HOMOTOPY FIXED POINTS FOR $L_{K(n)}(E_n \wedge X)$ USING THE CONTINUOUS ACTION[1]

DANIEL G. DAVIS[2]

Abstract. Let $K(n)$ be the nth Morava K-theory spectrum. Let E_n be the Lubin-Tate spectrum, which plays a central role in understanding $L_{K(n)}(S^0)$, the $K(n)$-local sphere. For any spectrum X, define $E^\vee(X)$ to be the spectrum $L_{K(n)}(E_n \wedge X)$. Let G be a closed subgroup of the profinite group G_n, the group of ring spectrum automorphisms of E_n in the stable homotopy category. We show that $E^\vee(X)$ is a continuous G-spectrum, with homotopy fixed point spectrum $(E^\vee(X))^{hG}$. Also, we construct a descent spectral sequence with abutment $\pi_*((E^\vee(X))^{hG})$.

1. Introduction

Let p be a fixed prime. For each $n \geq 0$, let $K(n)$ be the nth Morava K-theory spectrum, where $K(0)$ is the Eilenberg-Mac Lane spectrum $H\mathbb{Q}$, and, for $n \geq 1$, $K(n)_* = \mathbb{F}_p[v_n, v_n^{-1}]$, where the degree of v_n is $2(p^n - 1)$. Let X be a finite spectrum. There are maps $L_nX \to L_{n-1}X$, where L_n denotes Bousfield localization with respect to $K(0) \vee K(1) \vee \cdots \vee K(n)$. Then the chromatic convergence theorem [33, Theorem 7.5.7] says that $X_{(p)} \simeq \operatorname{holim}_{n \geq 0} L_nX$, where $X_{(p)}$ is the p-localization of X. Thus, to understand $X_{(p)}$, it is important that one understands each localization L_nX.

Henceforth, let $n \geq 1$, and let $\hat{L}$ denote Bousfield localization with respect to $K(n)$. Then there is a homotopy pullback square [15]

$$\begin{array}{ccc} L_nX & \longrightarrow & \hat{L}(X) \\ \downarrow & & \downarrow \\ L_{n-1}(X_{(p)}) & \longrightarrow & L_{n-1}\hat{L}(X), \end{array}$$

which shows that, to understand the localizations L_nX, it is very helpful to understand each $\hat{L}(X)$.

In attempting to understand $\hat{L}(X)$, one of the main tools is a certain spectral sequence, which we now recall. Let E_n be the Lubin-Tate spectrum with $E_{n*} =$

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$W[\![u_1,...,u_{n-1}]\!][u^{\pm 1}]$, where the degree of u is -2, and the complete power series ring over the Witt vectors $W = W(\mathbb{F}_{p^n})$ is in degree zero. Let G_n be the profinite group of ring spectrum automorphisms of E_n in the stable homotopy category [17, Thm. 1.4]. (There is an isomorphism $G_n \cong S_n \rtimes \mathrm{Gal}$, where S_n is the nth Morava stabilizer group - the automorphism group of the Honda formal group law Γ_n of height n over $\mathbb{F}_{p^n}$, and Gal is the Galois group $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ (see [38, Prop. 4]).) By [32] and [14, Prop. 7.4], Morava's change of rings theorem yields a spectral sequence

$$H^*_c(G_n; \pi_*(E_n \wedge X)) \Rightarrow \pi_* \hat{L}(X), \tag{1.1}$$

where the E_2-term is the continuous cohomology of G_n, with coefficients in the continuous G_n-module $\pi_*(E_n \wedge X)$ (see Definition 2.17). Thus, we see that it is critical to study the relationship between E_n and G_n.

The above action of G_n on $\pi_*(E_n \wedge X)$ is induced by a point-set level action of G_n on E_n (work of Goerss and Hopkins ([12], [9]), and Hopkins and Miller [34]). Let G be a closed subgroup of G_n. Using the G_n-action on E_n, Devinatz and Hopkins [5] construct spectra E_n^{dhG} with spectral sequences

$$H^s_c(G; \pi_t(E_n \wedge X)) \Rightarrow \pi_{t-s}(E_n^{dhG} \wedge X). \tag{1.2}$$

Also, they show that $E_n^{dhG_n} \simeq \hat{L}(S^0)$, so that $E_n^{dhG_n} \wedge X \simeq \hat{L}(X)$, and (1.1) is a special case of (1.2).

We compare the spectrum E_n^{dhG} and spectral sequence (1.2) with constructions for homotopy fixed point spectra. When K is a discrete group and Y is a K-spectrum of topological spaces, there is a homotopy fixed point spectrum $Y^{hK} = \mathrm{Map}_K(EK_+, Y)$, where EK is a free contractible K-space. Also, there is a descent spectral sequence

$$E_2^{s,t} = H^s(K; \pi_t(Y)) \Rightarrow \pi_{t-s}(Y^{hK}),$$

where the E_2-term is group cohomology [29, §1.1].

Now let K be a profinite group. If S is a K-set, then S is a *discrete K-set* if the action map $K \times S \to S$ is continuous, where S is given the discrete topology. Then, a *discrete K-spectrum* Y is a K-spectrum of simplicial sets, such that each simplicial set Y_k is a simplicial discrete K-set (that is, for each $l \geq 0$, $Y_{k,l}$ is a discrete K-set, and all the face and degeneracy maps are K-equivariant). Then, due to work of Jardine (e.g. [21], [22], [23], [24]) and Thomason [41], as explained in Sections 5 and 7, there is a homotopy fixed point spectrum Y^{hK} defined with respect to the continuous action of K, and, in nice situations, a descent spectral sequence

$$H^s_c(K; \pi_t(Y)) \Rightarrow \pi_{t-s}(Y^{hK}),$$

where the E_2-term is the continuous cohomology of K with coefficients in the discrete K-module $\pi_t(Y)$.

Notice that we use the notation E_n^{dhG} for the construction of Devinatz and Hopkins (which they denote as E_n^{hG} in [5]), and $(-)^{hK}$ for homotopy fixed points with respect to a continuous action, although henceforth, when K is finite and Y is a K-spectrum of topological spaces, we write $Y^{h'K}$ for $\mathrm{holim}_K Y$, which is an equivalent definition of the homotopy fixed point spectrum $\mathrm{Map}_K(EK_+, Y)$.

After comparing the spectral sequence for $E_n^{dhG} \wedge X$ with the descent spectral sequence for Y^{hK}, $E_n \wedge X$ appears to be a continuous G_n-spectrum with "descent" spectral sequences for "homotopy fixed point spectra" $E_n^{dhG} \wedge X$. Indeed, we apply [5] to show that $E_n \wedge X$ is a continuous G_n-spectrum; that is, $E_n \wedge X$ is the homotopy limit of a tower of fibrant discrete G_n-spectra. Using this continuous action, we define the homotopy fixed point spectrum $(E_n \wedge X)^{hG}$ and construct its descent spectral sequence.

In more detail, G_n acts on the $K(n)$-local spectrum E_n through maps of commutative S-algebras. The spectrum E_n^{dhG}, a $K(n)$-local commutative S-algebra, is referred to as a "homotopy fixed point spectrum" because it has the following desired properties: (a) spectral sequence (1.2), which has the form of a descent spectral sequence, exists; (b) when G is finite, there is a weak equivalence $E_n^{dhG} \rightarrow E_n^{h'G}$, and the descent spectral sequence for $E_n^{h'G}$ is isomorphic to spectral sequence (1.2) (when $X = S^0$) [5, Thm. 3]; and (c) E_n^{dhG} is an $N(G)/G$-spectrum, where $N(G)$ is the normalizer of G in G_n [5, pg. 5]. These properties suggest that G_n acts on E_n in a continuous sense.

However, in [5], the G_n-action on E_n is not proven to be continuous, and E_n^{dhG} is not defined with respect to a continuous G-action. Also, when G is profinite, homotopy fixed points should always be the total right derived functor of fixed points, in some sense, and [5] does not show that the "homotopy fixed point spectrum" E_n^{dhG} can be obtained through such a total right derived functor.

After introducing some notation, we state the main results of this paper. Let BP be the Brown-Peterson spectrum with $BP_* = \mathbb{Z}_{(p)}[v_1, v_2, ...]$, where the degree of v_i is $2(p^i - 1)$. The ideal $(p^{i_0}, v_1^{i_1}, ..., v_{n-1}^{i_{n-1}}) \subset BP_*$ is denoted by I; M_I is the corresponding generalized Moore spectrum $M(p^{i_0}, v_1^{i_1}, ..., v_{n-1}^{i_{n-1}})$, a spectrum with trivial G_n-action. Given I, M_I need not exist; however, enough exist for our constructions. Each M_I is a finite type n spectrum with $BP_*(M_I) \cong BP_*/I$. The set $\{i_0, ..., i_{n-1}\}$ of superscripts varies so that there is a family of ideals $\{I\}$. ([3, §4], [19, §4], and [27, Prop. 3.7] provide details for our statements about the spectra M_I.) The map $r\colon BP_* \rightarrow E_{n*}$ - defined by $r(v_i) = u_i u^{1-p^i}$, where $u_n = 1$ and $u_i = 0$, when $i > n$ - makes E_{n*} a BP_*-module. By the Landweber exact functor theorem, $\pi_*(E_n \wedge M_I) \cong E_{n*}/I$.

The collection $\{I\}$ contains a descending chain of ideals $\{I_0 \supset I_1 \supset I_2 \supset \cdots\}$, such that there exists a corresponding tower of generalized Moore spectra

$$\{M_{I_0} \leftarrow M_{I_1} \leftarrow M_{I_2} \leftarrow \cdots\}.$$

In this paper, the functors $\lim_I$ and holim_I are always taken over the tower of ideals $\{I_i\}$, so that $\lim_I$ and holim_I are really $\lim_{I_i}$ and $\operatorname{holim}_{I_i}$, respectively. Also, in this paper, if $\{X_\alpha\}_\alpha$ is a diagram of spectra (even if each X_α has additional structure), then $\operatorname{holim}_\alpha X_\alpha$ always denotes the version of the homotopy limit of spectra that is constructed levelwise in $\mathcal{S}$, the category of simplicial sets, as defined in [2] and [41, 5.6].

As in [5, (1.4)], let $G_n = U_0 \gneq U_1 \gneq \cdots \gneq U_i \gneq \cdots$ be a descending chain of open normal subgroups, such that $\bigcap_i U_i = \{e\}$ and the canonical map $G_n \rightarrow \lim_i G_n/U_i$ is a homeomorphism. We define

$$F_n = \operatorname*{colim}_i E_n^{dhU_i}.$$

Then the key to getting our work started is knowing that

$$E_n \wedge M_I \simeq F_n \wedge M_I,$$

and thus, $E_n \wedge M_I$ has the homotopy type of the discrete G_n-spectrum $F_n \wedge M_I$. This result (Corollary 6.5) is not difficult, thanks to [5].

Given a tower $\{Z_I\}$ of discrete G_n-spectra, there is a tower $\{(Z_I)_f\}$, with G_n-equivariant maps $Z_I \to (Z_I)_f$ that are weak equivalences, and $(Z_I)_f$ is a fibrant discrete G_n-spectrum (see Definition 4.1). For the remainder of this section, X is any spectrum with trivial G_n-action, and, throughout this paper,

$$E^\vee(X) = \hat{L}(E_n \wedge X).$$

We use $\cong$ to signify an isomorphism in the stable homotopy category.

Theorem 1.3. *As the homotopy limit of a tower of fibrant discrete G_n-spectra, $E_n \cong \operatorname{holim}_I (F_n \wedge M_I)_f$ is a continuous G_n-spectrum. Also, for any spectrum X, $E^\vee(X) \cong \operatorname{holim}_I (F_n \wedge M_I \wedge X)_f$ is a continuous G_n-spectrum.*

We define homotopy fixed points for towers of discrete G-spectra; we show that these homotopy fixed points are the total right derived functor of fixed points in the appropriate sense; and we construct the associated descent spectral sequence. This enables us to define the homotopy fixed point spectrum $(E^\vee(X))^{hG}$, using the continuous G_n-action, and construct its descent spectral sequence. More specifically, we have the following results.

Definition 1.4. Given a profinite group G, let $\mathcal{O}_G$ be the *orbit category* of G. The objects of $\mathcal{O}_G$ are the continuous left G-spaces G/K, for all K closed in G, and the morphisms are the continuous G-equivariant maps.

Let Spt be the model category $(\text{spectra})^{\text{stable}}$ of Bousfield-Friedlander spectra.

