{hK}\big)^{hG/K}$. Additionally, we obtain the following result.

Theorem 2.5. *Let E be a consistent profaithful k-local profinite G-Galois extension of A of finite vcd. If K is a closed normal subgroup of G such that G/K has finite vcd, then there is an equivalence*

$$\big((L_M(E))^{hK}\big)^{hG/K} \simeq (L_M(E))^{hG}.$$

The above theorem shows that, as desired, a sufficiently nice profinite Galois extension does indeed satisfy a non-local version of equivalence (2.4), when the quotient group G/K has finite vcd. The proof of Theorem 2.5 and the justification for the two conclusions that immediately precede it are placed at the end of this paper, in Section 4.

Remark 2.6. Under the hypotheses of Theorem 2.5, there is an equivalence

$$(L_M(E))^{hK} \simeq E^{h_kK},$$

by (4.2) and [1, Proposition 6.1.7, (3)], and the same argument shows that

$$(L_M(E))^{hG} \simeq E^{h_kG}.$$

Thus, the equivalence of Theorem 2.5 is exactly the equivalence of (2.4) above, but presented without using k-local homotopy fixed points.

In the following two examples, we describe some situations in which the cohomological conditions on G and G/K in Theorem 2.5 are satisfied.

Example 2.7. If a profinite group H is compact p-adic analytic, then it has finite vcd (an explanation is written out just after Lemma 2.9 in [3]) and any quotient group H/L, where L is a closed normal subgroup, is again a compact p-adic analytic group, by [8, Theorem 9.6]. Thus, if G is compact p-adic analytic, then it and the quotient G/K automatically have finite vcd.

Example 2.8. Suppose that G is a pro-p group of finite cohomological dimension. If K is nontrivial, topologically finitely generated, and free as a pro-p group, then G/K has finite vcd, by [9, Theorem 0.2]. If K is analytic pro-p of dimension $d \geq 1$, then G/K again has finite vcd, by [9, Theorem 0.7]. There are more such results in [9].

3. A comparison of spectral sequences

In this section, we give another application of Theorem 1.4. We will be referring to E_n, G_n, the tower $\{M_i\}_i$, and F_n, as defined in the Introduction. We note that given a spectrum X, we always use $\pi_*(X)$ to refer to the graded abelian group of maps in the stable homotopy category from sphere spectra to X.

Now, let K be any closed subgroup of G_n. In the Introduction, we mentioned that the commutative S-algebra E_n^{dhK} behaves like a K-homotopy fixed point spectrum. As an example of this behavior, by [7, Theorem 2, (ii)], for any spectrum Z with trivial K-action, where $Z \simeq \operatorname{hocolim}_\beta Z_\beta$ (as in equivalence (1.2); recall that each Z_β is a finite spectrum), there is a strongly convergent $K(n)_*$-local E_n-Adams spectral sequence

$$H_c^s\big(K; (E_n)^{-t}(Z)\big) \Rightarrow (E_n^{dhK})^{-t+s}(Z), \tag{3.1}$$

where

$$H_c^s\big(K; (E_n)^{-t}(Z)\big) = \lim_{\beta,i} H_c^s\big(K; \pi_t(E_n \wedge M_i \wedge DZ_\beta)\big), \tag{3.2}$$

with K acting trivially on each M_i and DZ_β. We note that, since G_n is a compact p-adic analytic group, it follows that K is also, by [8, Theorem 9.6], and hence, since each abelian group $\pi_t(E_n \wedge M_i \wedge DZ_\beta)$ is a finite discrete $\mathbb{Z}_p[\![K]\!]$-module, $H_c^s\big(K; \pi_t(E_n \wedge M_i \wedge DZ_\beta)\big)$ is a finite abelian group, by [12, Proposition 4.2.2].

Remark 3.3. To avoid any confusion, we point out that in (3.2) above, we are using the presentation of $H_c^s\big(K; (E_n)^{-t}(Z)\big)$ as an inverse limit that comes from the discussion between Corollary 3.4 and Lemma 3.5 in [6] and this same paper's Proposition 3.6, instead of the presentation that is given by [7, Remark 1.3]. These two (closely related) presentations are isomorphic, but the former is more suitable for our purposes.

