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OBTAINING INTERMEDIATE RINGS OF A LOCAL PROFINITE GALOIS EXTENSION WITHOUT LOCALIZATION

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Abstract

Let E_n be the Lubin-Tate spectrum and let G_n be the nth extended Morava stabilizer group. Then there is a discrete G_n-spectrum F_n, with $L_{K(n)}(F_n) \simeq E_n$, that has the property that $(F_n)^{hU} \simeq E_n^{hU}$, for every open subgroup U of G_n. In particular, $(F_n)^{hG_n} \simeq L_{K(n)}(S^0)$. More generally, for any closed subgroup H of G_n, there is a discrete H-spectrum $Z_{n,H}$, such that $(Z_{n,H})^{hH} \simeq E_n^{hH}$. These conclusions are obtained from results about consistent k-local profinite G-Galois extensions E of finite vcd, where $L_k(-)$ is $L_M(L_T(-))$, with M a finite spectrum and T smashing. For example, we show that $L_k(E^{hH}) \simeq E^{hH}$, for every open subgroup H of G.

1. Introduction

Let $n \geqslant 1$ and let p be a prime. Let $K(n)$ be the nth Morava K-theory spectrum and let $G_n = S_n \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ be the nth extended Morava stabilizer group. Also, let E_n be the nth Lubin-Tate spectrum, with $\pi_*(E_n) = W(\mathbb{F}_{p^n})[\![u_1, ..., u_{n-1}]\!][u^{\pm 1}]$, where $W(\mathbb{F}_{p^n})$ is the ring of Witt vectors with coefficients in the field $\mathbb{F}_{p^n}$, the degree of u is -2, and the complete power series ring is in degree zero.

By [**4**], the profinite group G_n acts continuously on E_n, so that for every closed subgroup H of G_n, E_n is a continuous H-spectrum, and the homotopy fixed point spectrum E_n^{hH} can be formed. Also, by [**5**] (see [**1**, Theorem 8.2.1] for a more efficient proof), there is an equivalence $E_n^{hH} \simeq E_n^{dhH}$, where E_n^{dhH} is the $K(n)$-local commutative S^0-algebra of [**6**] and, for open normal subgroups H, a key ingredient in building E_n as a continuous G_n-spectrum.

In more detail, as in [**1**, §5.2], let $\mathcal{Alg}$ be the model category of discrete commutative G_n-$L_{K(n)}(S^0)$-algebras: objects of $\mathcal{Alg}$ are discrete G_n-spectra that are also commutative $L_{K(n)}(S^0)$-algebras, and morphisms are G_n-equivariant maps of commutative $L_{K(n)}(S^0)$-algebras. Let $(-)_{\mathcal{F}} : \mathcal{Alg} \to \mathcal{Alg}$ be a fibrant replacement functor for the model category $\mathcal{Alg}$, and let $U <_o G_n$ denote an open subgroup of G_n. Since

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each E_n^{dhN} is a G_n/N-spectrum that is $K(n)$-local,

$$F_n := \operatorname*{colim}_{N \lhd_o G_n} (E_n^{dhN})_{\mathcal{F}}$$

is a discrete G_n-spectrum that is $E(n)$-local and, by [**4**, Lemma 6.7], *not* $K(n)$-local. Here, $E(n)$ is the Johnson-Wilson spectrum, with $\pi_*(E(n)) = \mathbb{Z}_{(p)}[v_1, ..., v_{n-1}][v_n^{\pm 1}]$.

Given a tower $M_0 \leftarrow M_1 \leftarrow \cdots \leftarrow M_i \leftarrow \cdots$ of generalized Moore spectra, such that for any $E(n)$-local spectrum Z, $L_{K(n)}(Z) \simeq \operatorname{holim}_i(Z \wedge M_i)$ (as in [**9**, §2]),

$$E_n \simeq L_{K(n)}(F_n) \simeq \operatorname*{holim}_i (F_n \wedge M_i),$$

where each $F_n \wedge M_i$ is a discrete G_n-spectrum. Hence, as in [**4**], for any closed subgroup H of G_n,

$$E_n^{hH} = (\operatorname*{holim}_i (F_n \wedge M_i))^{hH} = \operatorname*{holim}_i (F_n \wedge M_i)^{hH} \simeq L_{K(n)}((F_n)^{hH}),$$

where the last step uses that each M_i is a finite spectrum. In particular, when $H = G_n$, we have

$$L_{K(n)}(S^0) \simeq E_n^{hG_n} \simeq L_{K(n)}((F_n)^{hG_n}),$$

where the first equivalence is given by [**6**, Theorem 1, (iii)] and [**1**, Corollary 8.1.3]. When $p = 2$, G_2 has a finite subgroup $\mathcal{G}_{48}$ of order 48, and the Hopkins-Miller spectrum EO_2 is given by

$$EO_2 = E_2^{h\mathcal{G}_{48}} \simeq L_{K(2)}((F_2)^{h\mathcal{G}_{48}})$$

(for more on EO_2, see [**8**, Theorem 5.1]). Also, for each n and p, there is a finite subgroup $\mathbb{F}_{p^n}^\times$ of S_n such that, by setting $K_n = \mathbb{F}_{p^n}^\times \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$,

$$\widehat{E(n)} := L_{K(n)}(E(n)) \simeq E_n^{hK_n} \simeq L_{K(n)}((F_n)^{hK_n})$$

(see the proof of [**11**, Proposition 5.4.9, (a)]).

The equivalence $E_n^{hH} \simeq L_{K(n)}((F_n)^{hH})$ and the above three examples have the following pattern in common: the $K(n)$-local spectrum of interest is obtained by first taking the homotopy fixed points of the non-$K(n)$-local discrete G_n-spectrum F_n, and then $K(n)$-localizing. As seen above in the case of E_n^{hH}, this can be equivalently expressed by saying that the $K(n)$-local spectrum is obtained by taking the homotopy fixed points of the homotopy limit of the tower $\{F_n \wedge M_i\}_i$ of discrete G_n-spectra. The strength of this pattern is such that it might seem that the use of a tower of discrete G_n-spectra, instead of just a single discrete G_n-spectrum, is a necessary component of it. Similarly, one might conclude from the above discussion that the second step of $K(n)$-localizing the homotopy fixed points is an integral part of the pattern. However, in this paper, we show that such conclusions are not correct.

