FINITE CW-COMPLEXES AND THE CHROMATIC TOWER

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Abstract. This is an introduction to chromatic stable homotopy theory for a general audience of mathematicians. We begin by defining the homotopy groups of CW-complexes and consider the problem of computing these groups. After several reductions and simplifications in work on this problem, one is led to chromatic theory. We introduce some of the main ideas of this theory. Though no references are given, none of this theory is due to the author!

1. The homotopy groups of finite CW-complexes

Let X be an arbitrary CW-complex with basepoint. One of the fundamental problems in algebraic topology is the computation of

$$\pi_t(X) := [(S^t, *), (X, *)], \quad t \geq 0,$$

the set of pointed homotopy classes of maps. Notice that $\pi_0(X)$ is the set of path components of the space and $\pi_1(X)$ is the fundamental group. For $t > 0$, $\pi_t(X)$ is the tth homotopy group of X. Setting

$$X = \bigcup{}_{\alpha} X_\alpha,$$

the union of its finite subcomplexes, one has

$$\pi_t(X) = \varinjlim{}_{\alpha} \pi_t(X_\alpha).$$

Here the direct limit is like a union, designed for the case when the maps are not all inclusions. Thus, the computation of homotopy groups reduces to the problem of computing the homotopy groups of a finite CW-complex. Henceforth, let X denote a finite CW-complex with basepoint.

Nevertheless, $\pi_t(X)$ is still a difficult invariant to compute. For example, there is no non-contractible simply connected finite CW-complex for which all of its homotopy groups are known.

Now the suspension gives a sequence of maps:

$$\pi_t(X) \to \pi_{t+1}(\Sigma X) \to \pi_{t+2}(\Sigma^2 X) \to \cdots \to \pi_{t+k}(\Sigma^k X) \to \cdots .$$

By the Freudenthal suspension theorem, this map is an isomorphism whenever $k > t+1$. Thus, one defines the tth stable homotopy group

$$\pi_t^s(X) = \pi_{t+k}(\Sigma^k X), \quad k > t+1.$$

These are always finitely generated abelian groups and lots of machinery has been developed for their calculation, so that the stable homotopy groups are easier to calculate. Much interest is centered around computing $\pi_t^s(S^0)$, and it is known, for example, that

$$\pi_t^s(S^0) = \mathbb{Z},\ \mathbb{Z}/2,\ \mathbb{Z}/2,\ \mathbb{Z}/24,\ 0,\ 0,\ \mathbb{Z}/2,\ \mathbb{Z}/240, \quad t = 0, 1, ..., 7.$$

To facilitate their computation, one does homotopy theory with spectra, instead of topological spaces. A spectrum F is a collection of pointed spaces F_n, $n \in \mathbb{Z}$, and pointed maps $\Sigma F_n \to F_{n+1}$. For example, the sphere spectrum S^0 is the collection S^n, for $n \geq 0$, the point $*$ in negative degrees, and the maps $\Sigma S^n \to S^{n+1}$ are homeomorphisms. The suspension spectrum $\Sigma^\infty X$ is given by $(\Sigma^\infty X)_n = \Sigma^n X$. Its negative spaces are again just the point $*$. Also, the complex K-theory spectrum KU is given by $KU_{\text{even}} = BU \times \mathbb{Z}$ and $KU_{\text{odd}} = U$. The maps $\Sigma(BU \times \mathbb{Z}) \to U$ and $\Sigma U \to BU \times \mathbb{Z}$ are the adjoints of the maps $BU \times \mathbb{Z} \to \Omega U$ and $U \to \Omega(BU \times \mathbb{Z})$.

As with topological spaces, one can do homotopy theory with spectra. Roughly speaking, given spectra F and F', the abelian group $[F, F']_t$ is the set of homotopy classes of maps $f\colon F \to F'$, where

$$f = \{f_n\colon F_n \to F'_{n-t}\}_{n \in \mathbb{Z}}.$$

The tth homotopy group of F is defined to be

$$\pi_t(F) = [S^0, F]_t, \quad t \in \mathbb{Z}.$$

We write F_* for the graded abelian group $\{\pi_t(F)\}_{t \in \mathbb{Z}}$. We say that a map $f\colon F \to F'$ is a weak equivalence if $\pi_t(f)$ is an isomorphism for all t. It turns out that

$$\pi_t(\Sigma^\infty X) = \pi_t^s(X),$$

so that from now on, we write $\pi_t(X)$ for this group.

