Homotopy fixed points for profinite groups emulate homotopy fixed points for discrete groups

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ABSTRACT. If K is a discrete group and Z is a K–spectrum, then the homotopy fixed point spectrum Z^{hK} is $\mathrm{Map}_*(EK_+, Z)^K$, the fixed points of a familiar expression. Similarly, if G is a profinite group and X is a discrete G–spectrum, then X^{hG} is often given by $(\mathcal{H}_{G,X})^G$, where $\mathcal{H}_{G,X}$ is a certain explicit construction given by a homotopy limit in the category of discrete G–spectra. Thus, in each of two common equivariant settings, the homotopy fixed point spectrum is equal to the fixed points of an explicit object in the ambient equivariant category. We enrich this pattern by proving in a precise sense that the discrete G–spectrum $\mathcal{H}_{G,X}$ is just "a profinite version" of $\mathrm{Map}_*(EK_+, Z)$: at each stage of its construction, $\mathcal{H}_{G,X}$ replicates in the setting of discrete G-spectra the corresponding stage in the formation of $\mathrm{Map}_*(EK_+, Z)$ (up to a certain natural identification).

CONTENTS

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1. Introduction

1.1. Recalling a familiar scenario: homotopy fixed points for discrete groups. Let K be a discrete group and let Z be a (naive) K–spectrum, where, here and everywhere else in this paper (unless explicitly stated otherwise), “spectrum” means Bousfield-Friedlander spectrum of simplicial sets. Let EK be the usual simplicial set with n–simplices equal to the cartesian product K^{n+1}, for each $n \geq 0$; let EK_+ denote EK with a disjoint basepoint added; and let

$$(-)_f \colon \mathrm{Spt} \to \mathrm{Spt}, \quad Y \mapsto Y_f$$

be a fibrant replacement functor for the model category of spectra (with the usual stable structure). Also, given a pointed simplicial set L and any spectrum Y, let $\mathrm{Map}_*(L, Y)$ be the mapping spectrum with mth pointed simplicial set $\mathrm{Map}_*(L, Y)_m$ having n–simplices equal to

$$\mathrm{Map}_{\mathcal{S}_*}(L, Y_m)_n = \mathcal{S}_*(L \wedge \Delta[n]_+, Y_m),$$

where $\mathcal{S}_*$ is the category of pointed simplicial sets. Then the homotopy fixed point spectrum Z^{hK} is given explicitly by

$$Z^{hK} = \mathrm{Map}_*(EK_+, Z_f)^K.$$

One reason for the importance of the explicit construction $\mathrm{Map}_*(EK_+, Z_f)^K$ is that it makes it possible to build the descent spectral sequence

$$E_2^{s,t} = H^s(K; \pi_t(Z)) \Longrightarrow \pi_{t-s}(Z^{hK}).$$

1.2. Considering homotopy fixed points for profinite groups: a pattern emerges. Now let G be a profinite group, let Spt_G be the simplicial model category of discrete G–spectra (for details, we refer the reader to [4, Section 3] and [11, Remark 3.11]), and let $X \in \mathrm{Spt}_G$. We consider how to carry out the above constructions for K and Z in this profinite setting.

Remark 1.1. In the titles of this paper and §1.2, the phrase “homotopy fixed points for profinite groups” is meant for the setting of discrete G-spectra. We point out that there is a theory of homotopy fixed points for profinite G-spectra (see [22]) and our phrasing is not meant to be exclusionary.

As explained in [4, Definition 7.1], the functor

$$\mathrm{Map}_c(G, -) \colon \mathrm{Spt}_G \to \mathrm{Spt}_G, \quad X \mapsto \mathrm{Map}_c(G, X),$$

where each pointed simplicial discrete G–set $\mathrm{Map}_c(G, X)_m$ satisfies

$$(\mathrm{Map}_c(G, X)_m)_n = \mathrm{Map}_c(G, (X_m)_n)$$

(the set of continuous functions $G \to (X_m)_n$), forms a triple, and hence, there is a cosimplicial discrete G–spectrum $\mathrm{Map}_c(G^\bullet, X)$, whose l–cosimplices are obtained by applying $\mathrm{Map}_c(G, -)$ iteratively to X, $l+1$ times.

Thus, there is an isomorphism

$$\mathrm{Map}_c(G^\bullet, X)^l \cong \mathrm{Map}_c(G^{l+1}, X)$$

of discrete G–spectra. Also, by [7, Lemma 2.1], the map

$$X \xrightarrow{\cong} \operatorname*{colim}_{N \lhd_o G} X^N \to \operatorname*{colim}_{N \lhd_o G} (X^N)_f =: \widehat{X}$$

is a weak equivalence in Spt_G, with target $\widehat{X}$ fibrant in Spt.

We let X^{hG} denote the output of the total right derived functor of fixed points $(-)^G\colon \mathrm{Spt}_G \to \mathrm{Spt}$, when applied to X: the spectrum X^{hG} is more succinctly known as the homotopy fixed point spectrum of X with respect to the continuous action of G. Also, let holim^G denote the homotopy limit for Spt_G, as defined in [13, Definition 18.1.8], and let $H^s_c(G; M)$ be equal to the continuous cohomology of G with coefficients in the discrete G–module M. Then by [7, Theorem 7.2] and [5, Theorem 2.3, proof of Theorem 5.2], there is a weak equivalence

$$X^{hG} \xrightarrow{\simeq} \big(\operatorname*{holim}_{\Delta}{}^{G}\, \mathrm{Map}_c(G^\bullet, \widehat{X})\big)^G,$$

whenever any one of the following conditions holds:

(i) G has finite virtual cohomological dimension (that is, G contains an open subgroup U such that $H^s_c(U; M) = 0$, for all $s > u$ and all discrete U–modules M, for some integer u);

(ii) there exists a fixed integer p such that $H^s_c(N; \pi_t(X)) = 0$, for all $s > p$, all $t \in \mathbb{Z}$, and all $N \lhd_o G$; or

(iii) there exists a fixed integer r such that $\pi_t(X) = 0$, for all $t > r$.