Theorem 1.5. *There is a functor $P\colon (\mathcal{O}_{G_n})^{\text{op}} \to \text{Spt}$, defined by $P(G_n/G) = E_n^{hG}$, where G is any closed subgroup of G_n.*

We also show that the G-homotopy fixed points of $E^\vee(X)$ can be obtained by taking the $K(n)$-localization of the G-homotopy fixed points of the discrete G-spectrum $(F_n \wedge X)$. This result shows that the spectrum F_n is an interesting spectrum that is worth further study.

Theorem 1.6. *For any closed subgroup G and any spectrum X, there is an isomorphism $(E^\vee(X))^{hG} \cong \hat{L}((F_n \wedge X)^{hG})$. In particular, $E_n^{hG} \cong \hat{L}(F_n^{hG})$.*

Theorem 1.7. *Let G be a closed subgroup of G_n and let X be any spectrum. Then there is a conditionally convergent descent spectral sequence*

$$E_2^{s,t} \Rightarrow \pi_{t-s}((E^\vee(X))^{hG}). \tag{1.8}$$

If the tower of abelian groups $\{\pi_t(E_n \wedge M_I \wedge X)\}_I$ satisfies the Mittag-Leffler condition, for each $t \in \mathbb{Z}$, then $E_2^{s,t} \cong H^s_{\text{cont}}(G; \{\pi_t(E_n \wedge M_I \wedge X)\})$ (see Definition 2.15). If X is a finite spectrum, then (1.8) has the form

$$H^s_c(G; \pi_t(E_n \wedge X)) \Rightarrow \pi_{t-s}((E_n \wedge X)^{hG}), \tag{1.9}$$

where the E_2-term is the continuous cohomology of (1.2).

Also, Theorem 9.9 shows that, when X is finite, $(E_n \wedge X)^{hG} \cong E_n^{hG} \wedge X$, so that descent spectral sequence (1.9) has the same form as spectral sequence (1.2). It is natural to wonder if these two spectral sequences are isomorphic to each other. Also, the spectra E_n^{dhG} and E_n^{hG} should be the same. We plan to say more about the relationship between E_n^{dhG} and E_n^{hG} and their associated spectral sequences in future work.

While reading this Introduction (and taking a quick look at §9), the reader might notice that, due to the definition of F_n, we use the "homotopy fixed point spectra" $E_n^{dhU_i}$ to construct the homotopy fixed point spectra E_n^{hG}. We discuss the degree to which this method is circular. To obtain the results of this paper, we require a tower $\{E_n/I\}_I$ of discrete G_n-spectra such that $\operatorname{holim}_I(E_n/I)_f$ is the G_n-spectrum E_n, and, for each I, E_n/I and $E_n \wedge M_I$ have the same stable homotopy type. Any tower with the stated properties will work (and, given such a tower, one defines $F_n = \operatorname{colim}_i \operatorname{holim}_I(E_n/I)^{hU_i})$. We obtained such a tower by using the spectra $E_n^{dhU_i}$ to form the tower $\{F_n \wedge M_I\}_I$.

We believe that one could probably use obstruction theory to construct the tower $\{E_n/I\}_I$. This would yield the above results independently of [5], so that, presumably, [5] is not required to build $(E^\vee(X))^{hG}$. However, to date, no one has obtained the requisite tower using obstruction theory, and we suspect that such work would be quite difficult.

We outline the contents of this paper. In §2, we establish some notation and terminology, and we provide some background material. In §3, we study the model category of discrete G-spectra. In §4, we study towers of discrete G-spectra and give a definition of continuous G-spectrum. Homotopy fixed points for discrete G-spectra are defined in §5, and §6 shows that E_n is a continuous G_n-spectrum, proving the first half of Theorem 1.3. In §7, two useful models of the G-homotopy fixed point spectrum are constructed, when G has finite virtual cohomological dimension. In §8, we define homotopy fixed points for towers of discrete G-spectra, build a descent spectral sequence in this setting, and show that these homotopy fixed points are a total right derived functor, in the appropriate sense. In §9, we complete the proof of Theorem 1.3, study $(E^\vee(X))^{hG}$, and prove Theorems 1.5 and 1.6. In §10, we consider the descent spectral sequence for $(E^\vee(X))^{hG}$ and prove Theorem 1.7.

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2. Notation, Terminology, and Preliminaries

We begin by establishing some notation and terminology that will be used throughout the paper. **Ab** is the category of abelian groups. Outside of **Ab**, all groups are assumed to be profinite, unless stated otherwise. For a group G, we write $G \cong \lim_N G/N$, the inverse limit over the open normal subgroups. The notation $H <_c G$ means that H is a closed subgroup of G. We use G to denote arbitrary profinite groups and, specifically, closed subgroups of G_n.

Let $\mathcal{C}$ be a category. A tower $\{C_i\}$ of objects in $\mathcal{C}$ is a diagram in $\mathcal{C}$ of the form $\cdots \to C_i \to C_{i-1} \to \cdots \to C_1 \to C_0$. We always use Bousfield-Friedlander spectra [1], except when another category of spectra is specified. If $\mathcal{C}$ is a model category, then $\mathrm{Ho}(\mathcal{C})$ is its homotopy category. The phrase "stable category" always refers to $\mathrm{Ho}(\mathrm{Spt})$.

In $\mathcal{S}$, the category of simplicial sets, $S^n = \Delta^n/\partial\Delta^n$ is the n-sphere. Given a spectrum X, $X^{(0)} = S^0$, and for $j \geq 1$, $X^{(j)} = X \wedge X \wedge \cdots \wedge X$, with j factors.

Definition 2.1. [16, Def. 1.3.1] Let $\mathcal{C}$ and $\mathcal{D}$ be model categories. The functor $F\colon \mathcal{C} \to \mathcal{D}$ is a *left Quillen functor* if F is a left adjoint that preserves cofibrations and trivial cofibrations. The functor $P\colon \mathcal{D} \to \mathcal{C}$ is a *right Quillen functor* if P is a right adjoint that preserves fibrations and trivial fibrations. Also, if F and P are an adjoint pair and left and right Quillen functors, respectively, then (F, P) is a *Quillen pair* for the model categories $(\mathcal{C}, \mathcal{D})$.

Recall [16, Lemma 1.3.10] that a Quillen pair (F, P) yields total left and right derived functors $\mathbf{L}F$ and $\mathbf{R}P$, respectively, which give an adjunction between the homotopy categories $\mathrm{Ho}(\mathcal{C})$ and $\mathrm{Ho}(\mathcal{D})$.

We use $\mathrm{Map}_c(G, A) = \Gamma_G(A)$ to denote the set of continuous maps from G to the topological space A, where A is often a set, equipped with the discrete topology, or a discrete abelian group. Instead of $\Gamma_G(A)$, sometimes we write just $\Gamma(A)$, when the G is understood from context. Let $(\Gamma_G)^k(A)$ denote $(\Gamma_G\,\Gamma_G \cdots \Gamma_G)(A)$, the application of Γ_G to A, iteratively, $k+1$ times, where $k \geq 0$. Let G^k be the k-fold product of G and $G^0 = *$. Then, if A is a discrete set (discrete abelian group), there is a G-equivariant isomorphism $(\Gamma_G)^k(A) \cong \mathrm{Map}_c(G^{k+1}, A)$ of discrete G-sets (modules), where $\mathrm{Map}_c(G^{k+1}, M)$ has the G-action defined by $(g' \cdot f)(g_1, ..., g_{k+1}) = f(g_1 g', g_2, g_3, ..., g_{k+1})$. Also, we often write $\Gamma^k_G(A)$, or $\Gamma^k A$, for $\mathrm{Map}_c(G^k, A)$.

Let A be a discrete abelian group. Then $\mathrm{Map}^\ell_c(G^k_n, A)$ is the discrete G_n-module of continuous maps $G^k_n \to A$, with action defined by $(g' \cdot f)(g_1, ..., g_k) = f((g')^{-1} g_1, g_2, g_3, ..., g_k)$. Note that there is a G_n-equivariant isomorphism of discrete G_n-modules

$$p\colon \mathrm{Map}^\ell_c(G^k_n, A) \to \mathrm{Map}_c(G^k_n, A),$$

which is defined by $p(f)(g_1, g_2, ..., g_k) = f(g_1^{-1}, g_2, ..., g_k)$. $\mathrm{Map}^\ell_c(G^k_n, A)$ is also defined when A is an inverse limit of discrete abelian groups.

By a *topological G-module*, we mean an abelian Hausdorff topological group that is a G-module, with a continuous G-action. Note that if $M = \lim_i M_i$ is the limit of a tower $\{M_i\}$ of discrete G-modules, then M is a topological G-module.

For the remainder of this section, we recall some frequently used facts and discuss background material, to help get our work started.

As explained in [5], Goerss and Hopkins ([12], [9]), building on work by Hopkins and Miller [34], proved that the action of G_n on E_n is by maps of commutative S-algebras. Previously, Hopkins and Miller had shown that G_n acts on E_n by maps of A_∞-ring spectra. However, the continuous action presented here is not structured. As already mentioned, the starting point for the continuous action is the spectrum $F_n \wedge M_I$, which is not known to be an A_∞-ring object in the category of discrete G_n-spectra. Thus, we work in the unstructured category Spt of Bousfield-Friedlander spectra of simplicial sets, and the continuous action is simply by maps of spectra.

As mentioned above, [5] is written using E_∞, the category of commutative S-algebras, and $\mathcal{M}_S$, the category of S-modules (see [7]). However, [18, §4.2], [28, §14, §19], and [36, pp. 529-530] show that $\mathcal{M}_S$ and Spt are Quillen equivalent model categories [16, §1.3.3]. Thus, we can import the results of Devinatz and Hopkins from $\mathcal{M}_S$ into Spt. For example, [5, Thm. 1] implies the following result, where $R^+_{G_n}$ is the category whose objects are finite discrete left G_n-sets and G_n itself (a continuous profinite left G_n-space), and whose morphisms are continuous G_n-equivariant maps.

Theorem 2.2 (Devinatz, Hopkins)**.** *There is a presheaf of $K(n)$-local spectra $F\colon (R^+_{G_n})^{\mathrm{op}} \to \mathrm{Spt}$, such that (a) $F(G_n) = E_n$; (b) for U an open subgroup of G_n, $E_n^{dhU} := F(G_n/U)$; and (c) $F(*) = E_n^{dhG_n} \simeq \hat{L}S^0$.*

Now we define a spectrum that is essential to our constructions.

Definition 2.3. Let $F_n = \mathrm{colim}_i\, E_n^{dhU_i}$, where the direct limit is in Spt. Because $\mathrm{Hom}_{G_n}(G_n/U_i, G_n/U_i) \cong G_n/U_i$, F makes $E_n^{dhU_i}$ a G_n/U_i-spectrum. Thus, F_n is a G_n-spectrum, and the canonical map $\eta\colon F_n \to E_n$ is G_n-equivariant.

The following useful fact is stated in [5, pg. 9] (see also [38, Lemma 14]).

Theorem 2.4. *For $j \geq 0$, let X be a finite spectrum and regard $\hat{L}(E_n^{(j+1)} \wedge X)$ as a G_n-spectrum, where G_n acts only on the leftmost factor of the smash product. Then there is a G_n-equivariant isomorphism*

$$\pi_*(\hat{L}(E_n^{(j+1)} \wedge X)) \cong \mathrm{Map}^{\ell}_c(G_n^j, \pi_*(E_n \wedge X)).$$

We review some frequently used facts about the functor L_n and homotopy limits of spectra. First, L_n, defined earlier, can be equivalently defined as Bousfield localization with respect to the spectrum $E(n)$, where $E(n)_* = \mathbb{Z}_{(p)}[v_1, ..., v_{n-1}][v_n, v_n^{-1}]$. Also, L_n is smashing, e.g. $L_nX \simeq X \wedge L_nS^0$, for any spectrum X, and $E(n)$-localization commutes with homotopy direct limits [33, Thms. 7.5.6, 8.2.2]. Note that this implies that F_n is $E(n)$-local.

Definition 2.5. If $\cdots \to X_i \to X_{i-1} \to \cdots \to X_1 \to X_0$ is a tower of spectra such that each X_i is fibrant in Spt, then $\{X_i\}$ is a *tower of fibrant spectra*.

If $\{X_i\}$ is a tower of fibrant spectra, then there is a short exact sequence

$$0 \to \lim_i{}^1 \pi_{m+1}(X_i) \to \pi_m(\operatorname{holim}_i X_i) \to \lim_i \pi_m(X_i) \to 0.$$

Also, if each map in the tower is a fibration, the map $\lim_i X_i \to \mathrm{holim}_i\, X_i$ is a weak equivalence. If J is a small category and the functor $P\colon J \to \mathrm{Spt}$ is a diagram of spectra, such that P_j is fibrant for each $j \in J$, then $\mathrm{holim}_j\, P_j$ is a fibrant spectrum.