By [7] (see [3, Theorem 6.3, Corollary 6.5] for an explicit proof), there is an equivalence

$$E_n \wedge M_i \simeq F_n \wedge M_i,$$

for each i. Thus,

$$(E_n)^{-t}(Z) \cong \pi_t\big(F(Z, \operatorname{holim}_i (F_n \wedge M_i))\big),$$

where $\{F_n \wedge M_i\}_i$ is a continuous K-spectrum, and, since

$$E_n^{dhK} \simeq E_n^{hK},$$

by [1, Theorem 8.2.1], where

$$E_n^{hK} = \big(\operatorname*{holim}_i (F_n \wedge M_i)\big)^{hK},$$

we have

$$(E_n^{dhK})^{-t+s}(Z) \cong \pi_{t-s}\big(F(Z, E_n^{hK})\big) \cong \pi_{t-s}\big(F(Z, E_n)^{hK}\big),$$

where $F(Z, E_n)^{hK}$ is defined as in Definition 1.3 and the last isomorphism above is due to Theorem 1.4 and the fact that K has finite vcd (since G_n has finite vcd). The observations in the preceding sentence suggest that spectral sequence (3.1) ought to be isomorphic to a descent spectral sequence that has the form

$$H_c^s\big(K; \pi_t(F(Z, E_n))\big) \Rightarrow \pi_{t-s}\big(F(Z, E_n)^{hK}\big);$$

the following theorem shows that this suggestion is, in fact, correct.

Theorem 3.4. *Let K be a closed subgroup of G_n and let Z be a spectrum with trivial K-action. Then the strongly convergent Adams-type spectral sequence*

$$H_c^s\big(K; (E_n)^{-t}(Z)\big) \Rightarrow (E_n^{dhK})^{-t+s}(Z) \tag{3.5}$$

is isomorphic to the descent spectral sequence

$$H_c^s\big(K; \pi_t(F(Z, E_n))\big) \Rightarrow \pi_{t-s}\big(F(Z, E_n)^{hK}\big), \tag{3.6}$$

from the E_2-term onward.

Proof. Our first step is to show that the descent spectral sequence exists. Given a profinite group H and a discrete abelian group A, we let $\mathrm{Map}^c(H, A)$ denote the abelian group of continuous functions $H \to A$. Then

$$\lim_{\beta,i}{}^s \, \mathrm{Map}^c\big(K^m, \pi_q(F_n \wedge M_i \wedge DZ_\beta)\big) = 0,$$

for all $s > 0$, all $m \geq 0$, and all $q \in \mathbb{Z}$ (this follows from [7, Lemma 4.21, (i)], since the "G_n" that is in [7, Lemma 4.21, (i) and its proof] can be changed to any profinite group, without affecting the validity of the argument), and therefore, by [1, Section 4.6], there is a homotopy spectral sequence that has the form

$$H_{\mathrm{cts}}^s\big(K; \pi_t(F(Z, E_n))\big) \Rightarrow \pi_{t-s}\big(F(Z, E_n)^{hK}\big),$$

where $H_{\mathrm{cts}}^s\big(K; \pi_t(F(Z, E_n))\big)$ denotes the continuous cohomology of continuous cochains, with coefficients in the profinite $\mathbb{Z}_p[\![K]\!]$-module $\pi_t(F(Z, E_n))$. By [7, Remark 1.3], there is an isomorphism

$$H_{\mathrm{cts}}^s\big(K; \pi_t(F(Z, E_n))\big) \cong H_c^s\big(K; \pi_t(F(Z, E_n))\big),$$

so that the above homotopy spectral sequence is the desired descent spectral sequence.

Spectral sequence (3.5) is the inverse limit $\lim_{\beta,i} {}^{\mathcal{A}}E_r^{*,*}(\beta, i)$ of $K(n)_*$-local E_n-Adams spectral sequences ${}^{\mathcal{A}}E_r^{*,*}(\beta, i)$ that have the form

$${}^{\mathcal{A}}E_2^{s,t}(\beta, i) = H_c^s\big(K; \pi_t(E_n \wedge M_i \wedge DZ_\beta)\big) \Rightarrow \pi_{t-s}(E_n^{dhK} \wedge M_i \wedge DZ_\beta).$$

Similarly, spectral sequence (3.6) is the inverse limit $\lim_{\beta,i} {}^{\mathcal{D}}E_r^{*,*}(\beta, i)$ of descent spectral sequences ${}^{\mathcal{D}}E_r^{*,*}(\beta, i)$ that have the form

$${}^{\mathcal{D}}E_2^{s,t}(\beta, i) = H_c^s\big(K; \pi_t(F_n \wedge M_i \wedge DZ_\beta)\big) \Rightarrow \pi_{t-s}(E_n^{hK} \wedge M_i \wedge DZ_\beta),$$

where the abutment of spectral sequence $\lim_{\beta,i} {}^{\mathcal{D}}E_r^{*,*}(\beta,i)$ is identified by using the equivalence $F(Z, E_n^{hK}) \simeq F(Z, E_n)^{hK}$. By [1, proof of Theorem 8.2.5], for each β and i, there is an isomorphism

$$ {}^{\mathcal{A}}E_r^{*,*}(\beta,i) \cong {}^{\mathcal{D}}E_r^{*,*}(\beta,i) $$

of spectral sequences from the E_2-terms onward, completing the proof of the theorem. □

4. The proof of Theorem 2.5

In this section, we use the notation that was established in Section 2.