As in the case of Z^{hK}, one of the main reasons why the explicit construction

$$\big(\operatorname*{holim}_{\Delta}{}^{G}\, \mathrm{Map}_c(G^\bullet, \widehat{X})\big)^G = \Big(\operatorname*{colim}_{N \lhd_o G} \big(\operatorname*{holim}_{\Delta} \mathrm{Map}_c(G^\bullet, \widehat{X})\big)^N\Big)^G$$

(see [5, Theorem 2.3]; the "holim" denotes the homotopy limit for spectra) is important is that when X satisfies one of the above conditions, the construction makes it possible to build the descent spectral sequence

$$E_2^{s,t} = H^s_c(G; \pi_t(X)) \Longrightarrow \pi_{t-s}(X^{hG}) \tag{1.2}$$

(as in [4, Theorem 7.9], by using (1.5) below: given the context, this reference is the most immediate source for the derivation of (1.2), but the account in [4, Theorem 7.9] is just a particular case of the much earlier [25, Proposition 1.36], and, in the literature for "simplicial-set-based discrete G–objects," the references [14, Corollary 3.6], [12, Section 5], [15, (6.7)], and [26, Section 2.14] are earlier than [4, Theorem 7.9] and contain all of its key ingredients).

Given the above discussion, it is natural to make the following definition.

Definition 1.3. If the discrete G–spectrum X satisfies any one of the conditions (i), (ii), and (iii) above, then we say that X is a *concrete* discrete

G–spectrum, since X has a concrete model for its homotopy fixed point spectrum.

In practice, at least one of the above three conditions is usually satisfied. For example, as is common in chromatic homotopy theory, let Γ_h be equal to any formal group law of height h, with h positive, over k, a finite field of prime characteristic p that contains the field $\mathbb{F}_{p^h}$, and consider any closed subgroup H of the compact p-adic analytic group $G(k,\Gamma_h)$, the extended Morava stabilizer group (see [10, Section 7]). Then H is a profinite group with finite virtual cohomological dimension (see [20, Section 2.2.0]), and thus, the discrete H–spectrum

$$\Big(\operatorname*{colim}_{N \lhd_o G(k,\Gamma_h)} E(k,\Gamma_h)^{hN}\Big) \wedge F \simeq E(k,\Gamma_h) \wedge F \tag{1.4}$$

satisfies condition (i) above. In (1.4), $E(k,\Gamma_h)$ is the Morava E–theory associated to the pair (k,Γ_h) (again, see [10, Section 7]); the construction of the homotopy fixed point spectrum $E(k,\Gamma_h)^{hN}$ is described in [6, page 2895] (with substantial input from [9] and [3, Theorem 8.2.1]); F is any finite spectrum of type h; and the weak equivalence is due to [9] (the details are written out in [4, Theorem 6.3, Corollary 6.5]). Discrete H–spectra that have the form given by (1.4) are the building blocks for many of the continuous H–spectra that are studied in chromatic theory; for examples, see [1, Section 9], [2, Section 2.3], and [16, pages 153–155].

If Y is any spectrum and X is the discrete G–spectrum $\mathrm{Map}_c(G,Y)$, then by [27, Lemma 9.4.5], condition (ii) above is satisfied, with $p=0$. Such concrete discrete G–spectra arise in the theory [24] of Galois extensions for commutative rings in stable homotopy theory: for example, if T is a spectrum such that the Bousfield localization $L_T(-)$ is smashing, M is any finite spectrum, k is a spectrum such that $L_k(-) \simeq L_M L_T(-)$, and (for the remainder of this sentence, using symmetric spectra as needed) E is a k–local profinite G–Galois extension of a k–local commutative symmetric ring spectrum A, then

$$L_k(E \wedge_A E) \simeq L_k\big(\mathrm{Map}_c(G,E)\big),$$

by [3, Proposition 6.2.4].

We see that under hypotheses that are often satisfied, the homotopy fixed point spectrum X^{hG} can be obtained by taking the G–fixed points of the discrete G–spectrum

$$\operatorname*{holim}_{\Delta}{}^{G}\, \mathrm{Map}_c(G^\bullet, \widehat{X}) = \operatorname*{colim}_{N \lhd_o G} \big(\operatorname*{holim}_{\Delta} \mathrm{Map}_c(G^\bullet, \widehat{X})\big)^N$$

(this is the discrete G–spectrum $\mathcal{H}_{G,X}$ that is referred to in the abstract for this paper), and hence, the construction of X^{hG} follows a pattern that was seen before in the case of Z^{hK}: form the homotopy fixed point spectrum by taking the fixed points of an explicitly constructed spectrum that is an object in the equivariant category of spectra that is under consideration (Spt_G or K–spectra, respectively).