Definition 2.6. There is a functor $(-)_{\mathtt{f}}\colon \mathrm{Spt} \to \mathrm{Spt}$, such that, given Y in Spt, $Y_{\mathtt{f}}$ is a fibrant spectrum, and there is a natural transformation $\mathrm{id}_{\mathrm{Spt}} \to (-)_{\mathtt{f}}$, such that, for any Y, the map $Y \to Y_{\mathtt{f}}$ is a trivial cofibration. For example, if Y is a G-spectrum, then $Y_{\mathtt{f}}$ is also a G-spectrum, and the map $Y \to Y_{\mathtt{f}}$ is G-equivariant.

The following statement says that smashing with a finite spectrum commutes with homotopy limits.

Lemma 2.7 ([42, pg. 96])**.** *Let J be a small category, $\{Z_j\}$ a J-shaped diagram of fibrant spectra, and let Y be a finite spectrum. Then the composition*

$$(\operatorname{holim}_j Z_j) \wedge Y \to \operatorname{holim}_j (Z_j \wedge Y) \to \operatorname{holim}_j (Z_j \wedge Y)_{\mathtt{f}}$$

is a weak equivalence.

We recall the result that is used to build towers of discrete G-spectra.

Theorem 2.8 ([15, §2], [4, Remark 3.6]). *If X is an $E(n)$-local spectrum, then, in the stable category, there is an isomorphism*

$$\hat{L}X \cong \operatorname*{holim}_I (X \wedge M_I)_{\mathtt{f}}.$$

Lemma 2.9 ([19, Lemma 7.2]). *If X is any spectrum, and Y is a finite spectrum of type n, then $\hat{L}(X \wedge Y) \simeq \hat{L}(X) \wedge Y \simeq L_n(X) \wedge Y$.*

We recall some useful facts about compact p-adic analytic groups. Since S_n is compact p-adic analytic, and G_n is an extension of S_n by Gal, G_n is a compact p-adic analytic group [37, Cor. of Thm. 2]. Any closed subgroup of a compact p-adic analytic group is also compact p-adic analytic [6, Thm. 9.6]. Also, since the subgroup in S_n of strict automorphisms is finitely generated and pro-p, [35, pp. 76, 124] implies that all subgroups in G_n of finite index are open.

Let the profinite group G be a compact p-adic analytic group. Then G contains an open subgroup H, such that H is a pro-p group with finite cohomological p-dimension; that is, $\mathrm{cd}_p(H) = m$, for some non-negative integer m (see [25, 2.4.9] or the exposition in [39]). Since H is pro-p, $\mathrm{cd}_q(H) = 0$, whenever q is a prime different from p [45, Prop. 11.1.4]. Also, if M is a discrete H-module, then, for $s \geq 1$, $H^s_c(H; M)$ is a torsion abelian group [35, Cor. 6.7.4]. These facts imply that, for any discrete H-module M, $H^s_c(H; M) = 0$, whenever $s > m + 1$. We express this conclusion by saying that G has finite virtual cohomological dimension and we write $\mathrm{vcd}(G) \leq m$. Also, if K is a closed subgroup of G, $H \cap K$ is an open pro-p subgroup of K with $\mathrm{cd}_p(H \cap K) \leq m$, so that $\mathrm{vcd}(K) \leq m$, and thus, m is a uniform bound independent of K.

Now we state various results related to towers of abelian groups and continuous cohomology. The lemma below follows from the fact that an exact additive functor preserves images.

Lemma 2.10. *Let $F\colon \mathbf{Ab} \to \mathbf{Ab}$ be an exact additive functor. If $\{A_i\}_{i \geq 0}$ is a tower of abelian groups that satisfies the Mittag-Leffler condition, then so does the tower $\{F(A_i)\}$.*

Remark 2.11. Let G be a profinite group. The functor $\mathrm{Map}_c(G, -)\colon \mathbf{Ab} \to \mathbf{Ab}$, which sends A to $\mathrm{Map}_c(G, A)$, is defined by giving A the discrete topology. The isomorphism $\mathrm{Map}_c(G, A) \cong \mathrm{colim}_N \prod_{G/N} A$ shows that $\mathrm{Map}_c(G, -)$ is an exact additive functor. Later, we will use Lemma 2.10 with this functor.

The next lemma is a consequence of the fact that limits in $\mathbf{Ab}$ and in topological spaces are created in $\mathbf{Sets}$.

Lemma 2.12. *Let $M = \lim_\alpha M_\alpha$ be an inverse limit of discrete abelian groups, so that M is an abelian topological group. Let H be any profinite group. Then $\mathrm{Map}_c(H, M) \to \lim_\alpha \mathrm{Map}_c(H, M_\alpha)$ is an isomorphism of abelian groups.*

The lemma below follows from the fact that $\pi_t(E_n \wedge M_I) \cong \pi_t(E_n)/I$ is finite.

Lemma 2.13. *If X is a finite spectrum and t is any integer, then the abelian group $\pi_t(E_n \wedge M_I \wedge X)$ is finite.*

Corollary 2.14 ([14, pg. 116]). *If X is a finite spectrum, then $\pi_t(E_n \wedge X) \cong \lim_I \pi_t(E_n \wedge M_I \wedge X)$.*

We recall the definition of a second version of continuous cohomology [20].

Definition 2.15. Let C_G be the category of discrete G-modules and $\mathbf{tow}(C_G)$ the category of towers in C_G. Then $H^s_{\mathrm{cont}}(G;\{M_i\})$, the continuous cohomology of G with coefficients in the tower $\{M_i\}$, is the sth right derived functor of the left exact functor $\lim_i(-)^G \colon \mathbf{tow}(C_G) \to \mathbf{Ab}$, which sends $\{M_i\}$ to $\lim_i M_i^G$. By [20, Theorem 2.2], if the tower of abelian groups $\{M_i\}$ satisfies the Mittag-Leffler condition, then $H^s_{\mathrm{cont}}(G;\{M_i\}) \cong H^s_{\mathrm{cts}}(G;\lim_i M_i)$, for $s \geq 0$, where $H^s_{\mathrm{cts}}(G;M)$ is the cohomology of continuous cochains with coefficients in the topological G-module M (see [40, §2]).

Theorem 2.16 ([20, (2.1)])**.** *Let $\{M_i\}_{i\geq 0}$ be a tower of discrete G-modules satisfying the Mittag-Leffler condition. Then, for each $s \geq 0$, there is a short exact sequence*

$$0 \to \lim_i{}^1 H_c^{s-1}(G;M_i) \to H^s_{\mathrm{cont}}(G;\{M_i\}) \to \lim_i H^s_c(G;M_i) \to 0,$$

where $H_c^{-1}(G;-) = 0$.

Definition 2.17. Let G be a closed subgroup of G_n, let X be a finite spectrum, and let $I_n = (p, u_1, ..., u_{n-1}) \subset E_{n*}$. Then, by [5, Rk. 1.3],

$$\pi_t(E_n \wedge X) \cong \lim_k \pi_t(E_n \wedge X)/I_n^k \pi_t(E_n \wedge X)$$

is a profinite continuous $\mathbb{Z}_p[\![G]\!]$-module (since it is the inverse limit of finite discrete G-modules), and the definition of $H^s_c(G;\pi_t(E_n \wedge X))$ is given by

$$H^s_c(G;\pi_t(E_n \wedge X)) = \lim_k H^s_c(G;\pi_t(E_n \wedge X)/I_n^k \pi_t(E_n \wedge X)).$$

By [5, Rk. 1.3], for $s \geq 0$, there are isomorphisms

$$H^s_c(G;\pi_t(E_n\wedge X)) \cong H^s_{\mathrm{cts}}(G;\pi_t(E_n\wedge X)) \cong H^s_{\mathrm{cont}}(G;\{\pi_t(E_n\wedge X)/I_n^k\pi_t(E_n \wedge X)\}).$$

3. The model category of discrete G-spectra

In this section, we begin explaining the theory of homotopy fixed points for discrete G-spectra. We note that much of this theory (in this section and in Sections 5, 7, and 8, through Theorem 8.5) is already known, in some form, especially in the work of Jardine mentioned above, in the excellent article [29], by Mitchell (see also the opening remark of [31, §5]), and in Goerss's paper [11]. However, since the above theory has not been explained in detail before, using the language of homotopy fixed points for discrete G-spectra, we give a presentation of it.

A pointed simplicial discrete G-set is a pointed simplicial set that is a simplicial discrete G-set, such that the G-action fixes the basepoint.

Definition 3.1. A *discrete G-spectrum* X is a spectrum of pointed simplicial sets X_k, for $k \geq 0$, such that each simplicial set X_k is a pointed simplicial discrete G-set, and each bonding map $S^1 \wedge X_k \to X_{k+1}$ is G-equivariant (S^1 has trivial G-action). Let Spt_G denote the category of discrete G-spectra, where the morphisms are G-equivariant maps of spectra.

As with discrete G-sets, if $X \in \mathrm{Spt}_G$, there is a G-equivariant isomorphism $X \cong \mathrm{colim}_N X^N$. Also, a discrete G-spectrum X is a continuous G-spectrum since, for all $k, l \geq 0$, the set $X_{k,l}$ is a continuous G-space with the discrete topology, and all the face and degeneracy maps are (trivially) continuous.

Definition 3.2. As in [23, §6.2], let $G-\mathbf{Sets}_{df}$ be the canonical site of finite discrete G-sets. The pretopology of $G-\mathbf{Sets}_{df}$ is given by covering families of the form $\{f_\alpha\colon S_\alpha \to S\}$, a finite set of G-equivariant functions in $G-\mathbf{Sets}_{df}$ for a fixed $S \in G-\mathbf{Sets}_{df}$, such that $\coprod_\alpha S_\alpha \to S$ is a surjection.

Let **Shv** be the Grothendieck topos consisting of sheaves of sets on the site $G-\mathbf{Sets}_{df}$. The topos **Shv** has a unique point $u\colon \mathbf{Sets} \to \mathbf{Shv}$. The left and right adjoints, respectively, of the topos morphism u are

$$u^*\colon \mathbf{Shv} \to \mathbf{Sets}, \quad \mathcal{F} \mapsto \operatorname*{colim}_N \mathcal{F}(G/N), \quad \text{and}$$

$$u_*\colon \mathbf{Sets} \to \mathbf{Shv}, \quad X \mapsto \mathrm{Hom}_G(-, \mathrm{Map}_c(G, X))$$

[23, Rk. 6.25]. The G-action on the discrete G-set $\mathrm{Map}_c(G, X)$ is defined by $(g \cdot f)(g') = f(g'g)$, for g, g' in G, and f a continuous map $G \to X$, where X is given the discrete topology.

Recall that the topos **Shv** is equivalent to T_G, the category of discrete G-sets (see [23, Prop. 6.20] or [26, III-9, Thm. 1], for example). The functor $\mathrm{Map}_c(G, -)\colon \mathbf{Sets} \to T_G$ prolongs to the functor $\mathrm{Map}_c(G, -)\colon \mathrm{Spt} \to \mathrm{Spt}_G$. Thus, if X is a spectrum, then $\mathrm{Map}_c(G, X) \cong \mathrm{colim}_N \prod_{G/N} X$ is the discrete G-spectrum with $(\mathrm{Map}_c(G, X))_k = \mathrm{Map}_c(G, X_k)$, where $\mathrm{Map}_c(G, X_k)$ is a pointed simplicial set, with l-simplices $\mathrm{Map}_c(G, X_{k,l})$ and basepoint $G \to *$, where $X_{k,l}$ is regarded as a discrete set. The G-action on $\mathrm{Map}_c(G, X)$ is defined by the G-action on the sets $\mathrm{Map}_c(G, X_{k,l})$. It is not hard to see that $\mathrm{Map}_c(G, -)$ is right adjoint to the forgetful functor $U\colon \mathrm{Spt}_G \to \mathrm{Spt}$.

Note that if $X \in \mathrm{Spt}_G$, then $\mathrm{Hom}_G(-, X)\colon (G-\mathbf{Sets}_{df})^{\mathrm{op}} \to \mathrm{Spt}$ is a presheaf, such that, for $S \in G-\mathbf{Sets}_{df}$, $\mathrm{Hom}_G(S, X) \in \mathrm{Spt}$ satisfies $\mathrm{Hom}_G(S, X)_{k,l} = \mathrm{Hom}_G(S, X_{k,l})$, a pointed set with basepoint $S \to *$.