Since M is a finite spectrum, [2, Proposition 3.6] implies that, for any spectrum Y,

$$ L_M(Y) \simeq F({}^{M}S^0, Y), $$

where ${}^{M}S^0 \to S^0$ denotes the $[M, -]_*$-colocalization of the sphere spectrum S^0. Then, as in the Introduction, by writing

$$ {}^{M}S^0 \simeq \operatorname*{hocolim}_{\beta} W_\beta, $$

where the right-hand side is a homotopy colimit of a directed system $\{W_\beta\}_\beta$ of finite spectra, we obtain that

$$ L_M(Y) \simeq \operatorname*{holim}_{\beta} (Y \wedge DW_\beta). $$

Now let G be any profinite group (we are not assuming that G has finite vcd), let X be a discrete G-spectrum, and give ${}^{M}S^0$ and each DW_β trivial G-action. Then $\{X \wedge DW_\beta\}_\beta$ is a continuous G-spectrum and

$$ L_M(X) \simeq \operatorname*{holim}_{\beta} (X \wedge DW_\beta). $$

These conclusions motivate the following definition.

Definition 4.1. If G is any profinite group and X is a discrete G-spectrum, then it is natural to identify $L_M(X)$ with $F({}^{M}S^0, X)$, and hence, we define

$$ (L_M(X))^{hG} = F({}^{M}S^0, X)^{hG}, $$

so that, by Definition 1.3,

$$ (L_M(X))^{hG} = \operatorname*{holim}_{\beta} (X \wedge DW_\beta)^{hG}. $$

We are now ready to prove Theorem 2.5: we suppose that E is a consistent profaithful k-local profinite G-Galois extension of A that has finite vcd. Recall that E is a discrete G-spectrum, so that by Definition 4.1, for any closed subgroup H of G,

$$ \begin{aligned} (L_M(E))^{hH} &= F({}^{M}S^0, E)^{hH} \\ &= \operatorname*{holim}_{\beta} (E \wedge DW_\beta)^{hH}. \end{aligned} $$

Since G has finite vcd, K does too, and hence, Theorem 1.4 implies that

$$ (L_M(E))^{hK} \simeq F({}^{M}S^0, E^{hK}) \simeq L_M(E^{hK}). $$

For the next step in our argument, we make a few recollections from [1, Sections 2.4, 3.2]. Given any profinite group P, let

$$\mathrm{Map}^c(P, E) = \operatorname*{colim}_{N \lhd_o P} \mathrm{Map}(P/N, E) \cong \operatorname*{colim}_{N \lhd_o P} \prod_{P/N} E.$$

Then $\mathrm{Map}^c(K, -)$ is a coaugmented comonad on the category of spectra and we let $\mathrm{Map}^c(K^\bullet, E)$ be the associated cosimplicial spectrum (obtained through the comonadic cobar construction), which, in codegree k, satisfies the isomorphism

$$\big(\mathrm{Map}^c(K^\bullet, E)\big)^k \cong \mathrm{Map}^c(K^k, E).$$

By [1, Theorem 3.2.1],

$$E^{hK} \simeq \operatorname*{holim}_{\Delta} \mathrm{Map}^c(K^\bullet, E).$$

By [1, Remark 6.2.2], E is T-local, and hence, by [1, Lemma 6.1.4, (3)], the cosimplicial spectrum $\mathrm{Map}^c(K^\bullet, E)$ is T-local in each codegree. Thus, the homotopy limit $\operatorname*{holim}_{\Delta} \mathrm{Map}^c(K^\bullet, E)$ is T-local, implying that E^{hK} is also T-local, so that

$$(L_M(E))^{hK} \simeq L_M(E^{hK}) \simeq L_M(L_T(E^{hK})) \simeq L_k(E^{hK}). \tag{4.2}$$