But there is more to the above pattern than just the last observation: this is hinted at by the tandem facts that, as in [5, proof of Theorem 5.2], there is an isomorphism

$$(1.5) \qquad \big(\operatorname{holim}_{\Delta}{}^{G} \operatorname{Map}_c(G^\bullet, \widehat{X})\big)^G \cong \big(\operatorname{holim}_{\Delta} \operatorname{Map}_c(G^\bullet, \widehat{X})\big)^G$$

and the G–spectrum $\operatorname{holim}_\Delta \operatorname{Map}_c(G^\bullet, \widehat{X})$ on the right-hand side is often viewed as being "a profinite version" of the construction $\operatorname{Map}_*(EK_+, Z_f)$ (for example, see [19]). (Also, it is worth pointing out that if X is a concrete discrete G–spectrum, then X^{hG} has almost always been presented in the literature as being the G–fixed points of $\operatorname{holim}_\Delta \operatorname{Map}_c(G^\bullet, \widehat{X})$ (this G–spectrum is not, in general, a discrete G–spectrum: see the remark below for an example of when this happens), instead of as the G–fixed points of $\operatorname{holim}^G_\Delta \operatorname{Map}_c(G^\bullet, \widehat{X})$.) However, what the last assertion means has never been explained in a precise and systematic way, and further, as the above considerations make clear, it is rather $\operatorname{holim}^G_\Delta \operatorname{Map}_c(G^\bullet, \widehat{X})$, instead of $\operatorname{holim}_\Delta \operatorname{Map}_c(G^\bullet, \widehat{X})$, that we want to understand as a "profinite version" of $\operatorname{Map}_*(EK_+, Z_f)$. Thus, in this paper, we give a careful explanation of how $\operatorname{holim}^G_\Delta \operatorname{Map}_c(G^\bullet, \widehat{X})$ is indeed a profinite version of $\operatorname{Map}_*(EK_+, Z_f)$. Rather than cluttering our introduction with an excess of definitions, we refer the reader to Section 4 for the exact details of this explanation.

Remark 1.6. We pause to give an example of $\operatorname{holim}_\Delta \operatorname{Map}_c(G^\bullet, \widehat{X})$ failing to be a discrete G–spectrum. All of the following details are expanded upon in [6, Appendix A]. Given distinct primes p and q, set

$$G = \mathbb{Z}/p \times \mathbb{Z}_q$$

(a profinite group of finite virtual cohomological dimension) and let

$$X = \bigvee_{n \geq 0} \Sigma^n H(\mathbb{Z}/p[\mathbb{Z}/q^n]),$$

a discrete G–spectrum. Then suppose that $\operatorname{holim}_\Delta \operatorname{Map}_c(G^\bullet, \widehat{X})$ is a discrete G–spectrum: its $(\mathbb{Z}/p)$–fixed point spectrum $\operatorname{holim}_\Delta \big(\operatorname{Map}_c(G^\bullet, \widehat{X})\big)^{\mathbb{Z}/p}$ is a discrete $\mathbb{Z}_q$–spectrum, and hence, $\pi_0\big(\operatorname{holim}_\Delta \big(\operatorname{Map}_c(G^\bullet, \widehat{X})\big)^{\mathbb{Z}/p}\big)$ is a discrete $\mathbb{Z}_q$–module, a contradiction.

For now, we summarize our explanation with the following: it turns out that the "co-steps" in the construction of $\operatorname{holim}^G_\Delta \operatorname{Map}_c(G^\bullet, \widehat{X})$ are essentially identical to those involved in the construction of a certain K–spectrum $\widetilde{Z_K}$ that is equivalent to $\operatorname{Map}_*(EK_+, Z_f)$, except that when imitating the construction of $\widetilde{Z_K}$, at each co-step, if one obtains a G–spectrum that need not be, in general, a discrete G–spectrum, then one makes it a discrete G–spectrum in "the canonical way," by applying the discretization functor (see Definition 4.2).

In Section 2, we define the K–spectrum $\widetilde{Z_K}$ and show that it is equivalent to $\mathrm{Map}_*(EK_+, Z_f)$: this reduces our task to relating $\mathrm{holim}_\Delta^G \mathrm{Map}_c(G^\bullet, \widehat{X})$ to $\widetilde{Z_K}$. We do not claim any originality for Section 2 and we note that $\widetilde{Z_K}$ is closely related to the homotopy limit that is used in [19, second half of page 226] to describe Z^{hK}. (The main difference between our presentation and that of [19] is that the object in [19] that plays the role of our $K^\bullet$ (below, in Section 2) is defined differently.)

1.3. The pattern and the cases of compact Lie groups and profinite *G*-spectra. Let H be a discrete or profinite group and let Z be an object in the corresponding category $\mathtt{Sp}_H$ of H–spectra: if H is discrete, then $\mathtt{Sp}_H$ is the category of naive H–spectra considered at the beginning of this Introduction, and if H is profinite, then $\mathtt{Sp}_H$ is the full subcategory of Spt_H that consists of the concrete discrete H–spectra. In both cases, as recalled at the beginning and by our main result, respectively, there is the following pattern: the homotopy fixed point spectrum Z^{hH} can always be formed by taking the H–fixed points of some construction "$\mathrm{Map}_*(EH_+, Z)$" (the particular version of the spectrum "$\mathrm{Map}_*(EH_+, Z)$" that is used depends on the case) that is an object in the category $\mathtt{Sp}_H$.

Remark 1.7. It was just noted that when H is profinite, the appropriate version of "$\mathrm{Map}_*(EH_+, Z)$" is not just a discrete H–spectrum, but it is also concrete (that is, a concrete discrete H–spectrum). This can be justified as follows: because Z is concrete, the H–equivariant map

$$Z \xrightarrow{\simeq} \operatorname*{colim}_{N \lhd_o H} \big(\operatorname*{holim}_{\Delta} \mathrm{Map}_c(H^\bullet, \widehat{Z})\big)^N = \operatorname*{holim}_{\Delta}{}^H \mathrm{Map}_c(H^\bullet, \widehat{Z})$$

is a weak equivalence of spectra (by [5, proof of Theorem 4.2] and [7, Theorem 7.2]; see [5, page 145] for the definition of the map), and hence, the target of the weak equivalence (which is the appropriate version of "$\mathrm{Map}_*(EH_+, Z)$") is concrete (since the homotopy groups of the source and target of the weak equivalence are isomorphic as discrete H-modules), as desired.

