ROGNES'S THEORY OF GALOIS EXTENSIONS AND THE CONTINUOUS ACTION OF G_n ON E_n

DANIEL G. DAVIS

1. INTRODUCTION

In recent years, John Rognes has given various talks introducing his theory of Galois extensions for commutative S-algebras, and several manuscripts about this topic are available at his website. One family of Galois extensions that he has discussed are those that arise from the action of the extended Morava stabilizer group $G_n = S_n \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ on the Lubin-Tate spectrum E_n through automorphisms of commutative S-algebras. In considering this new theory, the author has found these E_n-related Galois extensions quite interesting.

In the author's 2003 thesis ([3] - see [2] for a short summary), for a profinite group G, he developed (making explicit ideas that were implicit in the literature, especially in the foundational work of Bob Thomason and Rick Jardine, and work by Paul Goerss and Steve Mitchell) the notions of continuous action and homotopy fixed points for discrete G-spectra and their towers. For a closed subgroup G in G_n, Ethan Devinatz and Mike Hopkins defined

$$E_n^{hG} := L_{K(n)}(\mathrm{colim}_i E_n^{hU_i G})$$

(see [5, Def. 1.5]). The author applied their work [5] and this definition to show that E_n is a continuous G_n-spectrum with homotopy fixed point spectra, defined using the continuous action, for closed subgroups G in G_n.

The above formula for E_n^{hG} follows a convention that is used throughout this paper: E_n^{hG} is used to denote both the homotopy fixed point spectra of Devinatz and Hopkins, and the homotopy fixed point spectra defined with respect to the continuous action of G on E_n (defined by the author in [3]), since the author showed that these constructions are isomorphic in the stable homotopy category (in [3]).

Since the author does not possess a detailed account of Rognes's ideas (nor has he had the fortune of hearing Rognes speak), and, believing that the machinery of his thesis could be useful for the theory of Galois extensions, the author wrote this paper, to help him precisely understand Rognes's examples and to see more clearly exactly what kind of Galois extensions arise in Lubin-Tate theory. Thus, the job of this paper is primarily to study the extensions that arise from E_n, with its G_n-action, and to consider what kinds of definitions of Galois extension are needed to fit the actual structures.

Part of our work in this paper depends on results that are not yet known to be true. Beginning in §3, we assume that the discrete G_n-module $\pi_*(E_n \wedge M_I)$ can be realized by a spectrum, abusively labeled $F_n \wedge M_I$, that is a discrete G_n-symmetric ring spectrum, that is, a discrete G_n-spectrum that is a ring object in the category

of symmetric spectra, and whose discrete G_n-action is by ring maps. We also assume the existence of certain model categories for various categories of discrete G-symmetric spectra. See Remark 3.1 for more details about our assumptions. The author hopes and believes that these assumptions are true.

If we only assume what is already known and if we modify Definition 7.2 (for "$K(n)$-local G-pro-Galois extension") in an obvious way, then our main results, stated as (4) and (6) below, are still true. We make the above assumptions because they allow us to paint a more coherent and structured picture than would be possible otherwise, and because we hope that the picture will turn out to be correct.

Summary of Main Results. To ease the notation, we write $\hat{L}$ in place of $L_{K(n)}$. We summarize the types of extensions and examples that are considered in this paper by listing our main results (given the above assumptions):

(1) Theorem 2.6: Given an open normal subgroup U of G_n, the map of commutative S-algebras $\alpha(U)\colon \hat{L}S^0 \to E_n^{hU}$ is a $K(n)$-local G_n/U-Galois extension.

(2) Theorem 3.5: The map $\gamma(U, I)\colon \hat{L}M_I \to (F_n \wedge M_I)^{hU} \cong E_n^{hU} \wedge M_I$ is an associative G_n/U-Galois extension.

(3) Theorem 5.4: The map $\gamma(I) = \operatorname{colim}_i \gamma(U_i, I)\colon \hat{L}M_I \to F_n \wedge M_I$ is an associative filtered G_n-Galois extension.

(4) Theorem 7.5: The map $\operatorname{holim}_I \gamma(I)\colon \hat{L}S^0 \to E_n$ is a $K(n)$-local G_n-pro-Galois extension. Also, we explain why $\hat{L}S^0 \to E_n$ is a strongly $K(n)$-local filtered G_n-pro-Galois extension.

(5) Theorem 8.2: Given any closed subgroup G of G_n, the directed system $\{\beta(G, i, I)\colon E_n^{hU_iG} \wedge M_I \to E_n^{hU_i} \wedge M_I\}_i$, of associative U_iG/U_i-Galois extensions, makes the map

$$\beta(G, I) = \operatorname{colim}_i \beta(G, i, I)\colon (E_n^{hG} \wedge M_I) \to F_n \wedge M_I \simeq E_n \wedge M_I$$

an associative filtered G-Galois extension.

(6) Theorem 8.4: The inverse system $\{\beta(G, I)\}_I$ of $K(n)$-local associative G-Galois extensions makes the map $\beta(G) = \operatorname{holim}_I \beta(G, I)\colon E_n^{hG} \to E_n$ a $K(n)$-local G-pro-Galois extension.

(7) Theorem 8.5: The map $\alpha(U) = \operatorname{holim}_I \gamma(U, I)$ is a $K(n)$-local G_n/U-pro-Galois extension.

Notation and Conventions. Often, when we use results from [5] and [3], we do not give references. Throughout this paper, U is an open normal subgroup of G_n. Sp is the model category $(\text{spectra})^{\text{stable}}$ of Bousfield-Friedlander spectra. We often use the symbol $\cong$ to denote isomorphism in the stable homotopy category. Whenever necessary, we assume that our commutative S-algebras are cell commutative, and, given an S-algebra R, that our R-modules are cell R-modules. All colimits of spectra are formed in S-modules, Sp, or Sp^Σ, the model category of symmetric spectra of simplicial sets; which category is used will be clear from context. Whenever necessary, we view an S-module as a (symmetric) spectrum of simplicial sets, and vice versa.

Given a profinite group G, if a colimit or limit is indexed by a collection $\{N\}$, then $\{N\}$ is a cofinal collection of open normal subgroups of G. Also, Sp_G is the category of discrete G-spectra, and, given $X \in Sp_G$, $X_{f,G}$ denotes the spectrum

obtained from factoring $X \to *$ as $X \to X_{f,G} \to *$, a trivial cofibration, followed by a fibration, in Sp_G. Then, by definition, $X^{hG} = (X_{f,G})^G$.