For any open subgroup U of G_n, it turns out that

$$E_n^{hU} \simeq (F_n)^{hU}.$$

Thus, we have

$$(F_n)^{hG_n} \simeq L_{K(n)}(S^0),$$

which is an equivalence that was first obtained by Mark Behrens, during discussions that the author had with Behrens about local homotopy fixed points (which are studied in [**1**, Section 6.1] and reviewed in Section 2). Hence, for any finite spectrum X, the equivalence $L_{K(n)}(X) \simeq L_{K(n)}(S^0) \wedge X$ implies that

$$L_{K(n)}(X) \simeq (F_n \wedge X)^{hG_n},$$

where $F_n \wedge X$ is a discrete G_n-spectrum, with G_n acting trivially on X. Therefore, the aforementioned tower/"$K(n)$-localizing at the end" is not necessary, and a single discrete G_n-spectrum (F_n or $F_n \wedge X$) suffices to yield the $K(n)$-local spectra E_n^{hU}, $L_{K(n)}(S^0)$, and $L_{K(n)}(X)$, for finite X.

Also, for any closed subgroup H of G_n, we show that there is a discrete H-spectrum $Z_{n,H}$ such that

$$E_n^{hH} \simeq (Z_{n,H})^{hH},$$

so that, for example,

$$EO_2 \simeq (Z_{2,\mathcal{G}_{48}})^{h\mathcal{G}_{48}} \quad \text{and} \quad \widehat{E(n)} \simeq (Z_{n,K_n})^{hK_n}.$$

Thus, once again, to obtain the $K(n)$-local spectrum of interest, it suffices to just take the homotopy fixed points of a single discrete H-spectrum.

We obtain the above results by considering the more general context of k-local profinite Galois extensions. Here, following [**1**, Assumption 1.0.3], k denotes a spectrum whose localization functor $L_k(-)$ is a composite $L_M(L_T(-))$ of localizations, where M is a finite spectrum and T is smashing. (Also, as explained in Section 2, we make the technical assumption that k is an S-cofibrant symmetric spectrum.) Examples of such spectra k include S^0, $H\mathbb{F}_p$, $E(n)$, and $K(n)$ (see [**2, 9**]).

Now let A be a k-local cofibrant commutative symmetric ring spectrum, E a commutative A-algebra, and G a profinite group. Then E is a *k-local profinite G-Galois extension* of A if

(a) there is a directed system of finite k-local G/U_α-Galois extensions E_α of A, where $\{U_\alpha\}_\alpha$ is a cofinal system of open normal subgroups of G (see [**11**, Section 4.1] for the definition of a finite k-local Galois extension);

(b) the commutative A-algebra E is given by

$$E = \operatorname*{colim}_{\alpha} (E_\alpha)_{fGA},$$

where

$$(-)_{fGA} \colon \mathrm{Alg}_{A,G} \to \mathrm{Alg}_{A,G}$$

is a fibrant replacement functor for the model category $\mathrm{Alg}_{A,G}$ of discrete commutative G-A-algebras (see [**1**, Section 5.2]);

(c) each of the maps $E_\alpha \to E_\beta$ is G-equivariant and is a cofibration of underlying commutative A-algebras; and

(d) for each $\alpha \leqslant \beta$, the natural map $E_\alpha \to (E_\beta)^{h(U_\alpha/U_\beta)}$ is a weak equivalence.

Also, the Galois extension E has *finite vcd* (finite virtual cohomological dimension) if G has finite vcd (that is, there exists an open subgroup U of G and a positive integer m such that the continuous cohomology $H^s_c(U; M) = 0$, for all $s > m$, whenever M is a discrete U-module).

Given a k-local profinite G-Galois extension E of A, the k-local Amitsur derived completion $A^{\wedge}_{k,E}$ is the homotopy limit of the cosimplicial spectrum

$$L_k(E) \rightrightarrows L_k(E \wedge_A E) \rightrightarrows L_k(E \wedge_A E \wedge_A E) \rightrightarrows \cdots$$

that is built from the unit map $A \to E$ and the multiplication $E \wedge_A E \to E$ (see [**11**, Definition 8.2.1] and [**3**]). Then, following [**1**, Definition 1.0.4, (1)], the extension E is *consistent* if the coaugmentation $A \to A^{\wedge}_{k,E}$ is a weak equivalence.

Suppose that E is a consistent k-local profinite G-Galois extension of A of finite vcd. Then in Theorem 5.1, for each open subgroup U of G, we show that

$$L_k(E^{hU}) \simeq E^{hU},$$

and, by Corollary 5.3,

$$A \simeq E^{hG}.$$

More generally, if H is any closed subgroup of G, then, by (5.4), there is a discrete H-spectrum $Z_{E,H}$ such that

$$L_k(E^{hH}) \simeq (Z_{E,H})^{hH}.$$

If the Galois extension E is profaithful (in the sense of [**1**]; see Definition 5.5), then the dependence of $Z_{E,H}$ on H can be lessened somewhat (see Theorem 5.6).

Since F_n is a consistent profaithful $K(n)$-local profinite G_n-Galois extension of $L_{K(n)}(S^0)$ of finite vcd (by [**1**, Proposition 8.1.2] and [**11**, Theorem 5.4.4, Proposition 5.4.9, (b)]), the results stated earlier for E_n follow immediately from the results described in the previous paragraph.

We briefly summarize the remaining portions of this paper. Let G be a profinite group and let H be a closed subgroup of G. In Section 2, we recall various definitions and results about (local) homotopy fixed points that are used later. In Section 3, we study several discrete H-spectra that are canonically associated to a discrete G-spectrum X and are useful for the results of Section 4, where we show that, when X is T-local and G has finite vcd, $L_k(X^{hH})$ can be obtained by just taking the homotopy fixed points of a single discrete H-spectrum.

We conclude the Introduction with some comments about the terminology that we use. All spectra are symmetric spectra of simplicial sets and we use $\Sigma\mathrm{Sp}$ to denote the model category of spectra (more precisely, the stable model category of symmetric spectra, as defined in [**10**, Section 3.4]).

Given a profinite group G, a discrete G-spectrum (following [**1**, Section 2.3]; see also [**4**, Section 3]) is a spectrum such that, for each $j \geqslant 0$, the simplicial set X_j is a pointed simplicial discrete $(G \times \Sigma_j)$-set, where Σ_j denotes the jth symmetric group, together with compatible $(G \times \Sigma_l \times \Sigma_j)$-equivariant maps

$$\sigma^l \colon S^l \wedge X_j \to X_{j+l}.$$

Here, $S^l = (S^1)^{\wedge l}$ has the trivial G-action and the factors in this smash product are permuted by the action of Σ_l.