The above pattern also occurs when H is a compact Lie group and $\mathtt{Sp}_H$ is the category of naive H–equivariant spectra (in the context of [17]): in this case, Z^{hH} is the H–fixed points of the naive H–equivariant spectrum $F(EH_+, Z)$.

Now we again let H be a profinite group and set $\mathtt{Sp}_H$ equal to the category of profinite H–spectra, as defined in [23]. Interestingly, we will see that in this case, the above pattern does not go through all the way. By [22, Remark 3.8, Definition 3.14], Z^{hH} is the H–fixed points of the explicit H–spectrum $\mathrm{Map}(EH, R_H Z)$, where here, EH is regarded as a simplicial profinite H–set and $R_H Z$ is a functorial fibrant replacement of Z in the stable model category $\mathtt{Sp}_H$. Also, the H–spectrum $\mathrm{Map}(EH, R_H Z)$ is defined as follows:

for each $m \geq 0$,

$$\mathrm{Map}(EH, R_H Z)_m = \mathrm{map}_{\hat{\mathcal{S}}_*}(EH_+, (R_H Z)_m),$$

where the right-hand side is an instance of the simplicial mapping space for the category $\hat{\mathcal{S}}_*$ of pointed simplicial profinite sets. Thus, in agreement with the pattern, the construction $\mathrm{Map}(EH, R_H Z)$ is indeed a version of "$\mathrm{Map}_*(EH_+, Z)$."

In contrast with the pattern, however, it turns out that $\mathrm{Map}(EH, R_H Z)$ is not, in general, a profinite H–spectrum. To see that this is true, suppose that $\mathrm{Map}(EH, R_H Z)$ is always a profinite H–spectrum. Then

$$Z^{hH} = \mathrm{Map}(EH, R_H Z)^H = \lim_H \mathrm{Map}(EH, R_H Z),$$

where the last expression is a limit in the category of spectra. Since the forgetful functor from profinite spectra to spectra is a right adjoint (see [23, Proposition 4.7]), limits in profinite spectra are formed in spectra, and thus, since $\mathrm{Map}(EH, R_H Z)$ is a profinite H–spectrum, the above limit can be regarded as a limit in the category of profinite spectra. It follows that Z^{hH} must be a profinite spectrum. But Z^{hH} is not always a profinite spectrum, by [22, page 194; Remark 3.16] (see also the helpful discussion between Proposition 2.15 and Theorem 2.16 in [21]), showing that $\mathrm{Map}(EH, R_H Z)$ is not always a profinite H–spectrum.

We continue to let H be profinite. With various properties of the theories of homotopy fixed points for discrete and profinite H–spectra laid out on the table, it is worth making the following observation: in these theories, abstract and explicit realizations of homotopy fixed points do not go easily together. In the world of discrete H–spectra, the homotopy fixed point spectrum is abstractly defined as the right derived functor of fixed points, but only when certain hypotheses are satisfied, is the homotopy fixed points known to be given by a concrete model. In the setting of profinite H–spectra, the situation is reversed: the homotopy fixed points are always given by an explicit model (that is, $\mathrm{Map}(EH, R_H Z)^H$, as considered above), but in general, the homotopy fixed points are not the right derived functor of fixed points. To see this last point, suppose that Z is a profinite H–spectrum with $Z^{hH} = (R_H Z)^H$. Then, by repeating an argument that was used above, $Z^{hH} = \lim_H R_H Z$ must be a profinite spectrum. Since Z^{hH} is not always a profinite spectrum (see above), Z^{hH} cannot in general be defined abstractly as the output of the right derived functor of fixed points.

We conclude the Introduction with a few comments about our notation. We use $\mathcal{S}$ to denote the category of simplicial sets. Given a set S, we let $c_\bullet(S)$ denote the constant simplicial set on S, and by a slight abuse of this notation, we use $c_\bullet(*)$ to denote the constant simplicial set on the set $\{*\}$ that consists of a single point. To avoid any possible confusion, we note that $c_\bullet(S)_+$ is $c_\bullet(S)$ with a disjoint basepoint added.

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2. K–spectrum $\widetilde{Z_K}$ is equivalent to $\mathrm{Map}_*(EK_+, Z_f)$

Recall from §1.1 the K–spectrum $\mathrm{Map}_*(EK_+, Z_f)$: for each $n \geq 0$, K acts diagonally on the n–simplices K^{n+1} of EK and the mapping spectrum has its K–action induced by conjugation on the level of sets (that is, by the formula

$$(k \cdot f_j)(k_1, k_2, ..., k_{j+1}) = k \cdot f_j(k^{-1} \cdot (k_1, k_2, ..., k_{j+1})),$$

where $k, k_1, ..., k_{j+1} \in K$ and

$$\{f_j \colon K^{j+1} \to \mathrm{hom}_*(\Delta[n], (Z_f)_m)_j\}_{j \geq 0} \in \mathcal{S}(EK, \mathrm{hom}_*(\Delta[n], (Z_f)_m)),$$

with

$$\mathcal{S}(EK, \mathrm{hom}_*(\Delta[n], (Z_f)_m)) \cong \mathrm{Map}_{\mathcal{S}_*}(EK_+, (Z_f)_m)_n,$$

$\mathrm{hom}_*(\Delta[n], (Z_f)_m)$ is a cotensor in $\mathcal{S}_*$, and K acts only on $(Z_f)_m$ in the expression $\mathrm{hom}_*(\Delta[n], (Z_f)_m))$.

Definition 2.1. Let $K^\bullet$ be the canonical bisimplicial set

$$K^\bullet \colon \Delta^{\mathrm{op}} \to \mathcal{S}, \quad [n] \mapsto (K^\bullet)_n = c_\bullet(K^{n+1}),$$

with $\mathrm{diag}(K^\bullet) = EK$, where $\mathrm{diag}(K^\bullet)$ is the diagonal of $K^\bullet$.