If X is a (pointed) discrete G-set, simplicial set, or spectrum, then $\text{Map}_c(G, X)$ is the (pointed) discrete G-set, simplicial set, or spectrum of continuous maps from G to X, and $\Gamma^\bullet_G(X)$ is the canonical cosimplicial (pointed) discrete G-set, simplicial set, or spectrum determined by the triple that is formed from $\text{Map}_c(G, -)$. We will use the fact that if the profinite group G has finite virtual cohomological dimension, then $\text{holim}_\Delta \left(\Gamma^\bullet_G(X_{f,G}) \right)^G$ is a model for X^{hG} (see [3]).

Note to Reader: The author wants to point out that besides the assumption of the validity of certain conjectural remarks, discussed above, the careful reader will notice that there are several other technical problems with this paper, which we now list. (a) We go back and forth between S-modules and spectra of simplicial sets frequently, and this movement is less than desirable. An ideal presentation of the various Galois extensions considered here would do everything in a single category of spectra. (b) Since the colimit in S-modules of S-algebras is not, in general, the colimit in the category of S-algebras (see [6, II, Prop. 7.4]), the colimit in Definition 5.3 should be handled in a better way. (c) At places where we would like to have point-set level maps of (commutative) S-algebras that are weak equivalences, we often have only isomorphisms in the stable homotopy category. Though the author has not ironed out these technicalities, he still believes that the ideas in this paper are essentially correct and worthwhile.

Relationship to Work of Rognes. The inspiration for this paper is the work of John Rognes. Some of the definitions and examples are originally due to him, and those that are not were motivated by his work. At the time of writing, the author does not know how much of the details of what he has written here is already known to Rognes. To make the relationship between this work and Rognes's clearer, the author notes the following. (a) Definition 2.1 is from [11, pg. 1]. (b) Definition 2.4 closely follows Rognes's definition of an E-local G-Galois extension (see [10]). (c) Definition 5.3 was motivated by Rognes's notion of a G-pro-Galois extension (see [11, pg. 1]). (d) Our main theorem, Theorem 7.5, is a slight modification of a result due to Rognes (see [11, pg. 2, (6)], [9, pg. 6, (4)], and [10]). The author readily acknowledges that he has perhaps omitted ways that this work is already known by, written up by, or indebted to Rognes.

2. Galois extensions for finite groups

Definition 2.1. Let G be a finite group. A map $A \to B$ of commutative S-algebras is a *G-Galois extension* if the following conditions hold:

(1) G acts on B through A-algebra maps.
(2) The natural map $A \to B^{hG}$ is a weak equivalence.

(3) There is a weak equivalence $B \wedge_A B \simeq \prod_G B$, where B is an A-module and $- \wedge_A -$ is the smash product in the category of A-modules.

Remark 2.2. The theorem below is due to Rognes ([9], [10]). Since the author knows of no written proof, he attempts one. The proof below is complete, except for the unjustified step marked with a question mark. The unjustified step is asking for a holim to commute with a colimit (specifically, a coequalizer). The author would like to know how to complete the proof, and then prove the same result when G is profinite, in which case the result could be used with Galois extensions that are considered later in this paper.

Theorem 2.3. *Let G be a finite group and let N be a normal subgroup. If $A \to B$ is a G-Galois extension, then $A \to B^{hN}$ is a G/N-Galois extension.*

Proof. We know that $B^{hG} \simeq A$ and $B \wedge_A B \simeq \prod_G B$. Regarding G as a profinite group, with N an open subgroup, $(B^{hN})^{hG/N} \simeq B^{hG} \simeq A$ (see Lemma 3.2). It remains only to show that $B^{hN} \wedge_A B^{hN} \simeq \prod_{G/N} B^{hN}$. We have (where N acts on B and $\mathbf{N}$ acts on $\mathbf{B}$):

$$B^{hN} \wedge_A B^{hN} = (\mathrm{holim}_N B) \wedge_A (\mathrm{holim}_N B) \overset{?}{\simeq} \mathrm{holim}_N \mathrm{holim}_{\mathbf{N}} (B \wedge_A \mathbf{B})$$
$$\simeq \mathrm{holim}_N \mathrm{holim}_{\mathbf{N}} \left(\prod_G \mathbf{B}\right) \cong \mathrm{holim}_N \prod_G (\mathrm{holim}_{\mathbf{N}} \mathbf{B})$$
$$= \mathrm{holim}_N \prod_G B^{hN} = \left(\prod_G B^{hN}\right)^{hN} = \left(\mathrm{Map}_c(G, B^{hN})\right)^{hN}.$$

For convenience, we switch to working in Sp, where we use homotopy fixed points for a profinite group. Since $\mathrm{Map}_c(G, B^{hN}) \cong \mathrm{Map}_c(N, \prod_{G/N} B^{hN})$ and $\prod_{G/N} B^{hN}$ is a fibrant spectrum, $\mathrm{Map}_c(G, B^{hN})$ is fibrant in Sp_N. Thus,

$$\left(\mathrm{Map}_c(G, B^{hN})\right)^{hN} = \left(\left(\mathrm{Map}_c(G, B^{hN})\right)_{f,N}\right)^N$$
$$= \left(\mathrm{Map}_c(G, B^{hN})\right)^N = \prod_{G/N} B^{hN}.$$

$\square$

We are also interested in an E-local version of the notion of Galois extension.

Definition 2.4. Let G be a finite group. A map $A \to B$ of E-local commutative S-algebras is an *E-local G-Galois extension* if the following conditions hold:

(1) G acts on B through A-algebra maps.
(2) The natural map $A \to B^{hG}$ is a weak equivalence.
(3) There is a weak equivalence $L_E(B \wedge_A B) \cong \prod_G B$.

Remark 2.5. Definition 2.4 is slightly different from the definition for the same term given in [10], where Rognes does not assume that A and B are E-local, but he requires $A \to B^{hG}$ and $B \wedge_A B \simeq \prod_G B$ to be E-equivalences.

The result below concerns the map $\alpha(U) = \mathbf{F}(G_n/U \to G_n/G_n) \colon E_n^{hG_n} \to E_n^{hU}$. Following [5, Thm. 1(iii)], we identify $E_n^{hG_n}$ with $\hat{L}S^0$.

Theorem 2.6. *The map $\alpha(U) \colon \hat{L}S^0 \to E_n^{hU}$ of commutative S-algebras is a $K(n)$-local G_n/U-Galois extension.*

Proof. As stated in [4, pg. 8], the finite group G_n/U acts on E_n^{hU} through maps of $\hat{L}S^0$-algebra maps. Also, by [5, Thm. 4], $\hat{L}S^0 \simeq \left(E_n^{hU}\right)^{hG_n/U}$. Also, by [4, Cor. 3.9], there is a weak equivalence

$$\hat{L}(E_n^{hU} \wedge_{LS^0} E_n^{hU}) \simeq \mathrm{Map}_c(G_n/U, E_n^{hU}) \simeq \prod{}_{G_n/U} E_n^{hU}.$$

$\square$

Remark 2.7. Note that (the author believes) $E_n^{hU} \wedge_{\hat{L}S^0} E_n^{hU}$ is not $K(n)$-local (see [6, VIII, Cor. 3.5]), so there is no weak equivalence $E_n^{hU} \wedge_{LS^0} E_n^{hU} \simeq \prod_{G_n/U} E_n^{hU}$. Therefore, $\alpha(U)$ is not a G_n/U-Galois extension.