We use $\Sigma\mathrm{Sp}_G$ to denote the category of discrete G-spectra, in which a morphism $f\colon X \to Y$ of discrete G-spectra is a collection of $(G \times \Sigma_j)$-equivariant maps $f_j\colon X_j \to Y_j$ of pointed simplicial sets which are compatible with the structure maps σ^1.

All filtered colimits in this paper are taken in the category $\Sigma\mathrm{Sp}$. This statement holds even for part (b) in the definition of k-local profinite G-Galois extension that we gave earlier, since filtered colimits in the category of commutative A-algebras are formed in $\Sigma\mathrm{Sp}$ (a reference for this fact is [**1**, Lemma 5.3.4]).

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2. Preliminaries on (local) homotopy fixed points

In this section, we recall various definitions and results about (local) homotopy fixed points that will be useful in the rest of the paper.

Let G be a profinite group. By [**1**, Theorem 2.3.2], there is a model category structure on $\Sigma\mathrm{Sp}_G$, where a morphism f is a weak equivalence (cofibration) if and only if f is a weak equivalence (cofibration) in $\Sigma\mathrm{Sp}$. Let

$$(-)_{fG}\colon \Sigma\mathrm{Sp}_G \to \Sigma\mathrm{Sp}_G$$

be a fibrant replacement functor, so that if X is a discrete G-spectrum, there is a natural map $X \to X_{fG}$ that is a trivial cofibration, with X_{fG} fibrant, in $\Sigma\mathrm{Sp}_G$. Since the fixed points functor

$$(-)^G\colon \Sigma\mathrm{Sp}_G \to \Sigma\mathrm{Sp}$$

is a right Quillen functor (by [**1**, Lemma 3.1.1]), the homotopy fixed points $(-)^{hG}$ are defined to be the right derived functor of $(-)^G$, so that, given $X \in \Sigma\mathrm{Sp}_G$,

$$X^{hG} = (X_{fG})^G$$

(see [**1**, Section 3.1]).

Now let F be any spectrum. Recall from [**10**, Section 5.3] that, by using cofibrant replacement in the S model structure on the category of symmetric spectra, there is a weak equivalence $F_c \to F$ in $\Sigma\mathrm{Sp}$ (with the usual model structure, as described near the end of the Introduction) such that the spectrum F_c is S-cofibrant. For this reason, we always assume that the original spectrum F itself is S-cofibrant. This assumption is used, for example, in the proof of Lemma 2.1 and the discussion after this proof.

The category of spectra, when equipped with the F-local model category structure, gives the model category $(\Sigma\mathrm{Sp})_F$ (considered, for example, in [**1**, Section 6.1]). In $(\Sigma\mathrm{Sp})_F$, a morphism f of spectra is a weak equivalence (cofibration) if and only if f is an F-local equivalence (cofibration) of spectra. Here, f is an F-local equivalence exactly when $F \wedge f$ is a weak equivalence in $\Sigma\mathrm{Sp}$. We define the Bousfield localization functor

$$L_F(-)\colon (\Sigma\mathrm{Sp})_F \to (\Sigma\mathrm{Sp})_F$$

by taking it to be a fibrant replacement functor for $(\Sigma\mathrm{Sp})_F$, so that, given a spectrum X, the natural map $X \to L_F(X)$ is a trivial cofibration, with $L_F(X)$ fibrant, in $(\Sigma\mathrm{Sp})_F$.

In [**1**, Section 6.1], it was shown that the category $\Sigma\mathrm{Sp}_G$ can be equipped with an F-local model category structure, to give the model category $(\Sigma\mathrm{Sp}_G)_F$, in which a morphism f of discrete G-spectra is a weak equivalence (cofibration) if and only if f is a weak equivalence (cofibration) in $(\Sigma\mathrm{Sp})_F$. (The idea for this F-local model structure has an antecedent in [**7**, Section 7], where such a model structure is put on the category of simplicial discrete G-sets.) Additionally, the fixed points functor

$$(-)^G\colon (\Sigma\mathrm{Sp}_G)_F \to (\Sigma\mathrm{Sp})_F$$

is a right Quillen functor (see [**1**, Section 6.1] for more detail). This implies, for example, that if X is a discrete G-spectrum that is fibrant in $(\Sigma\mathrm{Sp}_G)_F$, then X^G is fibrant in $(\Sigma\mathrm{Sp})_F$, and hence, X^G is an F-local spectrum.

Let

$$(-)_{f_F G}\colon (\Sigma\mathrm{Sp}_G)_F \to (\Sigma\mathrm{Sp}_G)_F$$

be a fibrant replacement functor, so that, given a discrete G-spectrum X, there is a natural map $X \to X_{f_F G}$ that is a trivial cofibration, with $X_{f_F G}$ fibrant, in $(\Sigma\mathrm{Sp}_G)_F$. Then, as in [**1**, Section 6.1], the F-local homotopy fixed points $(-)^{h_F G}$ are defined to be the right derived functor of $(-)^G$ with respect to the F-local model structure, so that, given a discrete G-spectrum X, the F-local homotopy fixed point spectrum $X^{h_F G}$ is given by

$$X^{h_F G} = (X_{f_F G})^G.$$

Notice that $X^{h_F G}$ is an F-local spectrum.

Though the following result was (quickly) noted in the proof of [**1**, Proposition 6.1.7, (3)], because of its usefulness, we present it as a lemma and give a proof.