Given a simplicial set L and a spectrum Y, we write Y^L for the cotensor in the simplicial model category Spt. It will be helpful to note that

$$Y^L = \mathrm{Map}_*(L_+, Y).$$

Definition 2.2. Notice that $\operatorname*{hocolim}_{\Delta^{\mathrm{op}}} K^\bullet \equiv \operatorname*{hocolim}_{[n] \in \Delta^{\mathrm{op}}} (K^\bullet)_n$. There is an isomorphism $(Z_f)^{(\mathrm{hocolim}_{\Delta^{\mathrm{op}}} K^\bullet)} \xrightarrow{\cong} \mathrm{holim}_\Delta\, (Z_f)^{(K^\bullet)}$ and the target of this map is defined to be the K–spectrum $\widetilde{Z_K}$. Thus, we have

$$\widetilde{Z_K} = \operatorname*{holim}_{\Delta}\, (Z_f)^{(K^\bullet)}.$$

As alluded to in the Introduction, the following result – or at least some version of it – seems to be well-known, but for the sake of completeness, we give a proof of the precise version that we need.

Theorem 2.3. *There is a canonical K–equivariant map*

$$\mathrm{Map}_*(EK_+, Z_f) \xrightarrow{\simeq} \widetilde{Z_K}$$

that is a weak equivalence in Spt.

Proof. Since $\mathrm{Map}_*(EK_+, Z_f)$ is the cotensor $(Z_f)^{EK}$, it suffices to construct a canonical K–equivariant map $(Z_f)^{EK} \to (Z_f)^{(\mathrm{hocolim}_{\Delta^{\mathrm{op}}} K^\bullet)}$ that is a weak equivalence of spectra. Notice that there is the composition

$$\widetilde{\phi_*}\colon \operatorname*{hocolim}_{\Delta^{\mathrm{op}}} K^\bullet \xrightarrow{\simeq} |K^\bullet| \xrightarrow{\cong} \mathrm{diag}(K^\bullet) = EK$$

of canonical K–equivariant maps, with the first map,

$$\phi_*\colon \operatorname*{hocolim}_{\Delta^{\mathrm{op}}} K^\bullet \xrightarrow{\simeq} |K^\bullet|$$

(our label for this map comes from [13, Corollary 18.7.5], where this map is referred to as "the Bousfield-Kan map"), and the second map equal to a weak equivalence and an isomorphism (as labeled above), respectively. Then the desired map is just $(Z_f)^{\widetilde{\phi_*}}$ and we only need to show that this map is a weak equivalence: to do this, since a strict weak equivalence of spectra is a (stable) weak equivalence, it suffices to show that for each $m \geq 0$, the map

$$\begin{aligned}&\mathrm{Map}_*(|K^\bullet|_+, Z_f)_m \\ &\qquad = \mathrm{Map}_{\mathcal{S}_*}(|K^\bullet|_+, (Z_f)_m) \to \mathrm{Map}_{\mathcal{S}_*}\big(\big(\operatorname*{hocolim}_{\Delta^{\mathrm{op}}} K^\bullet\big)_+, (Z_f)_m\big)\end{aligned}$$

is a weak equivalence in $\mathcal{S}$.

If L and L' are simplicial sets, then $L_+ \wedge (L')_+ \cong (L \times L')_+$, and hence, we only need to show that each map

$$\mathrm{Map}_{\mathcal{S}}(|K^\bullet|, (Z_f)_m) \to \mathrm{Map}_{\mathcal{S}}\big(\operatorname*{hocolim}_{\Delta^{\mathrm{op}}} K^\bullet, (Z_f)_m\big)$$

is a weak equivalence in $\mathcal{S}$: this follows from the fact that in $\mathcal{S}$, ϕ_* is a weak equivalence and $(Z_f)_m$ is fibrant. $\square$

The equivalence in Theorem 2.3 implies that to relate the discrete G–spectrum $\mathrm{holim}^G_\Delta \mathrm{Map}_c(G^\bullet, \widehat{X})$ to the K–spectrum $\mathrm{Map}_*(EK_+, Z_f)$, we can just as well compare $\mathrm{holim}^G_\Delta \mathrm{Map}_c(G^\bullet, \widehat{X})$ to $\widetilde{Z_K}$. To do this comparison, it will be helpful to write $\widetilde{Z_K}$ a little differently: there are isomorphisms

$$\begin{aligned}\widetilde{Z_K} &= \operatorname*{holim}_{[n]\in\Delta} (Z_f)^{(c_\bullet(K^{n+1}))} \\ &\cong \operatorname*{holim}_{[n]\in\Delta} (Z_f)^{(\prod_{\ell\in\{1,2,\ldots,n+1\}} c_\bullet(K))} \\ &\cong \operatorname*{holim}_{[n]\in\Delta} \underbrace{(\cdots(((}_{n+1} Z_f \underbrace{)^{c_\bullet(K)})^{c_\bullet(K)})\cdots)^{c_\bullet(K)}}_{n+1}.\end{aligned}$$

3. Building $\mathrm{Map}_c(G, X)$ from fixed points of cotensors

We begin this section by recalling that given $X \in \mathrm{Spt}_G$, the G–action on the discrete G–spectrum $\mathrm{Map}_c(G, X)$ is induced by the G–action on the level of sets that is defined by $(g \cdot (h_m)_n)(g') = (h_m)_n(g'g)$, where $g, g' \in G$ and, for each $m, n \geq 0$, $(h_m)_n \in \mathrm{Map}_c(G, (X_m)_n)$.