Also, a spectrum gives a generalized cohomology theory, which is like ordinary cohomology except that the dimension axiom is not satisfied. Thus,

$$F^t(X) := [\Sigma^\infty X, F]_{-t}, \text{ and } F_t(X) := [S^0, F \wedge X]_t.$$

Also, one defines the reduced homology $\bar{F}_t(X)$ to be the kernel of the map $F_t(X) \to F_t(*)$.

As far as I know, in the simple case of the sphere, $\pi_t(S^0)$ is completely known only for $t < 60$, so that the stable homotopy groups of finite complexes are still very difficult to compute.

So what do we do? Well, for any finite complex X and any prime p, there is a finite p-local complex $X_{(p)}$ with the property that

$$\pi_t(X_{(p)}) \cong \pi_t(X) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}.$$

Since $\pi_t(X)$ is a finitely generated abelian group, it can be completely recovered from knowledge of $\pi_t(X_{(p)})$ for each prime p. Thus, the problem of computing $\pi_t(X)$ reduces to the problem of computing $\pi_t(X_{(p)})$, for all primes p.

Much progress has been made in computing the homotopy of the finite p-local complex $X_{(p)}$. For example,

$$\pi_{999}(S^0_{(5)}) \cong \mathbb{Z}/5 \oplus \mathbb{Z}/625.$$

However, these computations are still difficult, and thus, other techniques have been developed.

2. The chromatic tower of a finite p-local complex

The best way of tackling the problem of computing $\pi_t(X_{(p)})$ is given by the chromatic tower. To define it, we need some background.

For each n, there is the Morava K-theory spectrum $K(n)$ with

$$K(0)_* = \mathbb{Q}, \text{ and for } n \geq 1, \; K(n)_* = \mathbb{Z}/p[v_n, v_n^{-1}], \; |v_n| = 2(p^n - 1).$$

Now given any spectrum F, there is a map

$$\eta\colon X \to L_F(X),$$

that satisfies certain properties. We say that a spectrum Z is F_*-local, if $F_*(W) = 0$ implies that $[W, Z]_* = 0$. The Bousfield localization $L_F(X)$ is F_*-local and it has universal properties with respect to maps $f\colon X \to W$, where W is F_*-local or $F_*(f)$ is an isomorphism.

There are two localizations we are especially interested in. The first one is

$$\eta\colon X \to L_{K(n)}(X),$$

and the second is

$$\eta\colon X \to L_n(X) \equiv L_{K(0)\vee K(1)\vee\cdots\vee K(n-1)\vee K(n)}(X).$$

Then the chromatic tower is the diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & L_2(X) \arrow[d] \\
 & L_1(X) \arrow[d] \\
X \arrow[r] \arrow[ur] \arrow[uur] & L_0(X).
\end{tikzcd}$$

Much of modern stable homotopy theory is concerned with trying to understand this tower.

A key result in understanding the chromatic tower is the following.

Theorem 2.1 (Hopkins/Ravenel)**.** *If X is a p-local finite complex, then*

$$\pi_*(X) \cong \varprojlim{}_n \, \pi_*(L_nX).$$

The inverse limit here is a generalization of the intersection $\bigcap A_i$ of the system $A_0 \supset A_1 \supset A_2 \supset \cdots$.