Let **ShvSpt** be the category of sheaves of spectra on the site $G-\mathbf{Sets}_{df}$. A sheaf of spectra $\mathcal{F}$ is a presheaf $\mathcal{F}\colon (G-\mathbf{Sets}_{df})^{\mathrm{op}} \to \mathrm{Spt}$, such that, for any $S \in G-\mathbf{Sets}_{df}$ and any covering family $\{f_\alpha\colon S_\alpha \to S\}$, the usual diagram (of spectra) is an equalizer. Equivalently, a sheaf of spectra $\mathcal{F}$ consists of pointed simplicial sheaves $\mathcal{F}^n$, together with pointed maps of simplicial presheaves $\sigma\colon S^1 \wedge \mathcal{F}^n \to \mathcal{F}^{n+1}$, for $n \geq 0$, where S^1 is the constant simplicial presheaf. A morphism between sheaves of spectra is a natural transformation between the underlying presheaves.

We equip the category **PreSpt** of presheaves of spectra on the site $G-\mathbf{Sets}_{df}$ with the stable model category structure (see [22], [23, §2.3]). Recall that, in this model category structure, a map $h\colon \mathcal{F} \to \mathcal{G}$ of presheaves of spectra is a weak equivalence if and only if the associated map of stalks $u^*(\mathcal{F}) \to u^*(\mathcal{G})$ is a weak equivalence of spectra, where $u^*(\mathcal{F}) = \mathrm{colim}_N \mathcal{F}(G/N)$, by prolongation.

Definition 3.3. In the stable model category structure, fibrant presheaves are often referred to as *globally fibrant*, and if $\mathcal{F} \to \mathcal{G}$ is a weak equivalence of presheaves, with $\mathcal{G}$ globally fibrant, then $\mathcal{G}$ is a *globally fibrant model* for $\mathcal{F}$.

We recall the following fact, which is especially useful when $S = *$.

Lemma 3.4. *Let $S \in G-\mathbf{Sets}_{df}$. The S-sections functor* **PreSpt** $\to$ Spt*, defined by $\mathcal{F} \mapsto \mathcal{F}(S)$, preserves fibrations, trivial fibrations, and weak equivalences between fibrant objects.*

Proof. The S-sections functor has a left adjoint, obtained by left Kan extension, that preserves cofibrations and weak equivalences. See [23, pg. 60] and [29, Cor. 3.16] for the details. □

Let $\mathcal{L}^2$ denote the sheafification functor for presheaves of sets, simplicial presheaves, and presheaves of spectra, so that $\mathcal{L}^2\mathcal{F} \cong \mathrm{Hom}_G(-, u^*(\mathcal{F}))$, by [23, Cor. 6.22]. Then $i\colon \mathbf{ShvSpt} \to \mathbf{PreSpt}$, the inclusion functor, is right adjoint to $\mathcal{L}^2$. By [10, Rk. 3.11], **ShvSpt** has the following model category structure. A map $h\colon \mathcal{F} \to \mathcal{G}$ of sheaves of spectra is a weak equivalence (fibration) if and only if $i(f)$ is a weak equivalence (fibration) of presheaves. Also, h is a cofibration of sheaves of spectra if the following holds:

(1) the map $h^0\colon \mathcal{F}^0 \to \mathcal{G}^0$ is a cofibration of simplicial presheaves; and
(2) for each $n \geq 0$, the canonical map $\mathcal{L}^2((S^1 \wedge \mathcal{G}^n) \cup_{S^1 \wedge \mathcal{F}^n} \mathcal{F}^{n+1}) \to \mathcal{G}^{n+1}$ is a cofibration of simplicial presheaves.

Since i preserves weak equivalences and fibrations, $(\mathcal{L}^2, i)$ is a Quillen pair for $(\mathbf{PreSpt}, \mathbf{ShvSpt})$. Thus, for $\mathcal{F} \in \mathbf{PreSpt}$, $\mathcal{F} \to \mathcal{L}^2\mathcal{F}$ is a weak equivalence, and $\mathrm{Ho}(\mathbf{PreSpt}) \cong \mathrm{Ho}(\mathbf{ShvSpt})$ is a Quillen equivalence.

Since $\mathbf{Shv} \cong T_G$, prolongation gives an equivalence of categories $\mathbf{ShvSpt} \cong \mathrm{Spt}_G$, via the functors $u^*\colon \mathbf{ShvSpt} \to \mathrm{Spt}_G$, and $R\colon \mathrm{Spt}_G \to \mathbf{ShvSpt}$, where $R(X) = \mathrm{Hom}_G(-, X)$.

Definition 3.5. For the remainder of this paper, if X is a discrete G-set, a simplicial discrete G-set, or a discrete G-spectrum, we let $u^*X = \mathrm{colim}_N\, X^N$.

Exploiting the above equivalence, we make Spt_G a model category in the following way. Define a map f of discrete G-spectra to be a weak equivalence (fibration) if and only if $\mathrm{Hom}_G(-, f)$ is a weak equivalence (fibration) of sheaves of spectra. Also, define f to be a cofibration if and only if f has the left lifting property with respect to all trivial fibrations. Thus, f is a cofibration if and only if $\mathrm{Hom}_G(-, f)$ is a cofibration in **ShvSpt**. Using this, it is immediate that Spt_G is a model category, and there is a Quillen equivalence $\mathrm{Ho}(\mathbf{ShvSpt}) \cong \mathrm{Ho}(\mathrm{Spt}_G)$.

In the theorem below, we define the model category structure of Spt_G without reference to sheaves of spectra. This extends the model category structure on the category S_G (the category of simplicial objects in T_G), that is given in [11, Thm. 1.12], to Spt_G.

Theorem 3.6. *Let $f\colon X \to Y$ be a map in Spt_G. Then f is a weak equivalence (cofibration) in Spt_G if and only if f is a weak equivalence (cofibration) in* Spt.

Proof. For weak equivalences, the statement is clearly true. Assume that f is a cofibration in Spt_G. Since $\mathrm{Hom}_G(-, X_0) \to \mathrm{Hom}_G(-, Y_0)$ is a cofibration of simplicial presheaves, evaluation at G/N implies that $X_0^N \to Y_0^N$ is a cofibration in $\mathcal{S}$. Thus, $X_0 \cong u^*X_0 \to u^*Y_0 \cong Y_0$ is a cofibration in $\mathcal{S}$.

Since colimits commute with pushouts,

$$\mathrm{Hom}_G(-, (S^1 \wedge Y_n) \cup_{S^1 \wedge X_n} X_{n+1}) \to \mathrm{Hom}_G(-, Y_{n+1})$$

is a cofibration of simplicial presheaves, and hence, the map of simplicial sets $u^*((S^1 \wedge Y_n) \cup_{S^1 \wedge X_n} X_{n+1}) \to Y_{n+1}$ is a cofibration.

Let W be a simplicial pointed discrete G-set. Then $S^1 \wedge W \cong \mathrm{colim}_N(S^1 \wedge W^N)$, so that $S^1 \wedge W$ is also a simplicial pointed discrete G-set. Since the forgetful functor

$U\colon T_G \to \mathbf{Sets}$ is a left adjoint, pushouts in T_G are formed in **Sets**, and thus, there is an isomorphism

$$u^*((S^1 \wedge Y_n) \cup_{S^1 \wedge X_n} X_{n+1}) \cong (S^1 \wedge Y_n) \cup_{S^1 \wedge X_n} X_{n+1}$$

of simplicial discrete G-sets. Hence, $(S^1 \wedge Y_n) \cup_{S^1 \wedge X_n} X_{n+1} \to Y_{n+1}$ is a cofibration in $\mathcal{S}$, and f is a cofibration in Spt.

The converse follows from the fact that if j is an injection of simplicial discrete G-sets, then $\mathrm{Hom}_G(-, j)$ is a cofibration of simplicial presheaves. □

The preceding theorem implies the following two corollaries.

Corollary 3.7. *If $f\colon X \to Y$ is a weak equivalence (cofibration) in Spt_G, then, for any $K <_c G$, f is a weak equivalence (cofibration) in Spt_K.*

Corollary 3.8. *The functors $(U, \mathrm{Map}_c(G,-))$ are a Quillen pair for the categories $(\mathrm{Spt}_G, \mathrm{Spt})$.*

Let $\mathrm{t}\colon \mathrm{Spt} \to \mathrm{Spt}_G$ give a spectrum trivial G-action, so that $\mathrm{t}(X) = X$. The right adjoint of t is the fixed points functor $(-)^G$. Clearly, t preserves all weak equivalences and cofibrations, giving the next result.

Corollary 3.9. *The functors $(\mathrm{t}, (-)^G)$ are a Quillen pair for $(\mathrm{Spt}, \mathrm{Spt}_G)$.*

We conclude this section with a few more useful facts about discrete G-spectra.

Lemma 3.10. *If $f\colon X \to Y$ is a fibration in Spt_G, then it is a fibration in Spt. In particular, if X is fibrant in Spt_G, then X is fibrant in Spt.*

Proof. Since $\mathrm{Hom}_G(-, f)$ is a fibration of presheaves of spectra, $\mathrm{Hom}_G(G/N, f)$ is a fibration of spectra for each open normal subgroup N. Thus, $\mathrm{colim}_N \mathrm{Hom}_G(G/N, f)$ is a fibration of spectra. Then the lemma follows from factoring f as $X \cong u^*X \to u^*Y \cong Y$. □

The next lemma and its corollary, whose elementary proofs are omitted, show that the homotopy groups of a discrete G-spectrum are discrete G-modules.

Lemma 3.11. *If X is a pointed Kan complex and a simplicial discrete G-set, then $\pi_n(X)$ is a discrete G-module, for all $n \geq 2$.*

Corollary 3.12. *If X is a discrete G-spectrum, then $\pi_n(X)$ is a discrete G-module for any integer n.*

The following observation says that certain elementary constructions with discrete G-spectra yield discrete G-spectra.

Lemma 3.13. *Given a profinite group $G \cong \lim_N G/N$, let $\{X_N\}_N$ be a directed system of spectra, such that each X_N is a G/N-spectrum and the maps are G-equivariant. Then $\mathrm{colim}_N X_N$ is a discrete G-spectrum. If Y is a spectrum with trivial G-action, then $(\mathrm{colim}_N X_N) \wedge Y \cong \mathrm{colim}_N(X_N \wedge Y)$ is a G-equivariant isomorphism of discrete G-spectra. Thus, if X is a discrete G-spectrum, then $X \wedge Y$ is a discrete G-spectrum.*

The corollary below is very useful later on.

Corollary 3.14. *The spectra F_n, $F_n \wedge M_I$, and $F_n \wedge M_I \wedge X$, for any spectrum X, are discrete G_n-spectra.*

4. Towers of discrete G-spectra and continuous G-spectra

Let $\mathbf{tow}(\mathrm{Spt}_G)$ be the category where a typical object $\{X_i\}$ is a tower

$$\cdots \to X_i \to X_{i-1} \to \cdots \to X_1 \to X_0$$

in Spt_G. The morphisms are natural transformations $\{X_i\} \to \{Y_i\}$, such that each $X_i \to Y_i$ is G-equivariant. Since Spt_G is a simplicial model category, [13, VI, Prop. 1.3] shows that $\mathbf{tow}(\mathrm{Spt}_G)$ is also a simplicial model category, where $\{f_i\}$ is a weak equivalence (cofibration) if and only if each f_i is a weak equivalence (cofibration) in Spt_G. By [13, VI, Rk. 1.5], if $\{X_i\}$ is fibrant in $\mathbf{tow}(\mathrm{Spt}_G)$, then each map $X_i \to X_{i-1}$ in the tower is a fibration and each X_i is fibrant, all in Spt_G.

Definition 4.1. Let $\{X_i\}$ be in $\mathbf{tow}(\mathrm{Spt}_G)$. Then $\{X'_i\}$ denotes the target of a trivial cofibration $\{X_i\} \to \{X'_i\}$, with $\{X'_i\}$ fibrant, in $\mathbf{tow}(\mathrm{Spt}_G)$.

Remark 4.2. Let $\{X_\alpha\}$ be a diagram in Spt_G. Since there is an isomorphism $\lim_\alpha^{\mathrm{Spt}_G} X_\alpha \cong u^*(\lim_\alpha^{\mathrm{Spt}} X_\alpha)$, limits in Spt_G are not formed in Spt. For the rest of this paper, lim is always formed in Spt and not in Spt_G.

