A PROOF OF [1, THEOREM 8.5]

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In [1, Theorem 8.5] and the paragraph that follows it, the author remarks that the E_2-term of the spectral sequence of Theorem 8.5 is given by

$$E_2^{s,t} = \lim^s{}_{\Delta\times\{i\}}\, \pi_t((\Gamma^\bullet_G((Z_i)_{f,G}))^G).$$

However, in [1], we did not provide the details of the proof of Theorem 8.5, since it is a special case of a more general result that appears with proof in [2, Proposition 3.1.2]. But in this note, we spell out a proof of Theorem 8.5, in case the reader is interested in a less general proof that deals only with the setting considered in [1].

We begin with a definition.

Definition 0.1. Let $\mathrm{DMod}(G)^{\mathbb{N}}$ denote the category of diagrams in discrete G-modules of the form $\cdots \to M_2 \to M_1 \to M_0$. Define $H^s_{\mathrm{cont}}(G;\{M_i\})$, the *continuous cohomology of G with coefficients in the tower $\{M_i\}$*, to be the sth right derived functor of the left exact functor

$$\varprojlim{}_i\,(-)^G\colon \mathrm{DMod}(G)^{\mathbb{N}} \to \mathrm{Ab}, \quad \{M_i\} \mapsto \varprojlim{}_i\, M_i^G.$$

(This version of continuous cohomology is developed in [3].)

The next two results are special cases of statements in [2, Lemma 3.1.3] and its proof, for the site $G-\mathbf{Sets}_{df}$. Recall that $\lim^s{}_{\Delta\times\mathbb{N}}(-)$ is the sth right derived functor of $\lim_{\Delta\times\mathbb{N}}(-)\colon \mathrm{Ab}^{\Delta\times\mathbb{N}} \to \mathrm{Ab}$.

Lemma 0.2. *If $I = \{I_i\}$ is an injective object in $\mathrm{DMod}(G)^{\mathbb{N}}$, then, for $s > 0$,*

$$\lim^s{}_{\Delta\times\mathbb{N}}\,(\Gamma^\bullet_G I_i)^G = 0.$$

Proof. Let (j,i) be a typical element of $\Delta\times\mathbb{N}$. The functors $\varprojlim_i(-)\colon \mathrm{Ab}^{\mathbb{N}} \to \mathrm{Ab}$ and

$$\lim{}_\Delta\,(-)\colon \mathrm{Ab}^{\Delta\times\mathbb{N}} \to \mathrm{Ab}^{\mathbb{N}}, \quad \{A_{j,i}\}_{j,i} \mapsto \{\lim{}_\Delta A_{j,i}\}_i$$

are left exact functors. Also, the functor $\lim_\Delta(-)$ is right adjoint to the diagonal functor $c\colon \mathrm{Ab}^{\mathbb{N}} \to \mathrm{Ab}^{\Delta\times\mathrm{N}}$ that sends $\{A_i\}$ in $\mathrm{Ab}^{\mathbb{N}}$ to the diagram with $A_{j,i} = A_i$, for every $j \in \Delta$. Since c is exact, $\lim_\Delta(-)$ preserves injectives. Thus, there is a Grothendieck spectral sequence

$$E_2^{s,t} = \varprojlim{}^s{}_i\left(\lim^t{}_\Delta\,(\Gamma^\bullet_G I_i)^G\right) \Rightarrow \lim^{s+t}{}_{\Delta\times\mathbb{N}}\,(\Gamma^\bullet_G I_i)^G.$$

Since I is an injective object, each I_i is injective in $\mathrm{DMod}(G)$, and every map $d_i\colon I_{i+1} \to I_i$ is split surjective, with section r_i [3, Proposition 1.1]. This implies that $d_i\colon I_{i+1}^G \to I_i^G$ is surjective for each i: if $m \in I_i^G$, then $m = d_i(r_i(m))$, and $g\cdot r_i(m) = r_i(g\cdot m) = r_i(m)$, for all $g \in G$, so that $r_i(m) \in I_{i+1}^G$. Thus, $\{I_i^G\}$ is a Mittag-Leffler tower of epimorphisms.