Lemma 2.1. *Let G be a profinite group and let X be a discrete G-spectrum. If X is fibrant in $(\Sigma\mathrm{Sp}_G)_F$, then X is also fibrant in $\Sigma\mathrm{Sp}_G$.*

Proof. Let f be a trivial cofibration in $\Sigma\mathrm{Sp}_G$, so that f is both a weak equivalence and a cofibration of spectra. Since F is S-cofibrant, [**10**, Corollary 5.3.10] implies that the functor $F \wedge (-)$ preserves weak equivalences in $\Sigma\mathrm{Sp}$, so that f is an F-local equivalence. Thus, f is a trivial cofibration in $(\Sigma\mathrm{Sp}_G)_F$. This conclusion implies that in $\Sigma\mathrm{Sp}_G$, the map $X \to *$ to the terminal object has the right lifting property with respect to all trivial cofibrations. □

Let X be a discrete G-spectrum. By Lemma 2.1, $X_{f_F G}$ is fibrant in $\Sigma\mathrm{Sp}_G$, and hence, there exists a morphism

$$f_X^G\colon X_{fG} \to X_{f_F G}$$

in $\Sigma\mathrm{Sp}_G$. Notice that since F is S-cofibrant, the map $X \to X_{fG}$ is an F-local equivalence, so that the map f_X^G also is an F-local equivalence. Taking the G-fixed points of f_X^G yields the map

$$X^{hG} = (X_{fG})^G \to (X_{f_F G})^G = X^{h_F G}.$$

Therefore, since $X^{h_F G}$ is fibrant in $(\Sigma\mathrm{Sp})_F$, the trivial cofibration $X^{hG} \to L_F(X^{hG})$ in $(\Sigma\mathrm{Sp})_F$ induces a map

$$L_F(X^{hG}) \to X^{h_F G}. \tag{2.2}$$

When $F = k$ (where k is defined as in the Introduction and is assumed to be S-cofibrant), the following result gives conditions under which the map in (2.2) is a weak equivalence.

Theorem 2.3 ([**1**, Proposition 6.1.7, (3)]). *Let G have finite vcd and suppose that X is a discrete G-spectrum. If X is T-local, then the map $L_k(X^{hG}) \to X^{h_k G}$ is a weak equivalence in $\Sigma\mathrm{Sp}$.*

Given an open subgroup U of G, let

$$\mathrm{Res}_G^U \colon \Sigma\mathrm{Sp}_G \to \Sigma\mathrm{Sp}_U, \quad \mathrm{Res}_G^U(X) = X,$$

be the functor that takes a discrete G-spectrum X and regards it as a discrete U-spectrum. The proof of [**1**, Proposition 3.3.1, (2)] shows that this functor preserves fibrant objects. Similarly, there is a functor

$$(\mathrm{Res}_G^U)_F \colon (\Sigma\mathrm{Sp}_G)_F \to (\Sigma\mathrm{Sp}_U)_F$$

that regards a discrete G-spectrum as a discrete U-spectrum, and the proof of [**1**, Proposition 6.1.7, (1)] shows that this functor preserves fibrant objects. We summarize these remarks in the following result.

Lemma 2.4 ([**1**]). *Let G be a profinite group, U an open subgroup of G, and X a discrete G-spectrum. If X is fibrant in $\Sigma\mathrm{Sp}_G$, then it is fibrant in $\Sigma\mathrm{Sp}_U$. Similarly, if X is fibrant in $(\Sigma\mathrm{Sp}_G)_F$, then it is fibrant in $(\Sigma\mathrm{Sp}_U)_F$.*

We conclude this section with several useful facts about spectra. Suppose that $\{g_\alpha \colon Y_\alpha \to Z_\alpha\}_\alpha$ is a filtered system of weak equivalences of spectra, such that each Y_α and Z_α is fibrant. By [**10**, Corollaries 3.4.13, 3.4.16], the spectra Y_α and Z_α consist of Kan complexes, and hence, the map $\pi_*(\mathrm{colim}_\alpha\, g_\alpha)$ is an isomorphism, so that $\mathrm{colim}_\alpha\, g_\alpha$ is a weak equivalence (by [**10**, Theorem 3.1.11]).

If f is a trivial cofibration in $\Sigma\mathrm{Sp}$, then f is also a trivial cofibration in $(\Sigma\mathrm{Sp})_F$. This fact implies that, given a spectrum X, the localization $L_F(X)$ is fibrant in $\Sigma\mathrm{Sp}$.

3. Useful discrete H-spectra associated to $X \in \Sigma\mathrm{Sp}_G$

In this section, we restrict ourselves to $F = k$, since this is all that is needed in later sections. As before, G is a profinite group, and we let X be a discrete G-spectrum. Given a closed subgroup H of G, we define and study two discrete H-spectra that are useful for our later results.

Let N and N' be proper open normal subgroups of G, with N a subgroup of N'. Since the map $X \to X_{fN'}$ is a trivial cofibration in $\Sigma\mathrm{Sp}_N$, the right lifting property in $\Sigma\mathrm{Sp}_N$ of the map $X_{fN} \to *$ yields an N-equivariant map

$$\lambda_{N,N'} \colon X_{fN'} \to X_{fN}.$$

The map $\lambda_{N,N'}$ induces the map

$$\lambda^N_{N'} \colon X^{hN'} \to X^{hN},$$

which is defined to be the composition

$$X^{hN'} = (X_{fN'})^{N'} \hookrightarrow (X_{fN'})^N \to (X_{fN})^N = X^{hN}.$$

Notice that the spectra X^N and $(X_{fG})^N$ have natural G/N-actions, and hence, natural G-actions, through the projection $G \to G/N$. However, in general, the spectrum $X^{hN} = (X_{fN})^N$ is not known to have a natural G/N-action, apart from the trivial one, since X_{fN} is not known, in general, to have a G-action. This fact implies that, unlike in the case of the diagram $\{(X_{fG})^N\}_{N \lhd_o G}$, it is not possible, in general, to assemble together the maps $\lambda^N_{N'}$ to form, in a non-degenerate way, a diagram $\{X^{hN}\}_{N \lhd_o G}$ of G-spectra and G-equivariant maps. Thus, in general, it is not possible to form, for example, $\mathrm{colim}_{N \lhd_o G} L_k(X^{hN})$ as a discrete G-spectrum in the desired way. Therefore, to get around this problem, we make the following considerations.

Definition 3.1. Let X be a discrete G-spectrum. If H is a closed subgroup of G and V is an open normal subgroup of H, then we define

$$X(H, V) = L_k((X_{fH})^V)$$

and

$$X(k, H, V) = L_k((X_{f_k H})^V).$$

Since $(X_{fH})^V$ has an H/V-action and $L_k(-)$ is a functor, $X(H,V)$ is an H/V-spectrum. Since the finite group H/V is naturally discrete, the canonical projection $H \to H/V$ makes $X(H,V)$ a discrete H-spectrum. Thus,

$$\operatorname*{colim}_{V \lhd_o H} X(H, V) = \operatorname*{colim}_{V \lhd_o H} L_k((X_{fH})^V)$$

is also a discrete H-spectrum. The same argument shows that

$$\operatorname*{colim}_{V \lhd_o H} X(k, H, V) = \operatorname*{colim}_{V \lhd_o H} L_k((X_{f_k H})^V)$$

is a discrete H-spectrum. Since the output of the functor $L_k(-)$ is always a fibrant spectrum, $\mathrm{colim}_{V \lhd_o H} X(H,V)$ and $\mathrm{colim}_{V \lhd_o H} X(k,H,V)$ are fibrant spectra.