The functor $\lim_i(-)^G \colon \mathbf{tow}(\mathrm{Spt}_G) \to \mathrm{Spt}$, given by $\{X_i\} \mapsto \lim_i X_i^G$, is right adjoint to the functor $\underline{\mathrm{t}} \colon \mathrm{Spt} \to \mathbf{tow}(\mathrm{Spt}_G)$ that sends a spectrum X to the constant diagram $\{X\}$, where X has trivial G-action. Since $\underline{\mathrm{t}}$ preserves all weak equivalences and cofibrations, we have the following fact.

Lemma 4.3. *The functors $(\underline{\mathrm{t}}, \lim_i(-)^G)$ are a Quillen pair for the categories $(\mathrm{Spt}, \mathbf{tow}(\mathrm{Spt}_G))$.*

This lemma implies the existence of the total right derived functor

$$\mathbf{R}(\lim_i(-)^G) \colon \mathrm{Ho}(\mathbf{tow}(\mathrm{Spt}_G)) \to \mathrm{Ho}(\mathrm{Spt}), \quad \{X_i\} \mapsto \lim_i (X'_i)^G.$$

Lemma 4.4. *If $\{X_i\}$ in $\mathbf{tow}(\mathrm{Spt}_G)$ is a tower of fibrant spectra, then there are weak equivalences* $\mathrm{holim}_i X_i \xrightarrow{p} \mathrm{holim}_i X'_i \xleftarrow{q} \lim_i X'_i$.

Proof. Since each $X_i \to X'_i$ is a weak equivalence between fibrant spectra, p is a weak equivalence. Since $X'_i \to X'_{i-1}$ is a fibration in Spt, for $i \geq 1$, q is a weak equivalence. □

We use this lemma to define the continuous G-spectra that we will study.

Definition 4.5. If $\{X_i\} \in \mathbf{tow}(\mathrm{Spt}_G)$, then the inverse limit $\lim_i X_i$ is a *continuous G-spectrum*. Also, if $\{X_i\} \in \mathbf{tow}(\mathrm{Spt}_G)$ is a tower of fibrant spectra, we call $\mathrm{holim}_i X_i$ a *continuous G-spectrum*, due to the zigzag of Lemma 4.4 relating $\mathrm{holim}_i X_i$ to the continuous G-spectrum $\lim_i X'_i$. Note that if $X \in \mathrm{Spt}_G$, then, using the constant tower on X, $X \cong \lim_i X$ is a continuous G-spectrum.

Remark 4.6. Sometimes we will use the term "continuous G-spectrum" more loosely. Let X be a continuous G-spectrum, as in Definition 4.5. If Y is a G-spectrum that is isomorphic to X, in the stable category, with compatible G-actions, then we call Y a continuous G-spectrum.

We make a few more comments about Definition 4.5. Though the definition is not as general as it could be, it is sufficient for our applications. The inverse limit is central to the definition since the inverse limit of a tower of discrete G-sets is a topological G-space.

Given any tower $X_* = \{X_i\}$ in Spt_G, $\mathrm{holim}_i X_i = \mathrm{Tot}(\prod^* X_*) \cong \lim_n \mathrm{T}(n)$, where $\mathrm{T}(n) = \mathrm{Tot}_n(\prod^* X_*)$. (See §5 for the definition of $\prod^* X_*$, and [2] is a reference for any undefined notation in this paragraph.) Then it is natural to ask if $T(n)$ is in Spt_G, so that $\mathrm{holim}_i X_i$ is canonically a continuous G-spectrum. For this to be true, it must be that, for any $m \geq 0$, the simplicial set

$$\mathrm{T}(n)_m = \mathrm{Map}_{c\mathcal{S}}(\mathrm{sk}_n\Delta[-], \prod^*(X_*)_m) \in \mathcal{S}_G.$$

If, for all $s \geq 0$, $\prod^s (X_*)_m \in \mathcal{S}_G$, then $\mathrm{T}(n)_m \in \mathcal{S}_G$, by [11, pg. 212]. However, the infinite product $\prod^s(X_*)_m$ need not be in $\mathcal{S}_G$, and thus, in general, $\mathrm{T}(n) \notin \mathrm{Spt}_G$. Therefore, $\mathrm{holim}_i X_i$ is not always identifiable with a continuous G-spectrum, in the above way.

5. Homotopy fixed points for discrete G-spectra

In this section, we define the homotopy fixed point spectrum for $X \in \mathrm{Spt}_G$. We begin by recalling the homotopy spectral sequence, since we use it often.

If J is a small category, $P\colon J \to \mathrm{Spt}$ a diagram of fibrant spectra, and Z any spectrum, then there is a conditionally convergent spectral sequence

$$E_2^{s,t} = \lim_J{}^{\mathrm{s}} [Z, P]_t \Rightarrow [Z, \mathrm{holim}_J P]_{t-s}, \tag{5.1}$$

where $\lim^{\mathrm{s}}$ is the sth right derived functor of $\lim_J\colon \mathbf{Ab}^J \to \mathbf{Ab}$ (see [2], [41, Prop. 5.13, Lem. 5.31]).

Associated to P is the cosimplicial spectrum $\prod^* P$, with $\prod^n P = \prod_{(B\Delta)_n} P(j_n)$, where the n-simplices of the nerve $B\Delta$ consist of all strings $[j_0] \to \cdots \to [j_n]$ of n morphisms in Δ. For any $k \geq 0$, $(\prod^* P)_k$ is a fibrant cosimplicial pointed simplicial set, and $\mathrm{holim}_J P = \mathrm{Tot}(\prod^* P)$.

If P is a cosimplicial diagram of fibrant spectra (a *cosimplicial fibrant spectrum*), then $E_2^{s,t} = \pi^s[Z,P]_t$, the sth cohomotopy group of the cosimplicial abelian group $[Z,P]_t$.

Definition 5.2. Let $X \in \mathrm{Spt}_G$. In Spt_G, factor $X \to *$ as $X \to X_{f,G} \to *$, the composition of a trivial cofibration in Spt_G, followed by a fibration in Spt_G. Then we define $X^{hG} = (X_{f,G})^G$, and X^{hG} is the *homotopy fixed point spectrum* of X with respect to the continuous action of G. We write X_f instead of $X_{f,G}$ when doing so causes no confusion.

Note that $X^{hG} = \mathrm{Hom}_G(*, X_{f,G})$, the global sections of the presheaf of spectra $\mathrm{Hom}_G(-, X_{f,G})$. This definition has been developed in other categories: see [11] for simplicial discrete G-sets, [21] for simplicial presheaves, and [23, Ch. 6] for presheaves of spectra.

As expected, the definition of homotopy fixed points for a profinite group generalizes the definition for a finite group. Let G be a finite group, X a G-spectrum, and let $X \to X_{\mathtt{f}}$ be a weak equivalence that is G-equivariant, with $X_{\mathtt{f}}$ a fibrant spectrum. Then, since G is finite, the homotopy fixed point spectrum $X^{h'G} = \mathrm{Map}_G(EG_+, X_{\mathtt{f}})$ can be defined to be $\mathrm{holim}_G X_{\mathtt{f}}$ (as in the Introduction). Note that there is a descent spectral sequence

$$E_2^{s,t} = \lim_G{}^{\mathrm{s}} \pi_t(X) \cong H^s(G; \pi_t(X)) \Rightarrow \pi_{t-s}(X^{h'G}).$$

Since G is profinite, X is a discrete G-spectrum, and $X_{f,G}$ is a fibrant spectrum, so that $X^{h'G} = \mathrm{holim}_G X_f$. Then, by [23, Prop. 6.39], the canonical map $X^{hG} = \lim_G X_f \to \mathrm{holim}_G X_f = X^{h'G}$ is a weak equivalence, as desired.

The next lemma follows from Corollary 3.9.

Lemma 5.3. *The homotopy fixed points functor $(-)^{hG} \colon \mathrm{Ho}(\mathrm{Spt}_G) \to \mathrm{Ho}(\mathrm{Spt})$ is the total right derived functor of the fixed points functor $(-)^G \colon \mathrm{Spt}_G \to \mathrm{Spt}$. In particular, if $X \to Y$ is a weak equivalence of discrete G-spectra, then $X^{hG} \to Y^{hG}$ is a weak equivalence.*

6. E_n is a continuous G_n-spectrum

We show that the Lubin-Tate spectrum E_n is a continuous G_n-spectrum by successively eliminating simpler ways of constructing a continuous action, and by applying the theory of the previous section.

First of all, since the profinite ring $\pi_0(E_n)$ is not a discrete G_n-module, Corollary 3.12 implies the following observation.

Lemma 6.1. *E_n is not a discrete G_n-spectrum.*

However, note that, for $k \in \mathbb{Z}$, $\pi_{2k}(E_n \wedge M_I)$ is a finite discrete G_n-module so that the action factors through a finite quotient G_n/U_I, where U_I is some open normal subgroup (see [35, Lem. 1.1.16]). Thus, $\pi_{2k}(E_n \wedge M_I)$ is a G_n/U_I-module, and one is led to ask if $E_n \wedge M_I$ is a G_n/U_I-spectrum. If so, then $E_n \wedge M_I$ is a discrete G_n-spectrum, and E_n is easily seen to be a continuous G_n-spectrum.

However, $E_n \wedge M_I$ is not a G_n/U-spectrum for all open normal subgroups U of G_n, as the lemma below shows. As far as the author knows, this lemma is due to Hopkins; the author learned the proof from Hal Sadofsky.

Lemma 6.2. *There is no open normal subgroup U of G_n such that the G_n-action on $E_n \wedge M_I$ factors through G_n/U.*

Proof. Suppose the G_n-spectrum $E_n \wedge M_I$ is a G_n/U-spectrum. Then the G_n-action on the middle factor of $E_n \wedge E_n \wedge M_I$ factors through G_n/U, so that $\pi_*(E_n \wedge E_n \wedge M_I)$ is a G_n/U-module. Note that $\pi_*(E_n \wedge E_n \wedge M_I) \cong \mathrm{Map}^{\ell}_c(G_n, E_{n*}/I)$.

Since the G_n-module $\mathrm{Map}^{\ell}_c(G_n, E_{n*}/I)$ is a G_n/U-module, there is an isomorphism of sets $\mathrm{Map}^{\ell}_c(G_n, E_{n,0}/I) = \mathrm{Map}^{\ell}_c(G_n, E_{n,0}/I)^U \cong \mathrm{Map}_c(G_n/U, E_{n,0}/I)$. But the first set is infinite and the last is finite, a contradiction. □

Since $\pi_*(E_n \wedge M_I)$ is a discrete G_n-module, one can still hope for a spectrum $E_n/I \simeq E_n \wedge M_I$, such that E_n/I is a discrete G_n-spectrum.

To produce E_n/I, we make the following observation. By [23, Remark 6.26], since U_i is an open normal subgroup of G_n, the presheaf $\mathrm{Hom}_{U_i}(-, (E_n/I)_{f,G_n})$ is fibrant in the model category of presheaves of spectra on the site $U_i\text{--}\mathbf{Sets}_{df}$. Thus, for each i, the map $E_n/I \to (E_n/I)_{f,G_n}$ is a trivial cofibration, with fibrant target, all in Spt_{U_i}, so that $((E_n/I)_{f,G_n})^{U_i} = (E_n/I)^{hU_i}$.

Combining this fact with the idea, discussed in §1, that $E_n \wedge M_I$ has homotopy fixed point spectra $(E_n \wedge M_I)^{hU_i} \simeq E_n^{dhU_i} \wedge M_I \simeq (E_n/I)^{hU_i}$, we have:

$$\begin{aligned} E_n/I &\simeq (E_n/I)_{f,G_n} \cong \operatorname*{colim}_i ((E_n/I)_{f,G_n})^{U_i} = \operatorname*{colim}_i (E_n/I)^{hU_i} \\ &\simeq \operatorname*{colim}_i (E_n^{dhU_i} \wedge M_I) \cong F_n \wedge M_I. \end{aligned}$$

This argument suggests that $E_n \wedge M_I$ has the homotopy type of the discrete G_n-spectrum $F_n \wedge M_I$. To show that this is indeed the case, we consider the spectrum F_n in more detail. The key result is the following theorem, due to Devinatz and Hopkins.