Remark 2.8. Following [6, Chp. VIII] and [4, §1], let L_E^A denote Bousfield localization with respect to E for A-modules, where E is an A-module. Note that the $K(n)$-local spectrum $K(n)$ is a module over $\hat{L}S^0$, the unit in the $K(n)$-local category. One can define a *strongly E-local G-Galois extension* to be as in Definition 2.4, except that in condition (3), the functor L_E is replaced with L_E^A. Then $\alpha(U)$ is a strongly $K(n)$-local G_n/U-Galois extension. (To verify this, we only need to show that

$$(2.9) \qquad L_{K(n)}^{\hat{L}S^0}(E_n^{hU} \wedge_{\hat{L}S^0} E_n^{hU}) \simeq \hat{L}(E_n^{hU} \wedge_{\hat{L}S^0} E_n^{hU}).$$

Let X be an arbitrary $\hat{L}S^0$-module. Then by [4, pg. 4], $\hat{L}X \simeq L_{\hat{L}S^0 \wedge K(n)}^{\hat{L}S^0}(X)$. Since $\hat{L}S^0 \wedge K(n)$ is $K(n)$-local,

$$\hat{L}S^0 \wedge K(n) \simeq \hat{L}(\hat{L}S^0 \wedge K(n)) \simeq \hat{L}(S^0 \wedge K(n)) \simeq K(n).$$

Thus, $\hat{L}X \simeq L_{K(n)}^{\hat{L}S^0}(X)$, and (2.9) is true.)

The lemma below implies that $\hat{L}(E_n^{hU} \wedge_{\hat{L}S^0} E_n^{hU})$, which is associated to the extension $\alpha(U)$, and $\hat{L}(E_n^{hU} \wedge E_n^{hU})$ are the same as S-modules.

Lemma 2.10. *There is a weak equivalence* $\hat{L}(E_n^{hU} \wedge E_n^{hU}) \simeq \prod_{G_n/U} E_n^{hU}$.

Proof. The finite product of $K(n)$-local spectra is $K(n)$-local, so both spectra under consideration are $K(n)$-local. Then it suffices to show that there is a weak equivalence $\hat{L}(E_n^{hU} \wedge E_n^{hU} \wedge E_n) \simeq \hat{L}((\prod_{G_n/U} E_n^{hU}) \wedge E_n)$, and this follows from $\hat{L}(E_n^{hU} \wedge E_n) \simeq \prod_{G_n/U} E_n$ [5, Cor. 5.5]. $\square$

3. Associative Galois extensions

In this section, we consider Galois extensions of S-algebras that are not necessarily commutative. We use the fact that if R is just an S-algebra and M and N are right and left R-modules, respectively, then the tensor product $M \wedge_R N$ is still defined, though it need not be an R-module [6, III, Def. 3.1].

Recall that $F_n = \mathrm{colim}_i E_n^{hU_i}$ is a discrete G_n-spectrum of simplicial sets. Also, [5, Def. 1.5, Thm. 3(i)] shows that $E_n \simeq \hat{L}(\mathrm{hocolim}_i^{\mathcal{E}} E_n^{hU_i})$, where $\mathrm{hocolim}^{\mathcal{E}}$ is the homotopy colimit in the model category $\mathcal{E}$ of commutative S-algebras. Furthermore, by [5, Lem. 6.2], $\mathrm{hocolim}_i^{\mathcal{E}} E_n^{hU_i} \simeq \mathrm{colim}_i E_n^{hU_i}$, where the colimit is in the category of S-modules. Thus, we can regard F_n as a commutative S-algebra.

Remark 3.1. In the next two paragraphs, all statements are unproven, except for the statements that are in *italics*, which are known to be true. We include the unproven assertions because, if true, they form an integral part of the story of how

Galois extensions appear in Lubin-Tate theory, as the rest of this paper shows. Also, the author believes the assertions are probably true, and he has worked on showing that $E_1 \wedge M(p^i)$ can be realized in $Sp^a_{G_1}$ (see below). We assume the unproven statements are true for the remainder of the paper.

For G, a profinite group, there is a model category Sp^c_G of discrete G-commutative symmetric ring spectra, that is, E_∞-objects in the category of symmetric spectra of simplicial sets that are also discrete G-spectra, such that the G-action is by E_∞-maps. Let Sp^Σ_G be the model category of discrete G-symmetric spectra, and let Sp^c be the model category of commutative symmetric ring spectra. Then the forgetful functor $Sp^c_G \to Sp^\Sigma_G$ and the G-fixed points functor $(-)^G \colon Sp^c_G \to Sp^c$ preserve all weak equivalences and fibrations. Also, if $X \in Sp^c_G$, then X^{hG} is a commutative symmetric ring spectrum.

Now we consider what kind of Galois extension arises for $B = F_n \wedge M_I \simeq E_n \wedge M_I$. It is widely believed that $F_n \wedge M_I$ cannot be a commutative S-algebra. However, it is thought that $F_n \wedge M_I$ is an S-algebra, since Andrew Baker proved that the closely related spectra $E(n)/I_n^k$ are S-algebras [1]. Further, we suppose that $F_n \wedge M_I$ is a discrete G_n-symmetric ring spectrum, that is, $F_n \wedge M_I$ is an object in $Sp^a_{G_n}$, the model category of A_∞-objects in the category of symmetric spectra that are discrete G_n-spectra with an action by A_∞-maps. As above, the forgetful functor $Sp^a_G \to Sp^\Sigma_G$ and the G-fixed points functor $(-)^G \colon Sp^a_G \to Sp^a$ preserve all weak equivalences and fibrations. (Sp^a is the model category of symmetric ring spectra.) Thus, if $X \in Sp^a_G$, then X^{hG} is a symmetric ring spectrum.

The next result is useful for verifying that certain maps are Galois extensions.