By Lemma 2.4, the spectrum $X_{f_k H}$ in Definition 3.1 is fibrant in $(\Sigma\mathrm{Sp}_V)_k$, so that $(X_{f_k H})^V$ is already k-local. However, in Definition 3.1, we are interested instead in the fibrant replacement $L_k((X_{f_k H})^V)$, because there is an H/V-equivariant map

$$L_k((f^H_X)^V) \colon X(H, V) \to X(k, H, V)$$

that will often be used later.

Let N be an open normal subgroup of G. The map $\lambda_{N,G}\colon X_{fG} \to X_{fN}$, a weak equivalence between fibrant objects in $\Sigma\mathrm{Sp}_N$, induces the weak equivalence $(X_{fG})^N \to (X_{fN})^N = X^{hN}$, and hence, there is a weak equivalence

$$X(G,N) = L_k((X_{fG})^N) \to L_k(X^{hN}).$$

Thus, we can think of the discrete G-spectrum $\mathrm{colim}_{N \lhd_o G} X(G,N)$ as a replacement for the typically undefinable object $\mathrm{colim}_{N \lhd_o G} L_k(X^{hN})$ that was discussed just before Definition 3.1.

Notice that the isomorphism $X \cong \operatorname*{colim}_{V \lhd_o H} X^V$ yields a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \operatorname*{colim}_{V \lhd_o H} X(k,H,V) \\ \operatorname*{colim}_{V \lhd_o H} X(H,V) \arrow[ur, bend right, "\operatorname*{colim}_{V \lhd_o H} L_k((f_X^H)^V)"'] & \end{tikzcd} \tag{3.2}$$

in $\Sigma\mathrm{Sp}_H$. The result below considers a case where the diagonal map in (3.2) is a weak equivalence.

Lemma 3.3. *Let G have finite vcd. If X is a discrete G-spectrum that is T-local, then the map*

$$\operatorname*{colim}_{N \lhd_o G} L_k((f_X^G)^N)\colon \operatorname*{colim}_{N \lhd_o G} X(G,N) \xrightarrow{\simeq} \operatorname*{colim}_{N \lhd_o G} X(k,G,N)$$

is a weak equivalence in $\Sigma\mathrm{Sp}_G$.

Proof. Let N be an open normal subgroup of G. By Lemma 2.1, $(X_{fG})_{f_k N}$ is fibrant in $\Sigma\mathrm{Sp}_N$, giving the commutative diagram

$$\begin{tikzcd} X_{fG} \arrow[r] \arrow[d] & (X_{fG})_{f_k N} \\ (X_{fG})_{fN} \arrow[ur] & \end{tikzcd}$$

in $\Sigma\mathrm{Sp}_N$. The N-fixed points of this diagram give the commutative diagram

$$\begin{tikzcd} (X_{fG})^N \arrow[r] \arrow[d] & ((X_{fG})_{f_k N})^N \ . \\ ((X_{fG})_{fN})^N \arrow[ur] & \end{tikzcd} \tag{3.4}$$

Since $X_{fG} \to (X_{fG})_{fN}$ is a weak equivalence between fibrant objects in $\Sigma\mathrm{Sp}_N$, the vertical map $(X_{fG})^N \to ((X_{fG})_{fN})^N$ in (3.4) is a weak equivalence, and hence, a k-local equivalence. Since N has finite vcd (because G has finite vcd) and X_{fG} is T-local (because X is T-local), Theorem 2.3 implies that the diagonal map in (3.4) is a k-local equivalence. Thus, the horizontal map in (3.4) is also a k-local equivalence.

This last k-local equivalence is the top edge in the commutative diagram

$$\begin{tikzcd} (X_{fG})^N \arrow[r] \arrow[d, "(f_X^G)^N"'] & ((X_{fG})_{f_kN})^N \arrow[d] \\ (X_{f_kG})^N \arrow[r] & ((X_{f_kG})_{f_kN})^N. \end{tikzcd} \tag{3.5}$$

Since X_{f_kG} is fibrant in $(\Sigma\mathrm{Sp}_N)_k$, the map $X_{f_kG} \to (X_{f_kG})_{f_kN}$ is a weak equivalence between fibrant objects in $(\Sigma\mathrm{Sp}_N)_k$, so that the bottom edge in (3.5) is a k-local equivalence. Also, since f_X^G is a k-local equivalence, the map

$$(f_X^G)_{f_kN} \colon (X_{fG})_{f_kN} \to (X_{f_kG})_{f_kN}$$

is a weak equivalence between fibrant objects in $(\Sigma\mathrm{Sp}_N)_k$, and hence, the right edge in (3.5) is a k-local equivalence. Therefore, the left edge in (3.5), $(f_X^G)^N$, is a k-local equivalence.

Applying $L_k(-)$ to the k-local equivalence $(f_X^G)^N$ yields the k-local equivalence

$$L_k((f_X^G)^N) \colon X(G,N) = L_k((X_{fG})^N) \to L_k((X_{f_kG})^N) = X(k,G,N).$$

Since $L_k((f_X^G)^N)$ is a k-local equivalence between k-local spectra, it is a weak equivalence, and since it is a weak equivalence between fibrant spectra, we can conclude that the desired map $\operatorname*{colim}_{N \lhd_o G} L_k((f_X^G)^N)$ is a weak equivalence of spectra. □

The following result, which will be used in the proof of Corollary 4.3, is useful for understanding the relationship between Theorem 4.1 and its simplification (in certain cases) in Corollary 4.3.

Lemma 3.6. *Let G be a profinite group and H an open subgroup of G. If X is a discrete G-spectrum, then there is a weak equivalence*

$$\operatorname*{colim}_{V \lhd_o H} X(k,H,V) \xrightarrow{\simeq} \operatorname*{colim}_{N \lhd_o G} X(k,G,N)$$

in $\Sigma\mathrm{Sp}_H$.