Theorem 6.3 ([5]). *There is a weak equivalence $E_n \simeq \hat{L}(F_n)$.*

Proof. By [5, Thm. 3], $E_n^{h'\{e\}} \simeq E_n^{dh\{e\}}$. (We remark that this weak equivalence is far from obvious.) By [5, Definition 1.5], $E_n^{dh\{e\}} = \hat{L}(\operatorname{hocolim}_i E_n^{dhU_i})$, where the homotopy colimit is in the category E_∞. Then, by [5, Remark 1.6, Lemma 6.2], $\operatorname{hocolim}_i E_n^{dhU_i} \simeq \operatorname{colim}_i E_n^{dhU_i}$, where the colimit is in $\mathcal{M}_S$. Thus, as spectra in Spt, $E_n^{dh\{e\}} \simeq \hat{L}(F_n)$, so that $E_n \simeq E_n^{h'\{e\}} \simeq \hat{L}(F_n)$. □

Corollary 6.4. *In the stable category, there are isomorphisms*

$$E_n \cong \operatorname*{holim}_I (F_n \wedge M_I)_{\mathfrak{f}} \cong \operatorname*{holim}_I (E_n \wedge M_I)_{\mathfrak{f}}.$$

The following result shows that $E_n \wedge M_I \simeq F_n \wedge M_I$. This weak equivalence and $\operatorname{vcd}(G_n) < \infty$ are the main facts that make it possible to construct the homotopy fixed point spectra of E_n.

Corollary 6.5. *If Y is a finite spectrum of type n, then the G_n-equivariant map $F_n \wedge Y \to E_n \wedge Y$ is a weak equivalence. In particular, $E_n \wedge M_I \simeq F_n \wedge M_I$.*

Proof. We have $E_n \wedge Y \simeq \hat{L}(F_n) \wedge Y \simeq L_n(F_n) \wedge Y \simeq F_n \wedge Y$. □

Now we show that E_n is a continuous G_n-spectrum.

Theorem 6.6. *There is an isomorphism $E_n \cong \operatorname{holim}_I (F_n \wedge M_I)_{f,G_n}$. Thus, E_n is a continuous G_n-spectrum.*

Proof. By Corollary 6.4, $E_n \cong \operatorname{holim}_I (F_n \wedge M_I)_{\mathfrak{f}}$. By functorial fibrant replacement, the map of towers $\{F_n \wedge M_I\} \to \{(F_n \wedge M_I)_{f,G_n}\}$ induces a map of towers

$$\{(F_n \wedge M_I)_{\mathfrak{f}}\} \to \{((F_n \wedge M_I)_{f,G_n})_{\mathfrak{f}}\}$$

and, hence, weak equivalences

$$\operatorname*{holim}_I (F_n \wedge M_I)_{f,G_n} \to \operatorname*{holim}_I ((F_n \wedge M_I)_{f,G_n})_{\mathfrak{f}} \leftarrow \operatorname*{holim}_I (F_n \wedge M_I)_{\mathfrak{f}}.$$

Thus, $\operatorname{holim}_I (F_n \wedge M_I)_{f,G_n}$ is isomorphic to $\operatorname{holim}_I (F_n \wedge M_I)_{\mathfrak{f}}$ and E_n. Since $\{(F_n \wedge M_I)_{f,G_n}\}$ is a tower of fibrant spectra, $\operatorname{holim}_I (F_n \wedge M_I)_{f,G_n}$ is a continuous G_n-spectrum. Then, by Remark 4.6, E_n is a continuous G_n-spectrum. □

We conclude this section with some observations about F_n.

Lemma 6.7. *The map $\eta\colon F_n \to E_n$ is not a weak equivalence and F_n is not $K(n)$-local.*

Proof. If η is a weak equivalence, then $\pi_0(\eta)$ is a G_n-equivariant isomorphism from a discrete G_n-module (with all orbits finite) to a non-finite profinite G_n-module, which is impossible. If F_n is $K(n)$-local, then $F_n \simeq \hat{L}(F_n) \simeq E_n$, and η is a weak equivalence, a contradiction. □

Lemma 6.8. *The maps $\hat{L}(F_n \wedge F_n) \to \hat{L}(E_n \wedge F_n) \to \hat{L}(E_n \wedge E_n)$ are weak equivalences.*

Proof. Since $F_n \wedge M_I \simeq E_n \wedge M_I$, $F_n \wedge F_n \wedge M_I \simeq E_n \wedge E_n \wedge M_I$. Since $F_n \wedge F_n$, $E_n \wedge F_n$ and $E_n \wedge E_n$ are $E(n)$-local, the result follows from Theorem 2.8. □

7. Homotopy fixed points when vcd(G) < ∞

In this section, G always has finite virtual cohomological dimension. Thus, there exists a uniform bound m, such that for all $K <_c G$, $\mathrm{vcd}(K) \leq m$. For $X \in \mathrm{Spt}_G$, we use this fact to give a model for X^{hG} that eases the construction of its descent spectral sequence. Also, this fact yields a second model for X^{hK} that is functorial in K.

Definition 7.1. Consider the functor

$$\Gamma_G = \mathrm{Map}_c(G, -) \circ U \colon \mathrm{Spt}_G \to \mathrm{Spt}_G, \quad X \mapsto \Gamma_G(X) = \mathrm{Map}_c(G, X),$$

where $\Gamma_G(X)$ has the G-action defined in §3. We write Γ instead of Γ_G, when G is understood from context. There is a G-equivariant monomorphism $i \colon X \to \Gamma X$ defined, on the level of sets, by $i(x)(g) = g \cdot x$. As in [44, 8.6.2], since U and $\mathrm{Map}_c(G, -)$ are adjoints, Γ forms a triple and there is a cosimplicial discrete G-spectrum $\Gamma^\bullet X$, with $(\Gamma^\bullet X)^k \cong \mathrm{Map}_c(G^{k+1}, X)$.

We recall the construction of Thomason's hypercohomology spectrum for the topos **Shv** of sheaves of sets on the site $G-\mathbf{Sets}_{df}$ (see Definition 3.2). ([41, 1.31-1.33] and [29, §1.3, §3.2] give more details about the hypercohomology spectrum.) Consider the functor $T = u_* u^* \colon \mathbf{ShvSpt} \to \mathbf{ShvSpt}$, which sends $\mathcal{F}$ to $\mathrm{Hom}_G(-, \mathrm{Map}_c(G, \mathrm{colim}_N\, \mathcal{F}(G/N)))$, obtained by composing the adjoints in the point of the topos. For $X \in \mathrm{Spt}_G$, $T(\mathrm{Hom}_G(-, X)) \cong \mathrm{Hom}_G(-, \mathrm{Map}_c(G, X))$. By iterating this isomorphism, the cosimplicial sheaf of spectra $T^\bullet \mathrm{Hom}_G(-, X)$ gives rise to the cosimplicial sheaf $\mathrm{Hom}_G(-, \Gamma^\bullet X)$.

Definition 7.2. Given $X \in \mathrm{Spt}_G$, the *presheaf of hypercohomology spectra of* $G-\mathbf{Sets}_{df}$ *with coefficients in* X is the presheaf of spectra

$$\mathbb{H}^\bullet(-; X) = \operatorname*{holim}_\Delta \mathrm{Hom}_G(-, \Gamma^\bullet X) \colon (G-\mathbf{Sets}_{df})^{\mathrm{op}} \to \mathrm{Spt},$$

and $\mathbb{H}^\bullet(S; X) = \mathrm{holim}_\Delta \mathrm{Hom}_G(S, \Gamma^\bullet X)$ is the hypercohomology spectrum of S with coefficients in X.

The map $X_f \to \Gamma^\bullet X_f$, induced by i, out of the constant cosimplicial diagram, and $\mathrm{Hom}_G(-, X_f) \to \lim_\Delta \mathrm{Hom}_G(-, X_f) \to \mathrm{holim}_\Delta \mathrm{Hom}_G(-, X_f)$ induce a canonical map $\mathrm{Hom}_G(-, X_f) \to \mathbb{H}^\bullet(-, X_f)$.

Now we show that $\mathbb{H}^\bullet(*, X_f)$ is a model for X^{hG}. Below, a *cosimplicial globally fibrant presheaf* is a cosimplicial presheaf of spectra that is globally fibrant at each level.

Lemma 7.3. *If* $\mathcal{F}^\bullet$ *is a cosimplicial globally fibrant presheaf, then* $\mathrm{holim}_\Delta \mathcal{F}^\bullet$ *is a globally fibrant presheaf.*

Proof. By [23, Rk. 2.35], this is equivalent to showing that, for each $n \geq 0$, (a) $\mathrm{holim}_\Delta (\mathcal{F}^\bullet)^n$ is a globally fibrant simplicial presheaf; and (b) the adjoint of the bonding map, the composition

$$\gamma \colon \operatorname*{holim}_\Delta (\mathcal{F}^\bullet)^n \to \Omega(\operatorname*{holim}_\Delta (\mathcal{F}^\bullet)^{n+1}) \cong \operatorname*{holim}_\Delta \Omega(\mathcal{F}^\bullet)^{n+1}$$

is a local weak equivalence of simplicial presheaves. Part (a), the difficult part of this lemma, is proven in [21, Prop. 3.3].

The map γ is a local weak equivalence if the map of stalks $u^*\gamma$ is a weak equivalence in $\mathcal{S}$, which is true if each $\gamma(G/N)$ is a weak equivalence. By [21, pg. 74], if P is a globally fibrant simplicial presheaf, then ΩP is too, so that $\sigma\colon (\mathcal{F}^\bullet)^n \to \Omega(\mathcal{F}^\bullet)^{n+1}$ is a cosimplicial diagram of local weak equivalences between globally fibrant simplicial presheaves. Thus, $\sigma(G/N)$ is a cosimplicial diagram of weak equivalences between Kan complexes, so that $\gamma(G/N) = \operatorname{holim}_\Delta \sigma(G/N)$ is indeed a weak equivalence. □

The following result is not original: it is basically a special case of [21, Prop. 3.3], versions of which appear in [11, §5], [29, Prop. 3.20], and [30, Prop. 6.1]. Since the result is central to our work, for the benefit of the reader, we give the details of the proof. For use now and later, we recall that, for any group G and any closed subgroup K, $H^s_c(K; \operatorname{Map}_c(G, A)) = 0$, when $s > 0$ and A is any discrete abelian group [45, Lemma 9.4.5]. Also, recall that $X_f = X_{f,G}$ (Definition 5.2).

Theorem 7.4. *Let G be a profinite group with $\operatorname{vcd}(G) \leq m$, and let X be a discrete G-spectrum. Then there are weak equivalences*

$$\operatorname{Hom}_G(-, X) \xrightarrow{\simeq} \operatorname{Hom}_G(-, X_f) \xrightarrow{\simeq} \operatorname*{holim}_\Delta \operatorname{Hom}_G(-, \Gamma^\bullet X_f),$$

and $\operatorname{holim}_\Delta \operatorname{Hom}_G(-, \Gamma^\bullet X_f)$ is a globally fibrant model for $\operatorname{Hom}_G(-, X)$. Thus, evaluation at $* \in G-\mathbf{Sets}_{df}$ gives a weak equivalence $X^{hG} \to \operatorname{holim}_\Delta (\Gamma^\bullet X_f)^G$.

Remark 7.5. The weak equivalence $X \cong \operatorname{colim}_N X^N \to \operatorname{colim}_N (X^N)_{\mathtt{f}}$, whose target is a discrete G-spectrum that is fibrant in Spt, induces a weak equivalence $X_{f,G} \to (\operatorname{colim}_N (X^N)_{\mathtt{f}})_{f,G}$. Thus, there are weak equivalences

$$\mathbb{H}^\bullet(*, X_f) \to \mathbb{H}^\bullet(*, (\operatorname*{colim}_N (X^N)_{\mathtt{f}})_{f,G}) \leftarrow \mathbb{H}^\bullet(*, \operatorname*{colim}_N (X^N)_{\mathtt{f}}),$$

so that $\mathbb{H}^\bullet(*, \operatorname{colim}_N (X^N)_{\mathtt{f}})$ is a model for X^{hG} that does not require the model category Spt_G for its construction.

Proof of Theorem 7.4. Since X_f is fibrant in Spt, ΓX_f is fibrant in Spt_G by Corollary 3.8. By iteration, $\operatorname{Hom}_G(-, \Gamma^\bullet X_f)$ is a cosimplicial globally fibrant presheaf, so that $\operatorname{holim}_\Delta \operatorname{Hom}_G(-, \Gamma^\bullet X_f)$ is globally fibrant, by Lemma 7.3. It only remains to show that $\lambda\colon \operatorname{Hom}_G(-, X) \to \operatorname{holim}_\Delta \operatorname{Hom}_G(-, \Gamma^\bullet X_f)$ is a weak equivalence.

By hypothesis, G contains an open subgroup H with $\operatorname{cd}(H) \leq m$. Then by [45, Lem. 0.3.2], H contains a subgroup K that is an open normal subgroup of G. Let $\{N\}$ be the collection of open normal subgroups of G. Let $N' = N \cap K$. Observe that $\{N'\}$ is a cofinal subcollection of open normal subgroups of G so that $G \cong \lim_{N'} G/N'$. Since $N' <_c H$, $\operatorname{cd}(N') \leq \operatorname{cd}(H)$. Thus, $H^s_c(N'; M) = 0$, for all $s > m + 1$, whenever M is a discrete N'-module. Henceforth, we drop the $'$ from N' to ease the notation: N is really $N \cap K$.