Once again, $\pi_*(L_nX)$ can be difficult to compute and so we must develop techniques to understand it. One way to understand $\pi_*(L_nX)$ is to look at the fiber M_nX of the map $L_nX \to L_{n-1}X$. Another way is to look at $L_{K(n)}X$ because it is, in some sense, the difference between

$$L_{n-1}X = L_{K(0)\vee\cdots\vee K(n-1)}X \text{ and } L_nX = L_{K(0)\vee\cdots\vee K(n)}X.$$

It turns out that these approaches are in some sense equivalent, because M_nX and $L_{K(n)}X$ determine each other in the following way:

$$L_{K(n)}M_nX = L_{K(n)}X, \text{ and } M_nL_{K(n)}X = M_nX.$$

Also, there is a cofiber sequence

$$F(L_{n-1}S^0, L_nX) \to L_nX \to L_{K(n)}X$$

that relates these two localizations. In general, the experts believe that a complete understanding of $\pi_*(L_{K(n)}X)$ gives complete knowledge about $\pi_*(L_nX)$, and hence, of the stable homotopy groups $\pi_*(X)$, for any finite complex X. But as yet, there is no precise formulation of how this in fact works.

3. Understanding $L_{K(n)}X$

Thus, it is very important to understand $L_{K(n)}X$. To show one way of doing this, we need some more background.

Let S_n be the nth Morava stabilizer group; that is, S_n is the automorphism group of the unique $p-$typical height n formal group law Γ_n with $[p]_{\Gamma_n}(x) = x^{p^n}$, over $\mathbb{F}_{p^n}$. Then we define

$$G_n = S_n \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p).$$

Now G_n is a profinite group and there exists a collection of open normal subgroups

$$G_n = U_0 \gneqq U_1 \gneqq \ldots \gneqq U_i \gneqq \ldots, \text{ with } \bigcap_i U_i = \{e\},$$

such that $G_n \to \varprojlim_i G_n/U_i$ is a homeomorphism, where G_n/U_i is a finite discrete set and the inverse limit, as a subset of the product, has the subspace topology. Thus, G_n has the profinite topology. When $n = 1$,

$$G_n = \mathbb{Z}_p^{\times}.$$

Let E_n be the Lubin-Tate spectrum, with

$$E_{n*} = \mathbb{Z}_p[\zeta][[u_1, \ldots, u_{n-1}]][u^{\pm 1}],$$

where ζ is a primitive $(p^n - 1)$st root of unity, $|u_i| = 0$, and $|u| = -2$.

When $n = 1$, E_n is well-understood. For each $i \geq 1$, consider the map

$$S^k \xrightarrow{p^i} S^k \to S^k \cup_{p^i} D^{k+1}.$$

The collection of spaces $\{S^k \cup_{p^i} D^{k+1}\}_{k \geq 0}$ forms the Moore spectrum $M(p^i)$. Then one has

$$E_1 \simeq \mathrm{holim}_i\, KU \wedge M(p^i),$$

where "holim" is similar to the inverse limit, but better suited for homotopy theory.

In general, there is a collection of ideals I in $\pi_0(E_n)$ of the form $(p^{i_0}, \ldots, u_{n-1}^{i_{n-1}})$, and generalized Moore spectra M_I, so that for any n,

$$E_n \simeq \mathrm{holim}_I\, (E_n \wedge M_I),$$

and furthermore,

$$\pi_0(E_n) \cong \varprojlim_I \pi_0(E_n)/I.$$

Notice that this makes $\pi_0(E_n)$ into a profinite topological space, so that $\pi_*(E_n) = E_{n*}$ is a graded topological space.

From the theory of formal groups laws, there is a continuous action of G_n on E_{n*} by ring automorphisms. The fact that E_{n*} is a continuous G_n-module plays an important role in the main tool for understanding

$L_{K(n)}X$. By work of Morava, Hopkins and Ravenel, there is a spectral sequence

$$E_2^{s,t} = H_c^s(G_n; \pi_t(E_n \wedge X)) \Rightarrow \pi_{t-s}(L_{K(n)}X),$$

where $E_2^{s,t}$ is the continuous cohomology of G_n with coefficients in the topological G_n-module $\pi_t(E_n \wedge X)$. There is a strong relationship between $E_2^{s,t}$ and a filtration on $\pi_*(L_{K(n)}X)$, so that one obtains information about $\pi_*(L_{K(n)}X)$. For example, when $n < p - 1$, this spectral sequence behaves in a particularly nice way.