Proof. Recall that if K is a profinite group and if U is an open subgroup of K, then U contains a subgroup U_K that is open and normal in K (see, for example, [**12**, Lemma 0.3.2]). This fact implies that if V is an open normal subgroup of H, then, since V is an open subgroup of G, V contains a subgroup V_G that is open and normal in G. Thus, $\{N \mid N \lhd_o G,\ N < H\}$ is a cofinal subcollection of $\{V \mid V \lhd_o H\}$, giving the isomorphism

$$c_1 \colon \operatorname*{colim}_{V \lhd_o H} X(k,H,V) = \operatorname*{colim}_{V \lhd_o H} L_k((X_{f_kH})^V) \xrightarrow{\cong} \operatorname*{colim}_{N \lhd_o G,\, N < H} L_k((X_{f_kH})^N).$$

By Lemma 2.4, X_{f_kG} is fibrant in $(\Sigma\mathrm{Sp}_H)_k$, so that there is a weak equivalence

$$\lambda_{k,G,H} \colon X_{f_kH} \to X_{f_kG}$$

in $(\Sigma\mathrm{Sp}_H)_k$. If N is an open normal subgroup of G that is contained in H, then $\lambda_{k,G,H}$ is a weak equivalence between fibrant objects in $(\Sigma\mathrm{Sp}_N)_k$, so that the map

$$(\lambda_{k,G,H})^N \colon (X_{f_kH})^N \to (X_{f_kG})^N$$

is a weak equivalence in $(\Sigma\mathrm{Sp})_k$. Hence, the map

$$L_k((\lambda_{k,G,H})^N)\colon X(k,H,N) = L_k((X_{f_kH})^N) \to L_k((X_{f_kG})^N) = X(k,G,N)$$

is a weak equivalence in $(\Sigma\mathrm{Sp})_k$ between k-localized objects, and thus, it is a weak equivalence between fibrant objects in $\Sigma\mathrm{Sp}$. Therefore, the filtered colimit

$$\operatorname*{colim}_{N\lhd_o G,\, N<H} L_k((\lambda_{k,G,H})^N)\colon \operatorname*{colim}_{N\lhd_o G,\, N<H} X(k,H,N) \xrightarrow{\simeq} \operatorname*{colim}_{N\lhd_o G,\, N<H} X(k,G,N)$$

is also a weak equivalence of spectra.

Now let N be any open normal subgroup of G: $N\cap H$ is an open subgroup of G, and hence, $N\cap H$ contains a subgroup $(N\cap H)_G$ that is open and normal in G. Thus, the collection $\{N \mid N \lhd_o G,\, N < H\}$ is a cofinal subcollection of $\{N \mid N \lhd_o G\}$. This implies that there is an isomorphism

$$c_2\colon \operatorname*{colim}_{N\lhd_o G,\, N<H} X(k,G,N) \xrightarrow{\cong} \operatorname*{colim}_{N\lhd_o G} X(k,G,N),$$

so that the composition

$$c_2 \circ \bigl(\operatorname*{colim}_{N\lhd_o G,\, N<H} L_k((\lambda_{k,G,H})^N)\bigr) \circ c_1\colon \operatorname*{colim}_{V\lhd_o H} X(k,H,V) \xrightarrow{\simeq} \operatorname*{colim}_{N\lhd_o G} X(k,G,N)$$

is a weak equivalence. □

4. Obtaining k-local spectra from discrete G-spectra

As in Section 3, we continue to let H be a closed subgroup of a profinite group G, and, as usual, we let X be a discrete G-spectrum. Then the following result says that under certain conditions, the two-step process of taking the H-homotopy fixed points of X and then k-localizing can be realized somewhat more directly by just taking the H-homotopy fixed points of the discrete H-spectrum $\operatorname*{colim}_{V\lhd_o H} X(H,V)$, so that no subsequent k-localization is needed.

Theorem 4.1. *Let G have finite vcd and suppose that X is a discrete G-spectrum that is T-local. If H is a closed subgroup of G, then there is a zigzag of weak equivalences*

$$L_k(X^{hH}) \xrightarrow{\simeq} (\operatorname*{colim}_{V\lhd_o H} X(k,H,V))^{hH} \xleftarrow{\simeq} (\operatorname*{colim}_{V\lhd_o H} X(H,V))^{hH}. \tag{4.2}$$

Proof. Since G has finite vcd, H does too, and hence, Theorem 2.3 implies that the map

$$f_1\colon L_k(X^{hH}) \xrightarrow{\simeq} X^{h_kH}$$

is a weak equivalence. Also, since X_{f_kH} is fibrant in $\Sigma\mathrm{Sp}_H$ (by Lemma 2.1), the map $X_{f_kH} \to (X_{f_kH})_{fH}$ is a weak equivalence between fibrant objects in $\Sigma\mathrm{Sp}_H$, so that the induced map

$$f_2\colon X^{h_kH} = (X_{f_kH})^H \xrightarrow{\simeq} ((X_{f_kH})_{fH})^H = (X_{f_kH})^{hH}$$

is a weak equivalence in $\Sigma\mathrm{Sp}$. Therefore, composing the weak equivalences f_1, f_2 gives the weak equivalence

$$\widehat{f} = f_2 \circ f_1\colon L_k(X^{hH}) \xrightarrow{\simeq} (X_{f_kH})^{hH}.$$

Let V be an open normal subgroup of H and consider the trivial cofibration

$$g_H^V\colon (X_{f_kH})^V \to L_k((X_{f_kH})^V) = X(k,H,V)$$

in $(\Sigma\mathrm{Sp})_k$ given by the fibrant replacement $L_k(-)$. Notice that the map g_H^V is also H/V-equivariant. By Lemma 2.4, X_{f_kH} is fibrant in $(\Sigma\mathrm{Sp}_V)_k$, so that $(X_{f_kH})^V$ is k-local, and hence, g_H^V is a k-local equivalence between k-local spectra, so that g_H^V is a weak equivalence of spectra. Since X_{f_kH} is fibrant in $(\Sigma\mathrm{Sp}_V)_k$, it is also fibrant in $\Sigma\mathrm{Sp}_V$, so that $(X_{f_kH})^V$ is a fibrant spectrum. This implies that g_H^V is a weak equivalence between fibrant spectra. Therefore, there is a weak equivalence

$$g\colon X_{f_kH} \xrightarrow{\cong} \operatorname*{colim}_{V \lhd_o H} (X_{f_kH})^V \xrightarrow{\simeq} \operatorname*{colim}_{V \lhd_o H} X(k,H,V)$$

in $\Sigma\mathrm{Sp}_H$.