Any presheaf of sets $\mathcal{F}$ has stalk $\operatorname{colim}_N \mathcal{F}(G/N)$, so that λ is a weak equivalence if $\lambda_u\colon X \cong \operatorname{colim}_N X^N \to \operatorname{colim}_N \operatorname{holim}_\Delta (\Gamma^\bullet X_f)^N$ is a weak equivalence. Since $\operatorname{Hom}_G(-, \Gamma^\bullet X_f)$ is a cosimplicial globally fibrant spectrum, the diagram $(\Gamma^\bullet X_f)^N$ is a cosimplicial fibrant spectrum. Then, for each N, there is a conditionally convergent spectral sequence

$$E_2^{s,t}(N) = \pi^s \pi_t((\Gamma^\bullet X_f)^N) \Rightarrow \pi_{t-s}(\operatorname*{holim}_\Delta (\Gamma^\bullet X_f)^N). \tag{7.6}$$

Because $\pi_t(X)$ is a discrete G-module, we have

$$\pi_t(\operatorname{Map}_c(G, X_f)^N) \cong \pi_t(\operatorname{Map}_c(G/N, X_f)) \cong \textstyle\prod_{G/N} \pi_t(X) \cong \operatorname{Map}_c(G, \pi_t(X))^N$$

and $\pi_t(\mathrm{Map}_c(G, X_f)) \cong \pi_t(\mathrm{colim}_N \prod_{G/N} X_f) \cong \mathrm{Map}_c(G, \pi_t(X))$. By iterating such manipulations, we obtain $\pi^s\pi_t((\Gamma^\bullet X_f)^N) \cong H^s((\Gamma^*\pi_t(X))^N)$. The cochain complex $0 \to \pi_t(X) \to \Gamma^*\pi_t(X)$ of discrete N-modules is exact (see e.g. [11, pp. 210-211]), and, for $k \geq 1$ and $s > 0$,

$$H^s_c(N; \Gamma^k\pi_t(X)) \cong H^s_c(N; \mathrm{Map}_c(G, \Gamma^{k-1}\pi_t(X))) = 0.$$

Thus, the above cochain complex is a resolution of $\pi_t(X)$ by $(-)^N$-acyclic modules, so that $E_2^{s,t}(N) \cong H^s_c(N; \pi_t(X))$. Taking a colimit over $\{N\}$ of (7.6) gives the spectral sequence

$$E_2^{s,t} = \operatorname*{colim}_N H^s_c(N; \pi_t(X)) \Rightarrow \pi_{t-s}(\operatorname*{colim}_N \operatorname*{holim}_\Delta (\Gamma^\bullet X_f)^N). \tag{7.7}$$

Since $E_2^{s,*}(N) = 0$ whenever $s > m+1$, the E_2-terms $E_2(N)$ are uniformly bounded on the right. Therefore, by [29, Prop. 3.3], the colimit of the spectral sequences does converge to the colimit of the abutments, as asserted in (7.7).

Finally,

$$E_2^{*,t} \cong H^*_c(\lim_N N; \pi_t(X)) \cong H^*(\{e\}; \pi_t(X)),$$

which is isomorphic to $\pi_t(X)$, concentrated in degree zero. Thus, (7.7) collapses and for all t, $\pi_t(\mathrm{colim}_N \, \mathrm{holim}_\Delta(\Gamma^\bullet X_f)^N) \cong \pi_t(X)$, and hence, λ_u is a weak equivalence. □

Remark 7.8. Because of Theorem 7.4, if $\mathrm{vcd}(G) < \infty$ and X is a discrete G-spectrum, we make the identification

$$X^{hG} = \operatorname*{holim}_\Delta (\Gamma^\bullet_G X_{f,G})^G = \mathbb{H}^\bullet(*, X_{f,G}).$$

In [43, §2.14], an expression that is basically equivalent to $\mathrm{holim}_\Delta(\Gamma^\bullet_G X_{f,G})^G$ is defined to be the homotopy fixed point spectrum X^{hG}, even if $\mathrm{vcd}(G) = \infty$. This approach has the disadvantage that $(-)^{hG}$ need not always be the total right derived functor of $(-)^G$. Thus, we only make the identification of Remark 7.8 when $\mathrm{vcd}(G) < \infty$.

Now it is easy to construct the descent spectral sequence. Note that if X is a discrete G-spectrum, the proof of Theorem 7.4 shows that

$$\pi^s\pi_t((\Gamma^\bullet_G X)^G) \cong \pi^s((\Gamma^\bullet_G \pi_t(X))^G) \cong H^s_c(G; \pi_t(X)).$$

Theorem 7.9. *If $\mathrm{vcd}(G) < \infty$ and X is a discrete G-spectrum, then there is a conditionally convergent descent spectral sequence*

$$E_2^{s,t} = H^s_c(G; \pi_t(X)) \Rightarrow \pi_{t-s}(X^{hG}). \tag{7.10}$$

Proof. As in Theorem 7.4, $(\Gamma^\bullet X_f)^G$ is a cosimplicial fibrant spectrum. Thus, we can form the homotopy spectral sequence for $\pi_*(\mathrm{holim}_\Delta(\Gamma^\bullet X_f)^G)$. □

Remark 7.11. Spectral sequence (7.10) has been constructed in other contexts: for simplicial presheaves, presheaves of spectra, and S_G, see [21, Cor. 3.6], [23, §6.1], and [11, §§4, 5], respectively. In several of these examples, a Postnikov tower provides an alternative to the hypercohomology spectrum that we use. In all of these constructions of the descent spectral sequence, some kind of finiteness assumption is required in order to identify the homotopy groups of the abutment as being those of the homotopy fixed point spectrum.

Let X be a discrete G-spectrum. We now develop a second model for X^{hK}, where K is a closed subgroup of G, that is functorial in K.

The map $X \to X_{f,G}$ in Spt_K gives a weak equivalence $X^{hK} \to (X_{f,G})^{hK}$. Composition with the weak equivalence $(X_{f,G})^{hK} \to \mathrm{holim}_\Delta(\Gamma^\bullet_K((X_{f,G})_{f,K}))^K$ gives a weak equivalence $X^{hK} \to \mathrm{holim}_\Delta(\Gamma^\bullet_K((X_{f,G})_{f,K}))^K$ between fibrant spectra. The inclusion $K \to G$ induces a morphism $\Gamma_G(X_{f,G}) \to \Gamma_K(X_{f,G})$, giving a map $\Gamma^\bullet_G(X_{f,G}) \to \Gamma^\bullet_K(X_{f,G})$ of cosimplicial discrete K-spectra.

Lemma 7.12. *There is a weak equivalence*

$$\rho\colon \operatorname*{holim}_\Delta(\Gamma^\bullet_G(X_{f,G}))^K \to \operatorname*{holim}_\Delta(\Gamma^\bullet_K(X_{f,G}))^K \to \operatorname*{holim}_\Delta(\Gamma^\bullet_K((X_{f,G})_{f,K}))^K.$$

Proof. Recall the conditionally convergent spectral sequence

$$H^s_c(K;\pi_t(X)) \cong H^s_c(K;\pi_t((X_{f,G})_{f,K})) \Rightarrow \pi_{t-s}(\operatorname*{holim}_\Delta(\Gamma^\bullet_K((X_{f,G})_{f,K}))^K).$$

We compare this spectral sequence with the homotopy spectral sequence for $\mathrm{holim}_\Delta(\Gamma^\bullet_G(X_{f,G}))^K$. Note that if Y is a discrete G-spectrum that is fibrant as a spectrum, then $\mathrm{Map}_c(G,Y) \cong \mathrm{colim}_N \prod_{G/N} Y$ and

$$\mathrm{Map}_c(G,Y)^K \cong \mathrm{Map}_c(G/K,Y) \cong \operatorname*{colim}_N \prod_{G/(NK)} Y$$

are fibrant spectra. Thus, $(\Gamma^\bullet_G(X_{f,G}))^K$ is a cosimplicial fibrant spectrum, and there is a conditionally convergent spectral sequence

$$E_2^{s,t} = H^s((\Gamma^*_G\pi_t(X_{f,G}))^K) \Rightarrow \pi_{t-s}(\operatorname*{holim}_\Delta(\Gamma^\bullet_G(X_{f,G}))^K).$$

As in the proof of Theorem 7.4, $0 \to \pi_t(X_{f,G}) \to \Gamma^*_G(\pi_t(X_{f,G}))$ is a $(-)^K$-acyclic resolution of $\pi_t(X_{f,G})$, and thus, we have $E_2^{s,t} \cong H^s_c(K;\pi_t(X_{f,G})) \cong H^s_c(K;\pi_t(X))$.

Since ρ is compatible with the isomorphism between the two E_2-terms, the spectral sequences are isomorphic and ρ is a weak equivalence. □

Remark 7.13. Lemma 7.12 gives the following weak equivalences between fibrant spectra:

$$X^{hK} = (X_{f,K})^K \to \operatorname*{holim}_\Delta(\Gamma^\bullet_K((X_{f,G})_{f,K}))^K \leftarrow \operatorname*{holim}_\Delta(\Gamma^\bullet_G(X_{f,G}))^K.$$

Thus, if $\mathrm{vcd}(G) < \infty$, X is a discrete G-spectrum, and K is a closed subgroup of G, then $\mathrm{holim}_\Delta(\Gamma^\bullet_G(X_{f,G}))^K$ is a model for X^{hK}, so that

$$X^{hK} = \operatorname*{holim}_\Delta(\Gamma^\bullet_G(X_{f,G}))^K$$

is another definition of the homotopy fixed points.

This discussion yields the following result.

Theorem 7.14. *If X is a discrete G-spectrum, with* $\mathrm{vcd}(G) < \infty$*, then there is a presheaf of spectra $P(X)\colon (\mathcal{O}_G)^{\mathrm{op}} \to \mathrm{Spt}$, defined by*

$$P(X)(G/K) = \operatorname*{holim}_\Delta(\Gamma^\bullet_G(X_{f,G}))^K = X^{hK}.$$

Proof. If Y is a discrete G-set, any morphism $f\colon G/H \to G/K$, in $\mathcal{O}_G$, induces a map

$$\mathrm{Map}_c(G,Y)^K \cong \mathrm{Map}_c(G/K,Y) \to \mathrm{Map}_c(G/H,Y) \cong \mathrm{Map}_c(G,Y)^H.$$

Thus, if $Y \in \mathrm{Spt}_G$, f induces a map $\mathrm{Map}_c(G,Y)^K \to \mathrm{Map}_c(G,Y)^H$, so that there is a map $P(X)(f)\colon \mathrm{holim}_\Delta(\Gamma^\bullet_G(X_{f,G}))^K \to \mathrm{holim}_\Delta(\Gamma^\bullet_G(X_{f,G}))^H$. It is easy to check that $P(X)$ is actually a functor. □

We conclude this section by pointing out a useful fact: smashing with a finite spectrum, with trivial G-action, commutes with taking homotopy fixed points. To state this precisely, we first define the relevant map.