Lemma 3.2. *Let X be a discrete G-spectrum and let N be an open normal subgroup of G. Then there is a weak equivalence $X^{hG} \to (X^{hN})^{hG/N}$ is a weak equivalence.*

Proof. The sheaf of spectra $\operatorname{Hom}_G(-, X_{f,G})$ is a globally fibrant presheaf of spectra. Then [7, Prop. 6.39] implies that there is a weak equivalence

$$X^{hG} \cong \operatorname{Hom}_G(*, X_{f,G}) \to \operatorname{holim}_{G/N} \operatorname{Hom}_G(G/N, X_{f,G}) \cong \operatorname{holim}_{G/N} X^{hN},$$

since $X_{f,G}$ is fibrant in Sp_N. Note that $(X_{f,G})^N$ is a G/N-spectrum. Since G/N is finite and the G/N-spectrum X^{hN} is fibrant in Sp, $\operatorname{holim}_{G/N} X^{hN} = (X^{hN})^{hG/N}$. $\square$

Assuming the hypothetical picture discussed above, the map

$$\gamma(U, I) \colon \hat{L}M_I \cong (F_n \wedge M_I)^{hG_n} \to (F_n \wedge M_I)^{hU} \cong E_n^{hU} \wedge M_I$$

is a map of S-algebras. By Lemma 3.2, $\left((F_n \wedge M_I)^{hU}\right)^{hG_n/U} \simeq (F_n \wedge M_I)^{hG_n}$.

Lemma 3.3. *There is a weak equivalence of S-modules*

$$(F_n \wedge M_I)^{hU} \wedge_{\hat{L}M_I} (F_n \wedge M_I)^{hU} \simeq \prod_{G_n/U} (F_n \wedge M_I)^{hU}.$$

Proof. By Lemma 2.10, there are weak equivalences

$$\prod_{G_n/U} (F_n \wedge M_I)^{hU} \simeq \prod_{G_n/U} (E_n^{hU} \wedge M_I) \simeq E_n^{hU} \wedge E_n^{hU} \wedge M_I.$$

Now we consider the left hand side of the desired weak equivalence. Note that

$$(F_n \wedge M_I)^{hU} \cong F_n^{hU} \wedge M_I \cong F_n^{hU} \wedge L_n(M_I) \cong F_n^{hU} \wedge \hat{L}M_I.$$

Thus, by [6, III, Prop. 3.6], we have:

$$(F_n \wedge M_I)^{hU} \wedge_{\hat{L}M_I} (F_n \wedge M_I)^{hU} \cong \left(F_n^{hU} \wedge \hat{L}M_I\right) \wedge_{\hat{L}M_I} \left(F_n^{hU} \wedge M_I\right)$$
$$\simeq F_n^{hU} \wedge F_n^{hU} \wedge M_I \cong E_n^{hU} \wedge E_n^{hU} \wedge M_I.$$

$\square$

This lemma motivates us to make the following definition. The theorem below follows immediately from the lemma and the definition.

Definition 3.4. Let G be a finite group. A map $A \to B$ of S-algebras is an *associative G-Galois extension* if the following conditions hold:

(1) G acts on B.
(2) There is an isomorphism $A \cong B^{hG}$ in the stable homotopy category.
(3) There is a weak equivalence $B \wedge_A B \simeq \prod_G B$, where B is a left and a right A-module.

Theorem 3.5. *The map $\gamma(U, I) \colon \hat{L}M_I \to (F_n \wedge M_I)^{hU} \cong E_n^{hU} \wedge M_I$ of S-algebras is an associative G_n/U-Galois extension.*

Remark 3.6. Without spelling out another definition, it is easy to see that $\gamma(U, I)$ is also a $K(n)$-local associative G_n/U-Galois extension, since $\prod_{G_n/U} (E_n^{hU} \wedge M_I)$ is $K(n)$-local.

4. Galois extensions for profinite groups

In this section we consider the notion of G-Galois extension for a profinite group G.

Definition 4.1. Let G be a profinite group, and let $A \to B$ be a map of commutative S-algebras. Also, let B be a discrete G-commutative symmetric ring spectrum, that is, $B \in Sp_G^c$. Then $A \to B$ is a *G-Galois extension* if the following conditions hold:

(1) There is a compatible G-action on B that is by A-algebra maps.
(2) There is an isomorphism $A \cong B^{hG}$ in the stable homotopy category.
(3) There is a weak equivalence $B \wedge_A B \simeq \operatorname{colim}_N \prod_{G/N} B$.

Remark 4.2. The A-algebra action on the commutative S-algebra B is in the world of S-modules, whereas the discrete G-action is in the world of symmetric spectra of simplicial sets. Since these categories are different, we only ask for compatibility in condition (1) above, instead of requiring that the discrete G-action on B be by A-algebra maps. For an example of what "compatible" means, see the related example mentioned in Remark 7.3 (1).

Remark 4.3. If B is a spectrum of simplicial sets, then there is an isomorphism $\operatorname{colim}_N \prod_{G/N} B \cong \operatorname{Map}_c(G, B)$. We use the former construction in the above definition, since the latter construction, in general, does not give the right spectrum, if B is a spectrum of topological spaces. (If B is an S-module and V is a finite dimensional subspace of $\mathbb{R}^\infty$, then BV, in general, is not a discrete space, and $\operatorname{Map}_c(G, BV) \neq \operatorname{Map}_c(G, BV_{\mathrm{dis}})$, where BV_{dis} is the set BV with the discrete topology.)

Remark 4.4. Let G be profinite and let $A \to B$ be a G-Galois extension. Now suppose that G is finite. Then it is known that $B^{hG} = (B_{f,G})^G$ and $\mathrm{Map}_G(EG_+, B)$ are weakly equivalent. Also, since B is a discrete G-spectrum,

$$\mathrm{colim}_N \prod_{G/N} B \cong \mathrm{Map}_c(G, B) = \prod_G B.$$

Thus, the Galois extension satisfies the conditions of Definition 2.1, so that Definition 4.1 includes Definition 2.1 as a special case, as desired.

We have the following definition for when A and B are only S-algebras.

Definition 4.5. Let G be a profinite group, and let $A \to B$ be a map of S-algebras. Also, let B be a discrete G-symmetric ring spectrum, that is, $B \in Sp_G^a$. Then $A \to B$ is an *associative G-Galois extension* if the following conditions hold:

(1) There is an isomorphism $A \cong B^{hG}$ in the stable homotopy category.
(2) There is a weak equivalence $B \wedge_A B \simeq \mathrm{colim}_N \prod_{G/N} B$.

5. Filtered Galois extensions

In this section, we introduce the notion of filtered Galois extension, which is essentially what Rognes calls a pro-Galois extension [11, pg. 1]. We reserve use of the prefix "pro" for later, when we consider Galois extensions that are inverse limits of Galois extensions.