The composition of weak equivalences

$$(g)^{hH} \circ \widehat{f}\colon L_k(X^{hH}) \xrightarrow{\simeq} (\operatorname*{colim}_{V \lhd_o H} X(k,H,V))^{hH}$$

gives the first weak equivalence in (4.2). Since H has finite vcd, the second weak equivalence in (4.2) is an immediate consequence of Lemma 3.3. □

Notice that in Theorem 4.1, the discrete H-spectrum $\operatorname{colim}_{V \lhd_o H} X(H,V)$, whose H-homotopy fixed points give $L_k(X^{hH})$, depends on H for its construction. However, the result below shows that if H is open in G, then $\operatorname{colim}_{V \lhd_o H} X(H,V)$ can be replaced with the discrete G-spectrum $\operatorname{colim}_{N \lhd_o G} X(G,N)$, which is independent of H.

Corollary 4.3. *Let G have finite vcd and suppose that X is a discrete G-spectrum that is T-local. If H is an open subgroup of G, then there is a zigzag of weak equivalences*

$$L_k(X^{hH}) \xrightarrow{\simeq} (\operatorname*{colim}_{N \lhd_o G} X(k,G,N))^{hH} \xleftarrow{\simeq} (\operatorname*{colim}_{N \lhd_o G} X(G,N))^{hH}. \tag{4.4}$$

Proof. By Lemma 3.6 and Theorem 4.1, there are weak equivalences

$$L_k(X^{hH}) \xrightarrow{\simeq} (\operatorname*{colim}_{V \lhd_o H} X(k,H,V))^{hH} \xrightarrow{\simeq} (\operatorname*{colim}_{N \lhd_o G} X(k,G,N))^{hH},$$

giving the first weak equivalence in (4.4). The second weak equivalence in (4.4) is an immediate consequence of Lemma 3.3. □

5. A few applications to profinite Galois extensions

As in the Introduction, let $E = \operatorname{colim}_\alpha (E_\alpha)_{fGA}$ be a consistent k-local profinite G-Galois extension of A of finite vcd. As noted in [**1**, Proposition 6.2.3], E is a discrete G-spectrum, and, by [**1**, Remark 6.2.2], E is not necessarily k-local, but it is T-local.

In the theory of k-local profinite Galois extensions, the k-localization plays a very important role. In particular, most results about these extensions are only known to be true when $L_k(-)$ is applied to the objects involved. This is evident from the near ubiquity of "$(-)_k$" in [**1**, Sections 6.2, 6.3, 7]. However, the next result, which shows that E^{hU} is k-local for every open subgroup U of G, gives an interesting example of when the k-localization is not necessary.

Theorem 5.1. *Let $E = \mathrm{colim}_\alpha (E_\alpha)_{fGA}$ be a consistent k-local profinite G-Galois extension of A of finite vcd. If U is an open subgroup of G, then*

$$L_k(E^{hU}) \simeq E^{hU}.$$

Proof. By Corollary 4.3, we have

$$L_k(E^{hU}) \simeq \big(\operatorname*{colim}_{N \lhd_o G} L_k((E_{fG})^N)\big)^{hU} \cong \big(\operatorname*{colim}_{\alpha} L_k((E_{fG})^{U_\alpha})\big)^{hU},$$

where the isomorphism is due to the fact that $\{U_\alpha\}_\alpha$ is a cofinal collection of open normal subgroups of G.

For each α, the canonical projection $G \to G/U_\alpha$ makes the commutative G/U_α-A-algebra E_α a discrete commutative G-A-algebra, with U_α acting trivially. Thus, by functoriality, U_α also acts trivially on $(E_\alpha)_{fGA}$, so that $(E_\alpha)_{fGA}$ is also a G/U_α-spectrum. Since the canonical map $(E_\alpha)_{fGA} \to E$ is G-equivariant, the trivial U_α-action on the source induces a G/U_α-equivariant map $(E_\alpha)_{fGA} \to E^{U_\alpha}$, which yields the G/U_α-equivariant map

$$\widehat{\varepsilon}_\alpha \colon (E_\alpha)_{fGA} \to E^{U_\alpha} \to (E_{fG})^{U_\alpha} \to L_k((E_{fG})^{U_\alpha}).$$

Now [**1**, Lemma 6.3.6] shows that there are equivalences

$$E_\alpha \xrightarrow[\simeq]{\varepsilon_\alpha} L_k((E_{fGA})^{U_\alpha}) \simeq L_k(E^{hU_\alpha}),$$

and it is easy to see that the map $\widehat{\varepsilon}_\alpha$ is a simple variant of the weak equivalence ε_α (see the discussion just before [**1**, Lemma 6.3.6]). Therefore, the weak equivalence $(E_{fG})^{U_\alpha} \xrightarrow{\simeq} E^{hU_\alpha}$ implies that the map $\widehat{\varepsilon}_\alpha$ is a weak equivalence.

Since $(E_\alpha)_{fGA}$ is a positive fibrant discrete G-spectrum (by [**1**, Theorem 5.2.3]), it is a positive fibrant spectrum, by [**1**, Corollary 5.3.3]. Also, because $L_k((E_{fG})^{U_\alpha})$ is fibrant in $\Sigma\mathrm{Sp}$, it is a positive fibrant spectrum. Thus, $\widehat{\varepsilon}_\alpha$ is a weak equivalence in $\Sigma\mathrm{Sp}$ between positive fibrant spectra, and hence,

$$\operatorname*{colim}_{\alpha} \widehat{\varepsilon}_\alpha \colon E = \operatorname*{colim}_{\alpha} (E_\alpha)_{fGA} \xrightarrow{\simeq} \operatorname*{colim}_{\alpha} L_k((E_{fG})^{U_\alpha})$$

is a weak equivalence in $\Sigma\mathrm{Sp}_G$, by [**1**, Lemma 5.3.1]. Then we have

$$L_k(E^{hU}) \simeq \big(\operatorname*{colim}_{\alpha} L_k((E_{fG})^{U_\alpha})\big)^{hU} \xleftarrow{\simeq} E^{hU},$$

giving the desired result. □

By [**1**, Corollary 6.3.2, Lemmas 5.2.5, 5.2.6], there is a zigzag of weak equivalences

$$A \xrightarrow{\simeq} L_k(E^{h_{\mathrm{Alg}}G}) \xleftarrow{\simeq} L_k(E^{h^+G}) \xrightarrow{\simeq} L_k(E^{hG}),$$

where $E^{h_{\mathrm{Alg}}G}$ is the "homotopy fixed point commutative A-algebra" and E^{h^+G} is the homotopy fixed point spectrum of E that is formed with respect to the positive stable model structure on $\Sigma\mathrm{Sp}_G$ (see [**1**, Section 5.2] for more detail). Thus, we obtain the equivalence

$$A \simeq L_k(E^{hG}). \tag{5.2}$$

However, the next result, which follows immediately from Theorem 5.1 (by setting $U = G$) and (5.2), shows that, in fact, no k-localization is necessary to obtain A from the homotopy fixed points of E.