Notice that there are natural G–equivariant isomorphisms

$$\begin{aligned}\mathrm{Map}_c(G, X) &\cong \operatorname*{colim}_{N \lhd_o G} \textstyle\prod_{G/N} X \\ &\cong \operatorname*{colim}_{N \lhd_o G} \mathrm{Map}_*(\textstyle\bigvee_{G/N} c_\bullet(*)_+, X) \\ &\cong \operatorname*{colim}_{N \lhd_o G} \mathrm{Map}_*(c_\bullet(G/N)_+, X),\end{aligned}$$

where the last expression above uses the following convention.

Definition 3.1. The spectrum $\mathrm{Map}_*(c_\bullet(G/N)_+, X)$ has a G/N–action that is determined by the formula $(g_1N \cdot f_j)(g_2N) = f_j(g_2g_1N)$, for $g_1, g_2 \in G$ and

$$\{f_j\}_{j \geq 0} \in \mathcal{S}(c_\bullet(G/N), \mathrm{hom}_*(\Delta[n], X_m))$$

(for example, see the beginning of Section 2).

We have shown that there is a natural isomorphism

$$\mathrm{Map}_c(G, X) \cong \operatorname*{colim}_{N \lhd_o G} \mathrm{Map}_*(c_\bullet(G/N)_+, X)$$

in Spt_G; this observation was made in [12, page 210] in the context of simplicial discrete G–sets.

Proposition 3.2. *If N is an open normal subgroup of G, then there is a natural G/N–equivariant isomorphism*

$$\mathrm{Map}_*(c_\bullet(G)_+, X)^N \cong \mathrm{Map}_*(c_\bullet(G/N)_+, X)$$

of G/N–spectra, where $\mathrm{Map}_*(c_\bullet(G)_+, X)$ has the G–action given by conjugation, $\mathrm{Map}_*(c_\bullet(G)_+, X)^N$ denotes the N–fixed point spectrum (and not a cotensor), and $\mathrm{Map}_*(c_\bullet(G/N)_+, X)$ has the G/N–action given in Definition 3.1.

Proof. To verify this result, it suffices to show that on the level of simplices there is a natural G/N–equivariant isomorphism

$$\big(\mathrm{Map}_{\mathcal{S}_*}(c_\bullet(G)_+, X_m)^N\big)_n \cong \mathrm{Map}_{\mathcal{S}_*}(c_\bullet(G/N)_+, X_m)_n,$$

and hence, we only need to show that there is a natural G/N–equivariant bijection

$$\mathcal{S}_*(c_\bullet(G)_+, \mathrm{hom}_*(\Delta[n], X_m))^N \cong \mathcal{S}_*(c_\bullet(G/N)_+, \mathrm{hom}_*(\Delta[n], X_m))$$

of sets, where the G–action on $\mathcal{S}_*(c_\bullet(G)_+, \mathrm{hom}_*(\Delta[n], X_m))$ is such that G only acts on X_m in the cotensor $\mathrm{hom}_*(\Delta[n], X_m)$.

Since the functor $(-)_+ \colon \mathcal{S} \to \mathcal{S}_*$ is left adjoint to the forgetful functor, our last assertion above is equivalent to there being a natural G/N–equivariant bijection

$$\mathcal{S}(c_\bullet(G), \mathrm{hom}_*(\Delta[n], X_m))^N \cong \mathcal{S}(c_\bullet(G/N), \mathrm{hom}_*(\Delta[n], X_m)).$$

The existence of this G/N–equivariant bijection follows from the fact that if W is any G–set, then, letting Sets denote the category of sets, the natural function

$$\lambda\colon \mathrm{Sets}(G,W)^N \to \mathrm{Sets}(G/N,W), \quad f \mapsto \Big[\lambda(f)\colon gN \mapsto g\cdot f(g^{-1})\Big]$$

is a G/N–equivariant isomorphism. Here, of course, G acts on $\mathrm{Sets}(G,W)$ by conjugation and the G/N–action on $\mathrm{Sets}(G/N,W)$ is defined by

$$(g_1N\cdot h)(g_2N) = h(g_2g_1N), \quad g_1,g_2\in G,\ h\in \mathrm{Sets}(G/N,W). \qquad \square$$

By Proposition 3.2 and the discussion that precedes it, we immediately obtain the following result.

Proposition 3.3. *There is an isomorphism*

$$\mathrm{Map}_c(G,X) \cong \operatorname*{colim}_{N\lhd_o G} \mathrm{Map}_*(c_\bullet(G)_+,X)^N$$

of discrete G–spectra.

Remark 3.4. For the duration of this remark, suppose that $G \neq \{e_{_G}\}$. The right-hand side of the isomorphism in Proposition 3.3 can be written as the discrete G–spectrum $\mathrm{colim}_{N\lhd_o G}\big(X^{c_\bullet(G)}\big)^N$, where $X^{c_\bullet(G)}$ is a cotensor for spectra. Interestingly, by [5, proof of Theorem 2.3], the cotensor $\big(X^{c_\bullet(G)}\big)_G$ for the simplicial model category Spt_G can also be written as $\mathrm{colim}_{N\lhd_o G}\big(X^{c_\bullet(G)}\big)^N$, where G acts on $X^{c_\bullet(G)}$ by acting only on X. However, despite their cosmetic similarity, $\mathrm{Map}_c(G,X)$ and $\big(X^{c_\bullet(G)}\big)_G$ are, in general, not isomorphic as discrete G–spectra, because of their different G–actions. For example, suppose that Y_1 and Y_2 are discrete G–spectra, with each having the trivial G–action. Then

$$\begin{aligned}\mathrm{Spt}_G\big(Y_1,\big((Y_2)^{c_\bullet(G)}\big)_G\big) &\cong \mathrm{Spt}_G\big(Y_1,\big((Y_2)^{(\coprod_{g\in G} c_\bullet(*))}\big)_G\big)\\ &\cong \textstyle\prod_{g\in G}\mathrm{Spt}_G(Y_1,Y_2)\\ &\cong \textstyle\prod_{g\in G}\mathrm{Spt}(Y_1,Y_2)\end{aligned}$$

and

$$\mathrm{Spt}_G(Y_1,\mathrm{Map}_c(G,Y_2)) \cong \mathrm{Spt}(Y_1,Y_2),$$

and hence, $\big((Y_2)^{c_\bullet(G)}\big)_G$ and $\mathrm{Map}_c(G,Y_2)$ are, in general, not isomorphic as discrete G–spectra.