Definition 5.1. Let $\{A \to B_\alpha\}_\alpha$ be a direct system of G_α-Galois extensions, with $\{G_\alpha\}_\alpha$ an inverse system of finite groups, such that each map $B_\alpha \to B_{\alpha'}$ is $G_{\alpha'}$-equivariant. Let $G = \lim_\alpha G_\alpha$ and let $B = \mathrm{colim}_\alpha B_\alpha$, so that G is a profinite group and $B \in Sp_G$. Henceforth, whenever we say *direct system of G_α-Galois extensions*, we are referring to a system with these properties. A *direct system of associative G_α-Galois extensions* is a direct system of G_α-Galois extensions, except we only require the Galois extensions to be associative.

Let $\{A \to B_\alpha\}_\alpha$ be a direct system of G_α-Galois extensions, such that G has finite virtual cohomological dimension. Recall that if K is profinite with $\mathrm{vcd}(K) < \infty$, then, if $Z \in Sp_K$, $Z^{hK} \simeq \mathrm{holim}_\Delta \left(\Gamma_K^\bullet (Z_{f,K}) \right)^K$. Then there are conditionally convergent descent spectral sequences

$$E_2^{s,t}(\alpha) = H^s(G_\alpha; \pi_t(B_\alpha)) \Rightarrow \pi_{t-s}(B_\alpha^{hG_\alpha}),$$

and

$$E_2^{s,t} = H_c^s(G; \pi_t(B)) \Rightarrow \pi_{t-s}(B^{hG}).$$

Taking a colimit of the spectral sequences $E_r^{*,*}(\alpha)$ yields the spectral sequence

$$\mathrm{colim}_\alpha E_2^{s,t}(\alpha) \cong H_c^s(G; \pi_t(B)) \Rightarrow \pi_{t-s}\Big(\mathrm{holim}_\Delta \mathrm{colim}_\alpha \left(\Gamma_{G_\alpha}^\bullet ((B_\alpha)_{f,G_\alpha}) \right)^{G_\alpha}\Big).$$

Thus, the isomorphism of spectral sequences $\mathrm{colim}_\alpha E_r^{*,*}(\alpha) \cong E_r^{*,*}$, for $r \geq 2$, implies that

$$(5.2) \qquad\qquad B^{hG} \cong \mathrm{holim}_\Delta \mathrm{colim}_\alpha \left(\Gamma_{G_\alpha}^\bullet ((B_\alpha)_{f,G_\alpha}) \right)^{G_\alpha}.$$

Observe that if, in (5.2), the colimit and the holim commute with each other (that is, if the spectral sequence $\mathrm{colim}_\alpha E_r^{*,*}(\alpha)$ converges to the colimit of the abutments $\pi_*(B_\alpha^{hG_\alpha})$), then

$$B^{hG} \cong \mathrm{colim}_\alpha B_\alpha^{hG_\alpha} \simeq \mathrm{colim}_\alpha A = A.$$

However, a strong hypothesis (e.g. the collection $\{E_2^{*,*}(\alpha)\}$ is uniformly bounded on the right - see [12, Lem. 5.50]) is needed for this to be true. Thus, in general, we believe that it need not be the case that, given a directed system $\{A \to B_\alpha\}_\alpha$ of G_α-Galois extensions, there is a weak equivalence $B^{hG} \simeq A$. Thus, $A \to B$ in general, is not automatically a G-Galois extension. This motivates the following definition.

Definition 5.3. Let $\{A \to B_\alpha\}_\alpha$ be a direct system of (associative) G_α-Galois extensions. As before, $G = \lim_\alpha G_\alpha$ is profinite and $B = \operatorname{colim}_\alpha B_\alpha \in Sp_G$. If $A \to B$ is a (associative) G-Galois extension, then $A \to B$ is called a *(associative) filtered G-Galois extension.*

Recall that in Theorem 3.5, we showed that $\{\gamma(U_i, I)\}_i = \{\hat{L}M_I \to E_n^{hU_i} \wedge M_I\}_i$ is a direct system of associative G_n/U_i-Galois extensions.

Theorem 5.4. *The map $\gamma(I) = \operatorname{colim}_i \gamma(U_i, I)\colon \hat{L}M_I \to F_n \wedge M_I$ is an associative filtered G_n-Galois extension.*

Proof of Theorem 5.4. We only have to show that $\gamma(I)$ is an associative G_n-Galois extension. Since $(F_n \wedge M_I)^{hG_n} \cong E_n^{hG_n} \wedge M_I \cong \hat{L}M_I$, it suffices to show that $(F_n \wedge M_I) \wedge_{\hat{L}M_I} (F_n \wedge M_I) \simeq \operatorname{colim}_i \prod_{G_n/U_i} (F_n \wedge M_I)$. Since F_n is $E(n)$-local, $F_n \wedge M_I \simeq F_n \wedge \hat{L}M_I$, so that

$$(F_n \wedge M_I) \wedge_{\hat{L}M_I} (F_n \wedge M_I) \simeq (F_n \wedge \hat{L}M_I) \wedge_{\hat{L}M_I} (F_n \wedge M_I) \simeq F_n \wedge F_n \wedge M_I$$
$$\simeq E_n \wedge E_n \wedge M_I \simeq \operatorname{Map}_c(G_n, F_n \wedge M_I)$$
$$\simeq \operatorname{colim}_i \prod_{G_n/U_i} (F_n \wedge M_I).$$

$\square$

Remark 5.5. The map $\gamma(I)$ is also a $K(n)$-local associative G_n-Galois extension, where we use the following definition.

Definition 5.6. Let G be a profinite group, and let $A \to B$ be a map of E-local S-algebras. Also, let B be a discrete G-symmetric ring spectrum, that is, $B \in Sp_G^a$. Then $A \to B$ is an *E-local associative G-Galois extension* if the following conditions hold:

(1) There is an isomorphism $A \cong B^{hG}$ in the stable homotopy category.
(2) There is a weak equivalence $L_E(B \wedge_A B) \simeq \operatorname{colim}_N \prod_{G/N} B$.

