A potential definition for the classifying space BG_n, from the chromatic perspective

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Let S_n be the nth Morava stabilizer group, and let

$$G_n = S_n \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p),$$

the extended Morava stabilizer group. One can consider the classifying space BG_n, with the hope that the consideration will be useful in understanding chromatic stable homotopy theory. But since BG_n ignores the profinite topology of G_n, maybe the standard definition of BG_n is not useful. Let

$$G_n = U_0 \gneqq U_1 \gneqq U_2 \gneqq \cdots \gneqq U_i \gneqq \cdots$$

be a descending chain of open normal subgroups of G_n, such that $\bigcap_i U_i = \{e\}$ and $G_n \cong \lim_i G_n/U_i$. Then, for the chromatic perspective, perhaps the appropriate definition of BG_n is the tower of classifying spaces

$$\{B(G_n/U_i)\}$$

of maps

$$B(G_n/U_0) \leftarrow B(G_n/U_1) \leftarrow B(G_n/U_2) \leftarrow \cdots \leftarrow B(G_n/U_i) \leftarrow \cdots.$$

Note that each group G_n/U_i, in the tower $\{B(G_n/U_i)\}$, is a finite group.

Here is a way that the tower $\{G_n/U_i\}$ appears in the chromatic perspective. Let E_n be the Lubin-Tate spectrum with coefficients

$$\pi_*(E_n) = W(\mathbb{F}_{p^n})[[u_1, \cdots, u_{n-1}]][u^{\pm 1}],$$

where the degree of u is -2. Recall from [1] that the $K(n)$-local E_n-Adams spectral sequence $E_r^{*,*}$ with abutment $\pi_*(L_{K(n)}(S^0))$ has the form

$$E_2^{s,t} \cong H_c^s(G_n; \pi_t(E_n)) \Rightarrow \pi_{t-s}(L_{K(n)}(S^0)).$$

Let $\{M_I\}$ be a tower of generalized Moore spectra such that

$$L_{K(n)}(S^0) \cong \mathrm{holim}_I\, L_{K(n)}(M_I).$$

Also, let $F_n = \operatorname{colim}_i E_n^{hU_i}$. Then we make the following observation:

$$\begin{aligned} H_c^s(G_n; \pi_t(E_n)) &\cong \lim_I H_c^s(G_n; \pi_t(E_n \wedge M_I)) \\ &\cong \lim_I H_c^s(G_n; \pi_t(F_n \wedge M_I)) \\ &\cong \lim_I H_c^s(\lim_i G_n/U_i; \operatorname{colim}_i \pi_t(E_n^{hU_i} \wedge M_I)) \\ &\cong \lim_I \operatorname{colim}_i H^s(G_n/U_i; \pi_t(E_n^{hU_i} \wedge M_I)). \end{aligned}$$

The last isomorphism uses the functoriality of the main construction of [1]: $E_n^{hU_i}$ is a (G_n/U_i)-spectrum, so the abelian group $\pi_t(E_n^{hU_i} \wedge M_I)$ is a (G_n/U_i)-module.

Note that $H^s(G_n/U_i; \pi_t(E_n^{hU_i} \wedge M_I))$ is the E_2-term of the descent spectral sequence $E_r^{*,*}(i, I)$ that has the form

$$E_2^{s,t}(i, I) = H^s(G_n/U_i; \pi_t(E_n^{hU_i} \wedge M_I)) \Rightarrow \pi_{t-s}((E_n^{hU_i} \wedge M_I)^{h(G_n/U_i)})$$

and, by [1],

$$\pi_*((E_n^{hU_i} \wedge M_I)^{h(G_n/U_i)}) \cong \pi_*(E_n^{hG_n} \wedge M_I) \cong \pi_*(L_{K(n)}(M_I)).$$

Thus, $E_r^{*,*}(i, I)$ is the spectral sequence

$$H^s(G_n/U_i; \pi_t(E_n^{hU_i} \wedge M_I)) \Rightarrow \pi_{t-s}(L_{K(n)}(M_I)). \tag{1}$$

Also, I think that

$$\lim_I \operatorname{colim}_i E_r^{*,*}(i, I) \cong E_r^{*,*}. \tag{2}$$

In conclusion, I wonder if the tower $\{B(G_n/U_i)\}$ of classifying spaces of finite groups might be interesting or useful in chromatic theory, due to a possible connection with the spectral sequences (1), and the relationship (2) between spectral sequences (1) and $\pi_*(L_{K(n)}(S^0))$. I don't know the answer to this question.

References

[1] Ethan S. Devinatz and Michael J. Hopkins. Homotopy fixed point spectra for closed subgroups of the Morava stabilizer groups. *Topology*, 43(1):1–47, 2004.