Then

$$\lim^t{}_\Delta\,(\Gamma^\bullet_G I_i)^G \cong H^t((\Gamma^*_G I_i)^G) \cong H^t_c(G;I_i),$$

which equals 0 whenever $t > 0$, and is I_i^G when $t = 0$. Thus, for $t > 0$, $E_2^{s,t} = 0$, so that, for $s > 0$,

$$\lim{}^s{}_{\Delta\times\mathbb{N}}\,(\Gamma_G^\bullet I_i)^G \cong E_2^{s,0} = \varprojlim{}^s{}_i\, I_i^G = 0.$$

□

Theorem 0.3. *Let $\{M_i\}$ be in* $\mathrm{DMod}(G)^{\mathbb{N}}$*. Then for all $s \geq 0$,*

$$\lim{}^s{}_{\Delta\times\mathbb{N}}\,(\Gamma_G^\bullet M_i)^G \cong H^s_{\mathrm{cont}}(G; \{M_i\}).$$

Proof. By definition,

$$H^s_{\mathrm{cont}}(G;\{M_i\}) \cong H^s\big((\varprojlim{}_i\,(-)^G)(I^0 \to I^1 \to \cdots)\big),$$

where $0 \to \{M_i\} \to I^0 \to I^1 \to \cdots$ is an injective resolution in $\mathrm{DMod}(G)^{\mathbb{N}}$. If $\{N_i\}$ is in $\mathrm{DMod}(G)^{\mathbb{N}}$, then

$$\begin{aligned}\lim{}_{\Delta\times\mathbb{N}}\,(\Gamma_G^\bullet N_i)^G &\cong \varprojlim{}_i\,\lim{}_\Delta\,(\Gamma_G^\bullet N_i)^G \cong \varprojlim{}_i\, H^0((\Gamma_G^* N_i)^G)\\ &\cong \varprojlim{}_i\, H^0_c(G; N_i) \cong \varprojlim{}_i\, N_i^G.\end{aligned}$$

Thus,

$$H^s_{\mathrm{cont}}(G;\{M_i\}) \cong H^s\big(\lim{}_{\Delta\times\mathbb{N}}\big((\Gamma_G^\bullet I^0)^G \to (\Gamma_G^\bullet I^1)^G \to \cdots\big)\big),$$

which is associated to the sequence

$$0 \to (\Gamma^\bullet\{M_i\})^G \to (\Gamma^\bullet I^0)^G \to (\Gamma^\bullet I^1)^G \to \cdots, \tag{0.4}$$

in $\mathrm{Ab}^{\Delta\times\mathbb{N}}$. Since $\mathrm{Ab}^{\Delta\times\mathbb{N}}$ is an abelian category with enough injectives, and the functor $\lim_{\Delta\times\mathbb{N}}(-)\colon \mathrm{Ab}^{\Delta\times\mathbb{N}} \to \mathrm{Ab}$ is left exact, if (0.4) is a $(\lim_{\Delta\times\mathbb{N}}(-))$-acyclic resolution of $(\Gamma^\bullet\{M_i\})^G$, then

$$H^s_{\mathrm{cont}}(G;\{M_i\}) \cong \lim{}^s{}_{\Delta\times\mathbb{N}}\,(\Gamma^\bullet M_i)^G.$$

Fix any $([j],k) \in \Delta\times\mathbb{N}$ and consider the sequence

$$0 \to (\Gamma^{j+1}M_k)^G \to (\Gamma^{j+1}I_k^0)^G \to (\Gamma^{j+1}I_k^1)^G \to \cdots. \tag{0.5}$$

Since $0 \to M_k \to I_k^0 \to I_k^1 \to \cdots$ is exact in $\mathrm{DMod}(G)$,

$$0 \to \Gamma^{j+1}M_k \to \Gamma^{j+1}I_k^0 \to \Gamma^{j+1}I_k^1 \to \cdots \tag{0.6}$$

is exact. Since $H^s_c(G;\Gamma^{j+1}I_k^m) = 0$, for any $m \geq 0$ and $s > 0$, (0.6) is a resolution of $\Gamma^{j+1}M_k$ by $(-)^G$-acyclics, and thus,

$$H^s\big((\Gamma^{j+1}I_k^0)^G \to (\Gamma^{j+1}I_k^1)^G \to \cdots\big) \cong H^s_c(G;\Gamma^{j+1}M_k) = 0,$$

for $s > 0$. Since $(-)^G$ is left exact, it follows that (0.5) is an exact sequence. Thus, (0.4) is an exact sequence in $\mathrm{Ab}^{\Delta\times\mathbb{N}}$.

The proof is finished by applying Lemma 0.2. □

References

[1] Daniel G. Davis. Homotopy fixed points for $L_{K(n)}(E_n \wedge X)$ using the continuous action. *J. Pure Appl. Algebra*, 206(3):322–354, 2006.

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