A profinite group G is *strongly complete* if it is isomorphic to its profinite completion, or equivalently, every subgroup of finite index is open. Now every known (topologically) finitely generated profinite group G is strongly complete, but it is an open question if all are. However, it is a standard theorem that a finitely generated pro-p group is strongly complete. We will use this fact to show that the Morava stabilizer group $\mathbb{S}_n$ is strongly complete.

Notation: We will use

$$H \leq_O G, H \leq_C G, N \lhd_O G, N \lhd_C G$$

to denote open, closed, open normal, and closed normal subgroups, respectively. Let $d(G)$ denote the minimal cardinality of a topological generating set for G. Let the *rank* of G be

$$rk(G) = sup\{\, d(H) \,|\, H \leq_C G\}.$$

Lemma 1. *If profinite group G is not strongly complete, then no open subgroup of G is strongly complete.*

Proof. There exists an $H < G$ of finite index that is not open. Suppose $K \leq_O G$ is strongly complete. Consider $H \cap K$. Then $[H : H \cap K] \leq [G : K]$, which is finite since K is open. Thus, $H \cap K$ has finite index in G; hence, also in K and thus, $H \cap K \leq_O K$. This implies that $H \cap K \leq_O G$. Then we can write

$$H = \bigcup_{h \in H} h(H \cap K).$$

Since the union is open, $H \leq_O G$, a contradiction. □

Now the Morava stabilizer group $\mathbb{S}_n$ is a compact p-adic analytic group. Furthermore, a topological group is a compact p-adic analytic group if and only if it is a finitely generated profinite group containing an open pro-p subgroup of finite rank. For $\mathbb{S}_n$, we denote this pro-p subgroup by H. Clearly, $rk(H) < \infty$ implies that $d(H)$ is finite. Then as a finitely generated pro-p subgroup, H is strongly complete and thus, by the Lemma, $\mathbb{S}_n$ is strongly complete.

The reader must suspect that we can identify H with S_n, the open normal pro-p subgroup of $\mathbb{S}_n$ of strict automorphisms. We confirm this suspicion by making use of the notions of *powerful* and *uniformly powerful*; see any standard text in profinite group theory for the definitions. Now a compact p-adic analytic group G can also be characterized as a topological group that contains a normal open uniformly powerful pro-p subgroup K of finite index. Since S_n is normal of index $p^n - 1$, it is the unique p-Sylow subgroup of $\mathbb{S}_n$. Since pro-p K is open, it is closed and lies in S_n. Since K is uniformly powerful, it is powerful. It is open in S_n since $K = S_n \cap K$. Since $\mathbb{S}_n$ is finitely generated, every open subgroup is also finitely generated, so that S_n is. Thus, S_n is a pro-p group that is finitely generated and contains the powerful open subgroup K. This implies that S_n has finite rank. This shows that we can take H to be S_n.

The books *Profinite Groups* by Ribes and Zaleskii and *Analytic Pro-p Groups* by Dixon, du Sautoy, Mann, and Segal contain the theory used above; the bracketed expressions below refer to results in the latter book.

Note that we have also shown that any compact p-adic analytic group is strongly complete. Furthermore, we have

Theorem 2. *Let U be an open subgroup of compact p-adic analytic group G. Then U is also compact p-adic analytic.*

Proof. This is a standard result, but the proof is unusual in that it proceeds by group-theoretic means instead of applying differential geometry. We apply the first characterization of analytic groups given above.

Since $U \leq_O G$, it is a closed profinite group in G. Since G is finitely generated and U is open, U is finitely generated. Let H be the open pro-p subgroup of G of finite rank guaranteed by the characterization. We will now consider $K := U \cap H$. We will show that K is an open pro-p subgroup of U of finite rank.

Since $H \leq_O G$, $K \leq_O U$. Similarly, $K \leq_O H$, implying $K \leq_C H$, which, by definition of $rk(H) < \infty$, shows that any closed subgroup of K is also closed in H and thus is finitely generated. Thus, $rk(K) \leq rk(H)$. The proof is completed by showing that K is a pro-p group.

Now $K \leq_C H$ implies that K is a profinite group. Hence,

$$K \cong \varprojlim_{N \lhd_O K} (K/N).$$

Since H is pro-p, every open normal subgroup has finite index equal to a power of p. Choose any $N \lhd_O K$. $K \leq_O H$ yields $N \leq_O H$, so that there exists $M \leq N$ such that $M \lhd_O H$. Thus, for some finite n,

$$p^n = [H : M] = [H : K][K : N][N : M],$$

implying that $[K : N]$ is a finite power of p. Thus, K/N is a finite p-group for any N, proving that K is pro-p. □

Now we prove an interesting structure theorem about strict S_n. The theorem actually applies to any pro-p group having finite rank. G is a *finite product of subgroups* means that

$$G = H_1 H_2 \ldots H_n = \{ h_1 h_2 \ldots h_n \mid h_i \in H_i \leq G, 1 \leq i \leq n \}.$$

Theorem 3. *S_n is a finite product of closed subgroups, each of which is isomorphic to $\mathbb{Z}_p$ or a finite p-group.*

Proof. We have already seen that S_n is a pro-p group with finite rank, which by [3.17], is equivalent to being a product of finitely many procyclic subgroups. (Since $K \leq_C H \leq_C G$ implies $K \leq_C G$ for any topological groups G, H, K, the proof of [3.17] also shows that these subgroups are all closed.) As we showed above, these closed procyclics are pro-p. Now we just apply [1.28]: if G is pro-p, then being procyclic is equivalent to G being either finite and cyclic or topologically isomorphic to $\mathbb{Z}_p$. □

Ravenel's *Nilpotence and Periodicity* says that S_n contains an element of order p^{i+1} if and only if n is divisible by $(p-1)p^i$. In light of the above structure theorem, we examine this statement. Clearly, S_n is torsion-free if and only if it contains no element of finite order p^{i+1}. Equivalently, $(p-1)p^i \nmid n$ for $i = 0, 1, 2, \ldots$. Thus, we have

Remark 4. *S_n is torsion-free (hence, isomorphic to a finite product of closed subgroups, each isomorphic to $\mathbb{Z}_p$) if and only if $p - 1 \nmid n$.*