Let (a) $\{A \to B_\alpha\}_\alpha$ be a direct system of (associative) G_α-Galois extensions, and (b) assume that $A \to B$ is a map of commutative S-algebras. Note that if X is a discrete G-set such that $X \cong \operatorname{colim}_N X(N)$, where each $X(N)$ is a G/N-set, then $\operatorname{Map}_c(G, X) \cong \operatorname{colim}_N \operatorname{Map}_c(G/N, X(N))$ (see e.g. [8, Lem. 6.5.4(a)]). Similarly,

$$\operatorname{Map}_c(G, B) \cong \operatorname{Map}_c(\lim_\alpha G_\alpha, \operatorname{colim}_\alpha B_\alpha) \cong \operatorname{colim}_\alpha \operatorname{Map}_c(G_\alpha, B_\alpha)$$
$$\cong \operatorname{colim}_\alpha \prod_{G_\alpha} B_\alpha \simeq \operatorname{colim}_\alpha (B_\alpha \wedge_A B_\alpha).$$

Let $\{\alpha'\}$ be a copy of the indexing set $\{\alpha\}$, so that $\alpha = \alpha'$. Then the set of pairs $\{(\alpha, \alpha)\}_\alpha$ is cofinal in the indexing set $\{(\alpha, \alpha')\}_{\alpha, \alpha'}$ of all pairs, so that

$$\operatorname{colim}_\alpha (B_\alpha \wedge_A B_\alpha) \cong \operatorname{colim}_{(\alpha, \alpha)} (B_\alpha \wedge_A B_\alpha) \cong \operatorname{colim}_{(\alpha, \alpha')} (B_\alpha \wedge_A B_{\alpha'}).$$

Since the construction $B_\alpha \wedge_A B_\alpha$ is a coequalizer,

$$\mathrm{colim}_{(\alpha,\alpha')} (B_\alpha \wedge_A B_{\alpha'}) \cong (\mathrm{colim}_\alpha B_\alpha) \wedge_A (\mathrm{colim}_{\alpha'} B_{\alpha'})$$
$$\cong B \wedge_A B.$$

Thus, $\mathrm{Map}_c(G, B) \simeq B \wedge_A B$, and we summarize this discussion in the remark below.

Remark 5.7. As stated in [11, pg. 1], (a) and (b) above are enough to imply the weak equivalence $B \wedge_A B \simeq \mathrm{colim}_N \prod_{G/N} B$. Thus, Definition 5.3 can be simplified by noting that the last condition in Definitions 4.1 and 4.5 can be ignored.

6. A CONSEQUENCE OF THEOREM 2.3, WHEN G IS PROFINITE

In this brief section, we assume that Theorem 2.3 is true, when G is profinite. Thus, we are assuming that if (i) G is profinite; (ii) $A \to B$ is a G-Galois extension of commutative S-algebras; and (iii) N is an open normal subgroup of G, then $A \to B^{hN}$ is a G/N-Galois extension.

Remark 6.1. In the theorem below, N is an open normal subgroup of G, B is a discrete G-commutative symmetric ring spectrum, and $B_{f,G}$ comes from factoring $B \to *$ in Sp_G^c, as $B \to B_{f,G} \to *$, a trivial cofibration followed by a fibration. Since the forgetful functor $Sp_G^c \to Sp_G^\Sigma$ preserves weak equivalences and fibrations, $B \to B_{f,G}$ is a weak equivalence in Sp_G^Σ, and $B_{f,G}$ is fibrant in Sp_G^Σ. Thus, in Sp_N^Σ, $B \to B_{f,G}$ is a weak equivalence and $B_{f,G}$ is fibrant, so that $(B_{f,G})^N$ is a model for B^{hN}.

Theorem 6.2. *Let $G \cong \lim_N G/N$ be profinite. If $A \to B$ is a G-Galois extension, then the direct system $\{A \to B^{hN}\}_N$, of G/N-Galois extensions, makes the map $A \to B_{f,G}$ a filtered G-Galois extension in a canonical way.*

Proof. Observe that $B_{f,G} = \bigcup_N (B_{f,G})^N = \mathrm{colim}_N B^{hN}$, as required. Also, $(B_{f,G})^{hG} \simeq B^{hG} \simeq A$, and $(B_{f,G}) \wedge_A (B_{f,G}) \simeq B \wedge_A B \simeq \mathrm{colim}_N \prod_{G/N} B \simeq \mathrm{colim}_N \prod_{G/N} B_{f,G}$. $\square$

Remark 6.3. This theorem says that every G-Galois extension is canonically a filtered G-Galois extension.

7. PRO-GALOIS EXTENSIONS

In this section, we define a notion of Galois extension for towers of discrete G-spectra. We are primarily interested in understanding the structure of the Galois extension $\hat{L}S^0 \to E_n$, which Rognes has referred to as a "$K(n)$-local G_n-pro-Galois extension" [10]. We begin by recalling that $\gamma(I) \colon \hat{L}M_I \to F_n \wedge M_I \xrightarrow{\simeq} E_n \wedge M_I$ is an associative filtered G_n-Galois extension. Thus, $\{\gamma(I)\}_I$ is an inverse system of associative filtered G_n-Galois extensions.

Remark 7.1. The definition below is only in the $K(n)$-local setting because this is all that is needed for our examples.

Definition 7.2. Let $J = \{\cdots \to i \to i-1 \to \cdots \to 1 \to 0\}$. Let $\{A_i \to B_i\}_i$ be a J-shaped tower of $K(n)$-local G-Galois extensions, such that $\{B_i\}$ is a tower in Sp_G, and the isomorphism $B_i^{hG} \cong A_i$ comes from a natural weak equivalence $A_i \to B_i^{hG}$. (Whenever B_i is viewed as an object of Sp_G, then it is assumed to be

fibrant there. Similarly, whenever A_i is viewed as an object of Sp, then it is assumed to be fibrant.) We allow any or all of the extensions $A_i \to B_i$ to be associative. Let $A = \operatorname{holim}_i A_i$, $B = \operatorname{holim}_i B_i$, and let $A \to B$ be the obvious map. Then $A \to B$ is a $K(n)$-*local G-pro-Galois extension* if the following conditions hold:

(1) The map $A \to B$ is a map of commutative S-algebras.
(2) The spectrum B is a continuous G-spectrum, G acts on B by maps of A-algebras, and these two G-actions are compatible.
(3) There is a weak equivalence $A \simeq B^{hG}$.
(4) There is a weak equivalence $\hat{L}(B \wedge_A B) \simeq \operatorname{holim}_i \left(\operatorname{colim}_N \prod_{G/N} B_i \right)$.

Remark 7.3. We make some remarks about this definition; in particular, we discuss what is and is not automatically entailed by the hypotheses of the definition.

(1) So that condition (2) above is actually met in practice, we do not require that the continuous action be by maps of A-algebras; we only require that the continuous action and the A-algebra action be compatible. For example, G_n acts on E_n by maps of $\hat{L}S^0$-algebras and this action yields the continuous action described in [3], but the continuous action is only (thus far, known to be) by maps of (unstructured) spectra.
(2) Since all the A_i and B_i are $K(n)$-local, the homotopy limits A and B are also $K(n)$-local.
(3) The hypotheses of the definition imply that B is automatically a continuous G-spectrum.
(4) By [3], $B^{hG} = \operatorname{holim}_i B_i^{hG} \xleftarrow{\simeq} \operatorname{holim}_i A_i = A$, so that the assumptions automatically imply that condition (3) holds.