4. At each co-step, $\mathbf{holim}^G_\Delta\, \mathbf{Map}_c(G^\bullet,\widehat{X})$ follows $\widetilde{Z_K}$ exactly, then makes the output into a discrete G–spectrum

In Section 2, we showed that there is a K-equivariant weak equivalence of spectra between $\mathrm{Map}_*(EK_+,Z_f)$ and

$$(4.1)\qquad \widetilde{Z_K} = \operatorname*{holim}_{[n]\in\Delta} \underbrace{(\cdots(((}_{n+1} Z_f \underbrace{)^{c_\bullet(K)})^{c_\bullet(K)})\cdots)^{c_\bullet(K)}}_{n+1}.$$

We remark that (4.1) contains a slight abuse of notation: the equality in (4.1) is actually a natural identification between isomorphic K–spectra. Identity (4.1) is key to understanding the main result of this paper, but to explain this result, we need one more tool, given in Definition 4.2 below. After some discussion of the functor recalled in this definition, we will explain the main result.

Throughout this section, G denotes an arbitrary profinite group.

4.1. The discretization functor for G–spectra. As noted in [8, Remark 2.2], the isomorphism

$$W \cong \operatorname*{colim}_{N \lhd_o G} W^N$$

satisfied by every $W \in \mathrm{Spt}_G$ is the basic fact behind the following.

Definition 4.2 ([8, Remark 2.2])**.** Let $G-\mathrm{Spt}$ be the category of (naive) G–spectra. The right adjoint of the forgetful functor $U_G \colon \mathrm{Spt}_G \to G-\mathrm{Spt}$ is the *discretization* functor

$$(-)_d \colon G-\mathrm{Spt} \to \mathrm{Spt}_G, \quad Y \mapsto (Y)_d = \operatorname*{colim}_{N \lhd_o G} Y^N;$$

$(Y)_d$ is "the discrete G–subspectrum" of the G–spectrum Y. The application of the functor $(-)_d$ is the canonical way to "convert" Y into a discrete G–spectrum (the author would like to mention that he learned part of this perspective on $(-)_d$ from [12, the brief discussion of (1.2.2)]). It goes without saying that if the G–spectrum Y already is a discrete G–spectrum, then $(Y)_d \cong Y$.

Since Spt is a combinatorial model category, the category $G-\mathrm{Spt}$, which is isomorphic to the diagram category of functors $\{*_G\} \to \mathrm{Spt}$ out of the one-object groupoid $\{*_G\}$ associated to G, has an injective model structure (for example, see [18, Proposition A.2.8.2]) in which a morphism of G–spectra is a weak equivalence (cofibration) if and only if it is a weak equivalence (cofibration) in Spt. Thus, the left adjoint $U_G \colon \mathrm{Spt}_G \to G-\mathrm{Spt}$ preserves weak equivalences and cofibrations, giving the next result, which gives some homotopical content to the fact that the discretization functor $(-)_d$ is the most natural way to convert a G–spectrum into a discrete G–spectrum.

Theorem 4.3. *The functors $(U_G, (-)_d)$ are a Quillen pair. In particular, if Y is a fibrant G–spectrum,* $\operatorname{colim}_{N \lhd_o G} Y^N$ *is a fibrant discrete G–spectrum.*

Remark 4.4. It is well-known that, as with most combinatorial model categories that consist of objects built out of simplicial presheaves on the canonical site of finite discrete G–sets, it is not easy to produce fairly explicit examples of fibrant discrete G–spectra (for example, see [11, page 1049] and [5, Introduction]), and thus, one example of the utility of Theorem 4.3 is that it provides a tool for doing this.

Remark 4.5. We make a well-known observation that is a preparatory comment for the next remark below. The left adjoint $\mathrm{Spt} \to G\mathrm{-Spt}$ that sends a spectrum to itself, but now regarded as a G–spectrum that is equipped with the trivial G–action, preserves weak equivalences and cofibrations, and hence, the right adjoint

$$\lim_{\{*_G\}} (-)\colon G\mathrm{-Spt} \to \mathrm{Spt}, \quad Y \mapsto \lim_{\{*_G\}} Y = Y^G$$

is a right Quillen functor. It follows that if $Y \to Y_{\mathtt{f}}$ is a trivial cofibration to a fibrant object, in $G\mathrm{-Spt}$, then

$$Y^{hG} = (Y_{\mathtt{f}})^G,$$

the right derived functor of fixed points $(-)^G\colon G\mathrm{-Spt} \to \mathrm{Spt}$ applied to Y, is the homotopy fixed point spectrum of Y.