Corollary 5.3. *If E is a consistent k-local profinite G-Galois extension of A of finite vcd, then*

$$A \simeq E^{hG}.$$

Now let H be a closed subgroup of G. By Theorem 5.1, $L_k(E^{hH}) \simeq E^{hH}$ whenever H is open in G. However, this equivalence need not hold for all H closed in G; this is seen, for example, when $H = \{e\}$, since $E^{h\{e\}} \simeq E$ is not necessarily k-local. However, it follows immediately from Theorem 4.1 that for every H closed in G, a slightly coarser version of Theorem 5.1 is still valid: by Theorem 4.1,

$$L_k(E^{hH}) \simeq \bigl(\operatorname*{colim}_{V \lhd_o H} L_k((E_{fH})^V)\bigr)^{hH}, \tag{5.4}$$

so that $L_k(E^{hH})$ can still always be obtained by taking the H-homotopy fixed points of a discrete H-spectrum, with no subsequent k-localization needed.

Notice that in the discrete H-spectrum $\operatorname{colim}_{V \lhd_o H} L_k((E_{fH})^V)$ of (5.4), whose H-homotopy fixed points yield $L_k(E^{hH})$, both the indexing set of the colimit and E_{fH} depend on H. Now we show that when the Galois extension E is profaithful (defined below), the second of these dependencies can be eliminated.

Definition 5.5 ([**1**, Definition 6.2.1])**.** Let $E = \operatorname{colim}_\alpha (E_\alpha)_{fGA}$ be a k-local profinite G-Galois extension of A. If the k-local G/U_α-Galois extension E_α is k-locally faithful (that is, following [**11**, Definition 4.3.1], if M is an A-module such that $L_k(M \wedge_A E_\alpha) \simeq *$, then $L_k(M) \simeq *$) for each α, then E is a *profaithful k-local* extension.

Theorem 5.6. *Let E be a consistent profaithful k-local profinite G-Galois extension of A of finite vcd. If H is a closed subgroup of G, then*

$$L_k(E^{hH}) \simeq \bigl(\operatorname*{colim}_{V \lhd_o H} L_k((E_{fG})^V)\bigr)^{hH}. \tag{5.7}$$

Remark 5.8. Let E and H be as in Theorem 5.6. By (5.4), $L_k(E^{hH})$ can be obtained by taking the homotopy fixed points of the discrete H-spectrum

$$L = \operatorname*{colim}_{N \lhd_o G} L_k((E_{fH})^{H \cap N}) \cong \operatorname*{colim}_{V \lhd_o H} L_k((E_{fH})^V),$$

where the isomorphism uses that $\{H \cap N \mid N \lhd_o G\}$ is a cofinal subcollection of $\{V \mid V \lhd_o H\}$. The proof of [**1**, Theorem 7.2.1, (1)] shows that L is a consistent profaithful k-local profinite H-Galois extension of $L_k(E^{hH})$ of finite vcd. Also, the proof below of Theorem 5.6 shows that there is a weak equivalence

$$\operatorname*{colim}_{V \lhd_o H} L_k((E_{fG})^V) \xrightarrow{\simeq} L \tag{5.9}$$

in $\Sigma\mathrm{Sp}_H$, where the source of this weak equivalence is the discrete H-spectrum that appears on the right-hand side in (5.7). The proof of [**1**, Theorem 7.2.1, (1)] shows that there is at least a zigzag of weak equivalences between the source and target of the map in (5.9), so the main value of the proof of Theorem 5.6 below is, in addition to pointing out the relevant details from [**1**], to show explicitly that there is a single weak equivalence.

Proof of Theorem 5.6. Let V be an open normal subgroup of H. The map

$$\operatorname*{colim}_{V < U <_o G} (E_{fG})^U \cong (E_{fG})^V \xrightarrow{\psi(V)} ((E_{fG})_{fH})^V$$

induces the map

$$\widehat{\psi}_V \colon L_k\big(\operatorname*{colim}_{V < U <_o G} (E_{fG})^U\big) \to L_k\big(((E_{fG})_{fH})^V\big).$$

By [**1**, Theorem 7.1.1], $\widehat{\psi}_V$ is a weak equivalence, and hence, the H/V-equivariant map

$$L_k(\psi(V)) \colon L_k((E_{fG})^V) \xrightarrow{\simeq} L_k\big(((E_{fG})_{fH})^V\big)$$

is a weak equivalence.

Since the composition $E \to E_{fG} \to (E_{fG})_{fH}$ is a trivial cofibration in $\Sigma\mathrm{Sp}_H$, there is a weak equivalence $(E_{fG})_{fH} \xrightarrow{p} E_{fH}$ in $\Sigma\mathrm{Sp}_H$. Hence, p is a weak equivalence between fibrant objects, in $\Sigma\mathrm{Sp}_V$, so that the H/V-equivariant map

$$p^V \colon ((E_{fG})_{fH})^V \xrightarrow{\simeq} (E_{fH})^V$$

is a weak equivalence. Therefore, the composition

$$L_k(p^V) \circ L_k(\psi(V)) \colon L_k((E_{fG})^V) \xrightarrow{\simeq} L_k\big(((E_{fG})_{fH})^V\big) \xrightarrow{\simeq} L_k((E_{fH})^V),$$

which is H/V-equivariant, is a weak equivalence, and since its source and target are fibrant spectra, there is a weak equivalence

$$\operatorname*{colim}_{V \lhd_o H} L_k((E_{fG})^V) \xrightarrow{\simeq} \operatorname*{colim}_{V \lhd_o H} L_k((E_{fH})^V)$$

in $\Sigma\mathrm{Sp}_H$, giving

$$\big(\operatorname*{colim}_{V \lhd_o H} L_k((E_{fG})^V)\big)^{hH} \xrightarrow{\simeq} \big(\operatorname*{colim}_{V \lhd_o H} L_k((E_{fH})^V)\big)^{hH} \simeq L_k(E^{hH}),$$

where the last equivalence is from (5.4). $\square$

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