Remark 7.4. We explain part of our motivation for condition (4) in Definition 7.2. Recall (from [3]) that the functor $\operatorname{Map}_c(G, -) \colon Sp \to Sp_G$ is a right Quillen functor. Let $X \in Sp_G$ be fibrant, so that X is also fibrant in Sp, and hence, $\operatorname{colim}_N \prod_{G/N} X \cong \operatorname{Map}_c(G, X)$ is fibrant in Sp_G. Then $\cdots \to X \to X$, the constant tower of fibrations of fibrant spectra in Sp, gives

$$\operatorname{holim}_i \left(\operatorname{colim}_N \prod_{G/N} X \right) \cong \operatorname{holim}_i \operatorname{Map}_c(G, X) \simeq \lim_i \operatorname{Map}_c(G, X)$$
$$\cong \operatorname{Map}_c(G, X) \cong \operatorname{colim}_N \prod_{G/N} X,$$

where the last spectrum, as desired, has the form of the right-hand side in condition (3) of Definition 4.1. Therefore, a $K(n)$-local G-pro-Galois extension is a generalization of a $K(n)$-local (associative) G-Galois extension from the setting of discrete G-spectra to that of towers of discrete G-spectra.

Theorem 7.5. *The map of commutative S-algebras* $\operatorname{holim}_I \gamma(I)$,

$$\hat{L}S^0 \cong \operatorname{holim}_I \hat{L}M_I \to \operatorname{holim}_I (E_n \wedge M_I) \cong E_n,$$

is a $K(n)$-local G_n-pro-Galois extension.

Proof. We only need to verify condition (4) of Definition 7.2: by [4, Cor. 3.9],

$$\hat{L}(E_n \wedge_{\hat{L}S^0} E_n) \simeq \operatorname{holim}_I \operatorname{Map}_c(G_n, F_n \wedge M_I) \simeq \operatorname{holim}_I \operatorname{colim}_i \prod_{G_n/U_i} (E_n \wedge M_I).$$

$\square$

Remark 7.6. Since $L_{K(n)}^{\hat{L}S^0}(E_n \wedge_{\hat{L}S^0} E_n) \simeq \hat{L}(E_n \wedge_{\hat{L}S^0} E_n)$, $\hat{L}S^0 \to E_n$ is a strongly $K(n)$-local G_n-pro-Galois extension.

Though we have shown that $\hat{L}S^0 \to E_n$ is a $K(n)$-local G_n-pro-Galois extension, this notion still does not capture all of the structure that is present in this extension, due to the extra structure that comes from the filtered extension $\gamma(I)$. We capture this additional structure in the following way.

Let $\{A \xrightarrow{f_\alpha} B_\alpha\}_\alpha$ be a direct system of, possibly $K(n)$-local, G_α-Galois extensions, with $\{G_\alpha\}_\alpha$ an inverse system of finite groups, such that each map $B_\alpha \to B_{\alpha'}$ is $G_{\alpha'}$-equivariant. As usual, let $G = \lim_\alpha G_\alpha$, and let $B = \hat{L}(\text{colim}_\alpha B_\alpha)$. If B is regarded as a spectrum of simplicial sets, then, letting $(-)_{\mathfrak{f}}$ denote functorial fibrant replacement in Sp, $B \cong \text{holim}_I \text{colim}_\alpha (B_\alpha \wedge L_n M_I)_{\mathfrak{f}}$ is a continuous G-spectrum, since $\text{colim}_\alpha (B_\alpha \wedge L_n M_I)_{\mathfrak{f}}$ is a discrete G-spectrum that is fibrant in Sp. Then, if $\hat{L}(\text{colim}_\alpha f_\alpha) \colon \hat{L}A \to B$ is a G-Galois extension, we call $\hat{L}(\text{colim}_\alpha f_\alpha)$ a $K(n)$-*local filtered G-Galois extension*.

Since the direct system $\{\alpha(U_i) \colon \hat{L}S^0 \to E_n^{hU_i}\}_i$, of $K(n)$-local G_n/U_i-Galois extensions, yields the extension $\hat{L}(\text{colim}_i \alpha(U_i)) \colon \hat{L}S^0 \to E_n$, we can refer to the map $\hat{L}S^0 \to E_n$ as a $K(n)$-*local filtered G_n-pro-Galois extension*.

8. More examples, for closed subgroups of G_n

In this section, for any closed subgroup G of G_n (so G is always profinite and not necessarily finite), we give two examples of Galois extensions. First of all, we slightly expand the definition of filtered G-Galois extension.

Definition 8.1. Let $\{A_\alpha \to B_\alpha\}_\alpha$ be a direct system of (associative) G_α-Galois extensions, with $\{G_\alpha\}$ an inverse system of finite groups, where each map $B_\alpha \to B_{\alpha'}$ is $G_{\alpha'}$-equivariant. Let G and B be defined as usual, and let $A = \text{colim}_\alpha A_\alpha$. If $A \to B$ is a (associative) G-Galois extension, then $A \to B$ is a *(associative) filtered G-Galois extension*.

Theorem 8.2. *The direct system* $\{\beta(G,i,I) \colon E_n^{hU_iG} \wedge M_I \to E_n^{hU_i} \wedge M_I\}_i$, *of associative U_iG/U_i-Galois extensions, makes the map*

$$\beta(G,I) = \text{colim}_i \beta(G,i,I) \colon (E_n^{hG} \wedge M_I) \cong (F_n \wedge M_I)^{hG} \to F_n \wedge M_I \simeq E_n \wedge M_I$$

an associative filtered G-Galois extension.

Proof. To make the notation more manageable, we use X/I to denote the spectrum $X \wedge M_I$. Since (we are assuming that) $F_n/I \in Sp_{G_n}^a$, $(F_n/I)^{hU_i} \cong E_n^{hU_i}/I$ and, similarly, $E_n^{hU_iG}/I$ are S-algebras. Now we show that

$$(8.3) \qquad (E_n^{hU_i}/I) \wedge_{(E_n^{hU_iG}/I)} (E_n^{hU_i}/I) \simeq \prod_{U_iG/U_i} (E_n^{hU_i}/I).$$

Note that $E_n^{hU_i}/I$ and $\prod_{U_iG/U_i} (E_n^{hU_iG}/I)$ are $K(n)$-local. Applying [5, Cor. 5.5],

$$\pi_*(E_n \wedge E_n^{hU_i}/I) \cong \pi_*(\prod_{G_n/U_i} (E_n/I)) \cong \prod_{G_n/U_i} \pi_*(E_n/I).$$

Similarly,

$$\pi_*(E_n \wedge (\prod_{U_iG/U_i} (E_n^{hU_iG}/I))) \cong \prod_{G_n/U_iG} \prod_{U_iG/U_i} \pi_*(E_n/I)$$
$$\cong \prod_{G_n/U_i} \pi_*(E_n/I).$$