Remark 4.6. Theorem 4.3 has the following curious consequence: if Y is any G–spectrum and $Y \to Y_{\mathtt{f}}$ is a trivial cofibration to a fibrant object, in $G\mathrm{-Spt}$, then $(Y_{\mathtt{f}})_d$ is a fibrant discrete G–spectrum, and hence, there is a weak equivalence

$$Y^{hG} = (Y_{\mathtt{f}})^G \xrightarrow{\cong} ((Y_{\mathtt{f}})_d)^G \xrightarrow{\simeq} (((Y_{\mathtt{f}})_d)_{fG})^G = ((Y_{\mathtt{f}})_d)^{hG}, \tag{4.7}$$

where the isomorphism is as in [5, proof of Theorem 2.3: top of page 141] and the weak equivalence is obtained by taking the G–fixed points of the natural trivial cofibration $(Y_{\mathtt{f}})_d \xrightarrow{\simeq} ((Y_{\mathtt{f}})_d)_{fG}$ in Spt_G that is associated to a fibrant replacement functor $(-)_{fG}\colon \mathrm{Spt}_G \to \mathrm{Spt}_G$. The weak equivalence in (4.7) shows that for any G–spectrum Y, the "discrete homotopy fixed point spectrum" Y^{hG} is equivalent to the "profinite homotopy fixed point spectrum" $((Y_{\mathtt{f}})_d)^{hG}$. This conclusion is a "discrete analogue" of the fact that the homotopy fixed point spectrum for an arbitrary continuous G–spectrum $\mathrm{holim}_i X_i$ is equivalent to the "profinite homotopy fixed points" $(\mathrm{holim}_i^G (X_i)_{fG})^{hG}$ of the discrete G–spectrum $\mathrm{holim}_i^G (X_i)_{fG}$ [8, Corollary 2.6].

4.2. The main result. Now we are ready to give the main result of this paper. Let X be any discrete G–spectrum. Notice that, by Proposition 3.3, there is an isomorphism

$$\mathrm{Map}_c(G, X) \cong \left(X^{c_\bullet(G)}\right)_d,$$

where $X^{c_\bullet(G)}$ is a G–spectrum with G–action given by conjugation. Also, we have

$$\operatorname*{holim}_{\Delta}{}^G \mathrm{Map}_c(G^\bullet, \widehat{X}) = \Big(\operatorname*{holim}_{[n]\in\Delta} \underbrace{\mathrm{Map}_c(G, ..., \mathrm{Map}_c(G, \mathrm{Map}_c(G,}_{n+1} \widehat{X} \underbrace{)) \cdots)}_{n+1}\Big)_d.$$

Repeated application of the first of the above two conclusions, to the second conclusion, yields an isomorphism

$$\operatorname*{holim}_{\Delta}{}^{G} \operatorname{Map}_c(G^\bullet, \widehat{X}) \cong \Big(\operatorname*{holim}_{[n]\in\Delta} \underbrace{\underbrace{\underbrace{\underbrace{((\cdots((((}_{2(n+1)-1}(\widehat{X})^{c_\bullet(G)})_d}_{\text{once}})^{c_\bullet(G)})_d}_{\text{twice}}\cdots)^{c_\bullet(G)})_d}_{(n+1)\ \text{times}}\Big)_d$$

of discrete G–spectra.

We recall (4.1) for the purpose of comparing it with the above isomorphism:

$$\widetilde{Z_K} = \operatorname*{holim}_{[n]\in\Delta} \underbrace{\underbrace{\underbrace{\underbrace{(\cdots(((}_{n+1} Z_f)^{c_\bullet(K)}}_{\text{once}})^{c_\bullet(K)})}_{\text{twice}}\cdots)^{c_\bullet(K)}}_{(n+1)\ \text{times}}. \tag{4.8}$$

Now the desired conclusion is clear: the construction of the discrete G–spectrum $\operatorname{holim}_\Delta^G \operatorname{Map}_c(G^\bullet, \widehat{X})$ – whose G–fixed points often (that is, whenever X is a concrete discrete G–spectrum) serve as a model for the homotopy fixed point spectrum X^{hG} – follows exactly the construction of the K–spectrum $\operatorname{Map}_*(EK_+, Z_f)$ (modulo a natural identification with the right-hand side of (4.8)), subject to the natural constraint that whenever following the construction of $\operatorname{Map}_*(EK_+, Z_f)$ yields a G–spectrum that is not necessarily in Spt_G (that is, after each formation of a cotensor that has the form $W^{c_\bullet(G)}$, for some discrete G–spectrum W, and after forming the homotopy limit in Spt), one applies the discretization functor $(-)_d$.

Remark 4.9. We consider the last observation above in slightly more detail. Recall that G is any profinite group and let W denote any object in Spt_G that is fibrant as a spectrum. Also, let I be the directed set of finite subsets of G, partially ordered by inclusion. For any integer t, there are G–equivariant isomorphisms

$$\begin{aligned}\pi_t\big(W^{c_\bullet(G)}\big) &\cong \pi_t\big(\textstyle\prod_G W\big) \cong \textstyle\prod_G \pi_t(W)\\ &\cong \lim_{(g_1,g_2,\ldots,g_k)\in I}\Big(\pi_t(W)_{g_1}\times\pi_t(W)_{g_2}\times\cdots\times\pi_t(W)_{g_k}\Big),\end{aligned}$$

where each $\pi_t(W)_{g_i}$ denotes a copy of $\pi_t(W)$ indexed by g_i. Since the finite product $\pi_t(W)_{g_1}\times\pi_t(W)_{g_2}\times\cdots\times\pi_t(W)_{g_k}$ in the category of abelian groups coincides with the product in the category of discrete G–modules, we see that the G–module $\pi_t\big(W^{c_\bullet(G)}\big)$ is an inverse limit of discrete G–modules. Note that if $W^{c_\bullet(G)}$ is a discrete G–spectrum (or even just weakly equivalent in G–Spt to a discrete G–spectrum), then the "pro-discrete" G–module $\pi_t\big(W^{c_\bullet(G)}\big)$ is a discrete G–module. For arbitrary G, the preceding conclusion is typically not true, and hence, $W^{c_\bullet(G)}$ is not, in general, a discrete G–spectrum, so that applying the functor $(-)_d$ to $W^{c_\bullet(G)}$ typically does not leave $W^{c_\bullet(G)}$ unchanged.

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