By [1, Corollary 7.1.3], there is an equivalence

$$L_k(E^{hK}) \simeq L_k((E_{fG})^K), \tag{4.3}$$

where $E \to E_{fG}$ is a trivial cofibration, with E_{fG} fibrant, in the model category $\Sigma\mathrm{Sp}_G$. Therefore, putting (4.2) and (4.3) together yields

$$(L_M(E))^{hK} \simeq L_k((E_{fG})^K).$$

Notice that

$$(E_{fG})^K \cong \operatorname*{colim}_{K < U <_o G} (E_{fG})^U.$$

By the proof of [1, Proposition 3.3.1, (3)], $(E_{fG})^U \simeq E^{hU}$. The argument above that showed that E^{hK} is T-local also shows that each E^{hU}, and hence, each $(E_{fG})^U$, is T-local. Since T is smashing, the filtered colimit $\operatorname*{colim}_{K < U <_o G} (E_{fG})^U$ is T-local, implying that $(E_{fG})^K$ is too. We conclude that

$$(L_M(E))^{hK} \simeq L_k((E_{fG})^K) \simeq L_M((E_{fG})^K) \tag{4.4}$$

and we note that $(E_{fG})^K$ is a discrete (G/K)-spectrum. Therefore, thanks to (4.4), we identify $(L_M(E))^{hK}$ with $L_M((E_{fG})^K)$, and hence,

$$\big((L_M(E))^{hK}\big)^{hG/K} = \big(L_M((E_{fG})^K)\big)^{hG/K}.$$

Thus, by Definition 4.1, we have

$$\big((L_M(E))^{hK}\big)^{hG/K} = F\big({}^M S^0, (E_{fG})^K\big)^{hG/K}.$$

Now suppose that G/K has finite vcd. Then the proof of Theorem 2.5 is completed by the equivalences

$$\begin{aligned} F\big({}^M S^0, (E_{fG})^K\big)^{hG/K} &\simeq F\Big({}^M S^0, \big((E_{fG})^K\big)^{hG/K}\Big) \\ &\simeq F\big({}^M S^0, E^{hG}\big) \\ &\simeq F\big({}^M S^0, E\big)^{hG} \\ &= (L_M(E))^{hG}, \end{aligned}$$

where the first equivalence is due to Theorem 1.4, the second equivalence follows from [1, Proposition 3.5.1], and the third equivalence comes from another application of Theorem 1.4.

References

[1] Mark Behrens and Daniel G. Davis. The homotopy fixed point spectra of profinite Galois extensions. *Trans. Amer. Math. Soc.*, 362(9):4983–5042, 2010.

[2] A. K. Bousfield. The localization of spectra with respect to homology. *Topology*, 18(4):257–281, 1979.

[3] Daniel G. Davis. Homotopy fixed points for $L_{K(n)}(E_n \wedge X)$ using the continuous action. *J. Pure Appl. Algebra*, 206(3):322–354, 2006.

[4] Daniel G. Davis. Iterated homotopy fixed points for the Lubin-Tate spectrum (with an appendix by D.G. Davis and B. Wieland). *Topology Appl.*, 156(17):2881–2898, 2009.

[5] Daniel G. Davis. Obtaining intermediate rings of a local profinite Galois extension without localization. *J. Homotopy Relat. Struct.*, 5(1):253–268, 2010.

[6] Ethan S. Devinatz. A Lyndon-Hochschild-Serre spectral sequence for certain homotopy fixed point spectra. *Trans. Amer. Math. Soc.*, 357(1):129–150 (electronic), 2005.

[7] Ethan S. Devinatz and Michael J. Hopkins. Homotopy fixed point spectra for closed subgroups of the Morava stabilizer groups. *Topology*, 43(1):1–47, 2004.

[8] J. D. Dixon, M. P. F. du Sautoy, A. Mann, and D. Segal. *Analytic pro-p groups*, volume 61 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1999.

[9] A. Engler, D. Haran, D. Kochloukova, and P. A. Zalesskii. Normal subgroups of profinite groups of finite cohomological dimension. *J. London Math. Soc. (2)*, 69(2):317–332, 2004.

[10] Mark Hovey. Bousfield localization functors and Hopkins' chromatic splitting conjecture. In *The Čech Centennial (Boston, MA, 1993)*, pages 225–250. Amer. Math. Soc., Providence, RI, 1995.

[11] John Rognes. Galois extensions of structured ring spectra. In *Galois extensions of structured ring spectra/Stably dualizable groups,* in *Mem. Amer. Math. Soc.,* vol. 192 (898), pages 1–97, 2008.

[12] Peter Symonds and Thomas Weigel. Cohomology of p-adic analytic groups. In *New horizons in pro-p groups*, pages 349–410. Birkhäuser Boston, Boston, MA, 2000.

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