Thus, $\pi_*(E_n \wedge E_n^{hU_i}/I) \cong \pi_*(E_n \wedge (\prod_{U_iG/U_i} (E_n^{hU_iG}/I)))$, showing that

$$E_n^{hU_i}/I \cong \prod_{U_iG/U_i} (E_n^{hU_iG}/I).$$

12

This implies that

$$(E_n^{hU_i}/I) \wedge_{(E_n^{hU_iG}/I)} (E_n^{hU_i}/I) \cong \left(\prod_{U_iG/U_i} (E_n^{hU_iG}/I)\right) \wedge_{(E_n^{hU_iG}/I)} (E_n^{hU_i}/I)$$

$$\cong \prod_{U_iG/U_i} \left((E_n^{hU_iG}/I) \wedge_{(E_n^{hU_iG}/I)} (E_n^{hU_i}/I)\right)$$

$$\simeq \prod_{U_iG/U_i} (E_n^{hU_i}/I),$$

verifying (8.3). This shows that $\beta(G, i, I)$ is an associative U_iG/U_i-Galois extension.

Note that E_n/I and $\operatorname{colim}_i \prod_{G/(U_i \cap G)} (E_n^{hG}/I)$ are $K(n)$-local, and there is an isomorphism $\pi_*(E_n \wedge E_n/I) \cong \operatorname{Map}_c(G_n, \pi_*(E_n/I))$. Also, as abelian groups,

$$\pi_*(E_n \wedge (\operatorname{colim}_i \prod_{G/(U_i \cap G)} (E_n^{hG}/I))) \cong \operatorname{colim}_i \prod_{G/(U_i \cap G)} \pi_*(E_n \wedge E_n^{hG}/I),$$

which, by [5, Prop. 6.3], is isomorphic to

$$\operatorname{colim}_i \prod_{G/(U_i \cap G)} \operatorname{colim}_j \operatorname{Map}_c(G_n/U_jG, \pi_*(E_n/I)).$$

This last abelian group is isomorphic to

$$\operatorname{colim}_i \prod_{G/(U_i \cap G)} \operatorname{Map}_c(G_n/G, \pi_*(E_n/I)) \cong \operatorname{Map}_c(G \times G_n/G, \pi_*(E_n/I))$$

$$\cong \operatorname{Map}_c(G_n, \pi_*(E_n/I)).$$

Thus, $E_n/I \cong \operatorname{colim}_i \prod_{G/(U_i \cap G)} (E_n^{hG}/I)$, and therefore,

$$(E_n/I) \wedge_{(E_n^{hG}/I)} (E_n/I) \cong \operatorname{colim}_i \prod_{G/(U_i \cap G)} \left((E_n^{hG}/I) \wedge_{(E_n^{hG}/I)} (E_n/I)\right)$$

$$\cong \operatorname{colim}_i \prod_{G/(U_i \cap G)} (E_n/I),$$

completing the proof. $\qquad\square$

Theorem 8.4. *The inverse system $\{\beta(G, I)\}_I$ of associative G-Galois extensions makes the map*

$$\beta(G) = \operatorname{holim}_I \beta(G, I) \colon E_n^{hG} \to E_n$$

a $K(n)$-local G-pro-Galois extension.

Proof. Using the preceding theorem, it is easy to see that each $\beta(G, I)$ is a $K(n)$-local associative G-Galois extension, since $E_n \wedge M_I, E_n^{hG} \wedge M_I$, and

$$\operatorname{colim}_i \prod_{G/(U_i \cap G)} (E_n \wedge M_I) \cong (\operatorname{colim}_i \prod_{G/(U_i \cap G)} E_n) \wedge M_I$$

are all $K(n)$-local.

By [4, Cor. 3.9],

$$\pi_*(\hat{L}(E_n \wedge_{E_n^{hG}} E_n)) \cong \operatorname{Map}_c(G, \pi_*(E_n)) \cong \lim_I \operatorname{Map}_c(G, \pi_*(E_n \wedge M_I)).$$

This implies that

$$\hat{L}(E_n \wedge_{E_n^{hG}} E_n) \cong \operatorname{holim}_I \operatorname{Map}_c(G, (F_n \wedge M_I)_{f,G})$$

$$\cong \operatorname{holim}_I \operatorname{colim}_i \prod_{G/(U_i \cap G)} (F_n \wedge M_I),$$

where the second expression only occurs in Sp. $\qquad\square$

Our last result follows from the last line of the proof of Theorem 2.6.

Theorem 8.5. *The map $\alpha(U) = \operatorname{holim}_I \gamma(U, I)$ is a $K(n)$-local G_n/U-pro-Galois extension.*

References

[1] Andrew Baker. A_∞ structures on some spectra related to Morava K-theories. *Quart. J. Math. Oxford Ser. (2)*, 42(168):403–419, 1991.

[2] Daniel G. Davis. The Lubin-Tate spectrum and its homotopy fixed point spectra. *5 pp., Available at www.math.purdue.edu/~ dgdavis*, 2004.

[3] Daniel G. Davis. The Lubin-Tate Spectrum and its Homotopy Fixed Point Spectra. *Ph.D. thesis, 106 pp., Northwestern University*, May 9, 2003.

[4] Ethan S. Devinatz. A Lyndon-Hochschild-Serre spectral sequence for certain homotopy fixed point spectra. *Available at the Hopf topology archive at http://hopf.math.purdue.edu*, 2002.

[5] Ethan S. Devinatz and Michael J. Hopkins. Homotopy fixed point spectra for closed subgroups of the Morava stabilizer groups. *Topology*, 43(1):1–47, 2004.

[6] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May. *Rings, modules, and algebras in stable homotopy theory*. American Mathematical Society, Providence, RI, 1997. With an appendix by M. Cole.

[7] J. F. Jardine. *Generalized étale cohomology theories*. Birkhäuser Verlag, Basel, 1997.

[8] Luis Ribes and Pavel Zalesskii. *Profinite groups*. Springer-Verlag, Berlin, 2000.

[9] John Rognes. Galois extensions of "Brave New Rings". *Manuscript*, 2002.

[10] John Rognes. Abstract for Oberwolfach talk "Galois theory of commutative S-algebras". *Report No. 13/2003, March 16th - March 22nd, 2003, pg. 8. Available at www.mfo.de*, 2003.

[11] John Rognes. Algebraic K-theory of finitely presented ring spectra. *Available at www.math.uio.no/~ rognes/lectures.html*, September 29, 2000.

[12] R. W. Thomason. Algebraic K-theory and étale cohomology. *Ann. Sci. École Norm. Sup. (4)*, 18(3):437–552, 1985.