Let X be a discrete G-spectrum and let Y be any spectrum with trivial G-action. Then there is a map

$$(\operatorname*{holim}_{\Delta}(\Gamma^\bullet_G X_f)^G) \wedge Y \to \operatorname*{holim}_{\Delta}((\Gamma^\bullet_G X_f)^G \wedge Y) \to \operatorname*{holim}_{\Delta}((\Gamma^\bullet_G X_f) \wedge Y)^G.$$

Also, there is a natural G-equivariant map $\mathrm{Map}_c(G, X) \wedge Y \to \mathrm{Map}_c(G, X \wedge Y)$ that is defined by the composition

$$(\operatorname*{colim}_{N} \textstyle\prod_{G/N} X) \wedge Y \cong \operatorname{colim}_N((\prod_{G/N} X) \wedge Y) \to \operatorname{colim}_N \prod_{G/N} (X \wedge Y),$$

by using the isomorphism $\mathrm{Map}_c(G, X) \cong \operatorname{colim}_N \prod_{G/N} X$. This gives a natural G-equivariant map $(\Gamma_G \Gamma_G X) \wedge Y \to \Gamma_G((\Gamma_G X) \wedge Y) \to \Gamma_G \Gamma_G(X \wedge Y)$. Thus, iteration gives a G-equivariant map $(\Gamma^\bullet X) \wedge Y \to \Gamma^\bullet(X \wedge Y)$ of cosimplicial spectra. Hence, if $\mathrm{vcd}(G) < \infty$, $X^{hG} \wedge Y \to (X_f \wedge Y)^{hG}$ is a canonical map that is defined by composing $X^{hG} \wedge Y \to \operatorname{holim}_\Delta((\Gamma^\bullet X_f) \wedge Y)^G$, from above, with the map

$$\operatorname*{holim}_{\Delta}((\Gamma^\bullet X_f) \wedge Y)^G \to \operatorname*{holim}_{\Delta}(\Gamma^\bullet(X_f \wedge Y))^G \to \operatorname*{holim}_{\Delta}(\Gamma^\bullet(X_f \wedge Y)_f)^G.$$

Lemma 7.15 ([29, Prop. 3.10])**.** *If $\mathrm{vcd}(G) < \infty$, $X \in \mathrm{Spt}_G$, and Y is a finite spectrum with trivial G-action, then $X^{hG} \wedge Y \to (X_{f,G} \wedge Y)^{hG}$ is a weak equivalence.*

Remark 7.16. By Lemma 7.15, when Y is a finite spectrum, there is a zigzag of natural weak equivalences $X^{hG} \wedge Y \to (X_{f,G} \wedge Y)^{hG} \leftarrow (X \wedge Y)^{hG}$. We refer to this zigzag by writing $X^{hG} \wedge Y \cong (X \wedge Y)^{hG}$.

8. Homotopy fixed points for towers in Spt_G

In this section, $\{Z_i\}$ is always in $\mathbf{tow}(\mathrm{Spt}_G)$ (except in Definition 8.7). For $\{Z_i\}$ a tower of fibrant spectra, we define the homotopy fixed point spectrum $(\operatorname{holim}_i Z_i)^{hG}$ and construct its descent spectral sequence. Also, recall from §5 that, if G is finite and $X \in \mathrm{Spt}_G$, then $X^{h'G} = \operatorname{holim}_G X_{\mathtt{f}}$, where $X \to X_{\mathtt{f}}$ is a weak equivalence that is G-equivariant, with $X_{\mathtt{f}}$ fibrant in Spt.

Definition 8.1. If $\{Z_i\}$ in $\mathbf{tow}(\mathrm{Spt}_G)$ is a tower of fibrant spectra, we define $Z = \operatorname{holim}_i Z_i$, a continuous G-spectrum. The homotopy fixed point spectrum Z^{hG} is defined to be $\operatorname{holim}_i Z_i^{hG}$, a fibrant spectrum.

We make some comments about Definition 8.1. Let H be a closed subgroup of G. Then the map $\operatorname{holim}_i((Z_i)_f)^H \to \operatorname{holim}_i \operatorname{holim}_\Delta(\Gamma^\bullet_H(Z_i)_f)^H$ and the map $\operatorname{holim}_i \operatorname{holim}_\Delta(\Gamma^\bullet_G(Z_i)_{f,G})^H \to \operatorname{holim}_i \operatorname{holim}_\Delta(\Gamma^\bullet_H((Z_i)_{f,G})_{f,H})^H$ are weak equivalences. Thus, in Definition 8.1, each of our three definitions for homotopy fixed points (Definition 5.2, Remarks 7.8, 7.13) can be used for Z_i^{hH}.

In Definition 8.1, suppose that not all the Z_i are fibrant in Spt. Then the map $Z = \operatorname{holim}_i Z_i \to \operatorname{holim}_i(Z_i)_{f,\{e\}} = \operatorname{holim}_i Z_i^{h\{e\}} = Z^{h\{e\}}$ need not be a weak equivalence. Thus, for an arbitrary tower in Spt_G, Definition 8.1 can fail to have the desired property that $Z \to Z^{h\{e\}}$ is a weak equivalence.

Below, Lemmas 8.2 and 8.3, and Remark 8.4, show that when G is a finite group, $Z^{hG} \simeq Z^{h'G}$, and, for any G, Z^{hG} can be obtained by using a total right derived functor that comes from fixed points. Thus, Definition 8.1 generalizes the notion of homotopy fixed points to towers of discrete G-spectra.

Lemma 8.2. *Let G be a finite group and let $\{Z_i\}$ in* $\mathbf{tow}(\mathrm{Spt}_G)$ *be a tower of fibrant spectra. Then there is a weak equivalence $Z^{hG} \to Z^{h'G}$.*

Proof. It is not hard to see that the map $Z^{hG} \to Z^{h'G}$ can be defined to be

$$\operatorname*{holim}_i \lim_G (Z_i)_f \to \operatorname*{holim}_i \operatorname*{holim}_G (Z_i)_f \cong \operatorname*{holim}_G \operatorname*{holim}_i (Z_i)_f,$$

which is easily seen to be a weak equivalence. □

In the lemma below, whose elementary proof is omitted, the functor

$$\mathbf{R}(\lim_i (-)^G) \colon \mathrm{Ho}(\mathbf{tow}(\mathrm{Spt}_G)) \to \mathrm{Ho}(\mathrm{Spt})$$

is the total right derived functor of the functor $\lim_i (-)^G \colon \mathbf{tow}(\mathrm{Spt}_G) \to \mathrm{Spt}$.

Lemma 8.3. *If $\{Z_i\}$ is an arbitrary tower in* Spt_G, *then*

$$\operatorname*{holim}_i ((Z_i)_f)^G \xrightarrow{\simeq} \operatorname*{holim}_i ((Z_i)'_f)^G \xleftarrow{\simeq} \lim_i ((Z_i)'_f)^G = \mathbf{R}(\lim_i (-)^G)(\{Z_i\}).$$

Remark 8.4. By Lemma 5.3, if $X \in \mathrm{Spt}_G$, then $X^{hG} = (\mathbf{R}(-)^G)(X)$. Also, by Lemma 8.3, if $\{Z_i\}$ in $\mathbf{tow}(\mathrm{Spt}_G)$ is a tower of fibrant spectra, then

$$Z^{hG} = \operatorname*{holim}_i Z_i^{hG} = \operatorname*{holim}_i ((Z_i)_f)^G \cong \mathbf{R}(\lim_i (-)^G)(\{Z_i\}).$$

Thus, the homotopy fixed point spectrum Z^{hG} is again given by the total right derived functor of an appropriately defined functor involving G-fixed points.

Given any tower in Spt_G of fibrant spectra, there is a descent spectral sequence whose E_2-term is a version of continuous cohomology.

Theorem 8.5. *If* $\mathrm{vcd}(G) < \infty$ *and $\{Z_i\}$ in* $\mathbf{tow}(\mathrm{Spt}_G)$ *is a tower of fibrant spectra, then there is a conditionally convergent descent spectral sequence*

$$H^s_{\mathrm{cont}}(G; \{\pi_t(Z_i)\}) \Rightarrow \pi_{t-s}(Z^{hG}). \tag{8.6}$$

We omit the proof of Theorem 8.5, since it is a special case of [8, Prop. 3.1.2], and also because (8.6) is not our focus of interest. However, we point out that spectral sequence (8.6), whose construction goes back to the ℓ-adic descent spectral sequence of algebraic K-theory ([41], [29]), is the homotopy spectral sequence

$$E_2^{s,t} = \lim_{\Delta \times \{i\}}{}^{s}\, \pi_t((\Gamma^\bullet_G((Z_i)_{f,G}))^G) \Rightarrow \pi_{t-s}(\operatorname*{holim}_{\Delta \times \{i\}} (\Gamma^\bullet_G((Z_i)_{f,G}))^G).$$

For our applications, instead of spectral sequence (8.6), we are more interested in descent spectral sequence (8.9) below. Spectral sequence (8.9), a homotopy spectral sequence for a particular cosimplicial spectrum, is more suitable for comparison with the $K(n)$-local E_n-Adams spectral sequence (see [5, Prop. A.5]), when (8.9) has abutment $\pi_*((E_n \wedge X)^{hG})$, where X is a finite spectrum.

Definition 8.7. If $\{Z_i\}$ is a tower of spectra such that $\{\pi_t(Z_i)\}$ satisfies the Mittag-Leffler condition for every $t \in \mathbb{Z}$, then $\{Z_i\}$ is a *Mittag-Leffler tower* of spectra.

Theorem 8.8. *If* $\mathrm{vcd}(G) < \infty$ *and $\{Z_i\}$ in* $\mathbf{tow}(\mathrm{Spt}_G)$ *is a tower of fibrant spectra, then there is a conditionally convergent descent spectral sequence*

$$E_2^{s,t} = \pi^s \pi_t(\operatorname*{holim}_i (\Gamma^\bullet_G (Z_i)_f)^G) \Rightarrow \pi_{t-s}(Z^{hG}). \tag{8.9}$$

If $\{Z_i\}$ is a Mittag-Leffler tower, then $E_2^{s,t} \cong H^s_{\mathrm{cont}}(G; \{\pi_t(Z_i)\})$.

Remark 8.10. In Theorem 8.8, when $\{Z_i\}$ is a Mittag-Leffler tower, spectral sequence (8.9) is identical to (8.6). However, in general, spectral sequences (8.6) and (8.9) are different. For example, if $G = \{e\}$, then in (8.6), $E_2^{0,t} = \lim_i \pi_t(Z_i)$, whereas in (8.9), $E_2^{0,t} = \pi_t(\mathrm{holim}_i\, Z_i)$.

Proof of Theorem 8.8. Note that $Z^{hG} \cong \mathrm{holim}_\Delta\, \mathrm{holim}_i(\Gamma^\bullet_G(Z_i)_f)^G$, and the diagram $\mathrm{holim}_i(\Gamma^\bullet_G(Z_i)_f)^G$ is a cosimplicial fibrant spectrum.

Let $\{Z_i\}$ be a Mittag-Leffler tower. For $k \geq 0$, Lemma 2.10 and Remark 2.11 imply that $\lim^1{}_i \mathrm{Map}_c(G^k, \pi_{t+1}(Z_i)) = 0$. Therefore,

$$\pi_t(\mathrm{holim}_i(\mathrm{Map}_c(G^{k+1}, (Z_i)_f))^G) \cong \lim_i \mathrm{Map}_c(G^{k+1}, \pi_t(Z_i))^G,$$

and hence, $\pi_t(\mathrm{holim}_i(\Gamma^\bullet(Z_i)_f)^G) \cong \lim_i(\Gamma^\bullet \pi_t(Z_i))^G$. Thus,

$$E_2^{s,t} \cong \pi^s(\lim_i(\Gamma^\bullet \pi_t(Z_i))^G) \cong H^s(\lim_i(-)^G \{\Gamma^* \pi_t(Z_i)\}_i).$$

Consider the exact sequence $\{0\} \to \{\pi_t(Z_i)\} \to \{\Gamma^*\pi_t(Z_i)\}$ in $\mathbf{tow}(C_G)$. Note that, for $s, k > 0$, by Theorem 2.16,

$$H^s_{\mathrm{cont}}(G; \{\Gamma^k \pi_t(Z_i)\}) \cong \lim_i H^s_c(G; \Gamma^k \pi_t(Z_i)) = 0,$$

since the tower $\{\Gamma^k\pi_t(Z_i)\}$ satisfies the Mittag-Leffler condition, and, for each i, $\Gamma^k\pi_t(Z_i) \cong \mathrm{Map}_c(G, \Gamma^{k-1}\pi_t(Z_i))$ is $(-)^G$-acyclic. Thus, the above exact sequence is a $(\lim_i(-)^G)$-acyclic resolution of $\{\pi_t(Z_i)\}$, so that we obtain the isomorphism $E_2^{s,t} \cong H^s_{\mathrm{cont}}(G; \{\pi_t(Z_i)\})$. □

By Remark 8.4, we can rewrite spectral sequence (8.9), when $\{Z_i\}$ is a Mittag-Leffler tower, in a more conceptual way:

$$R^s(\lim_i(-)^G)\{\pi_t(Z_i)\} \Rightarrow \pi_{t-s}(\mathbf{R}(\lim_i(-)^G)(\{Z_i\})).$$

Spectral sequence (8.6) can always be written in this way.

9. Homotopy fixed point spectra for $E^\vee(X)$

Recall that $E^\vee(X) = \hat{L}(E_n \wedge X)$. In this section, for any spectrum X and for $G <_c G_n$, we define the homotopy fixed point spectrum $(E^\vee(X))^{hG}$, using the continuous action of G.

Let X be an arbitrary spectrum with trivial G_n-action. By Corollary 6.5, there is a weak equivalence $F_n \wedge M_I \wedge X \to E_n \wedge M_I \wedge X$. Then, by functorial fibrant replacement, there is a map $\{(F_n \wedge M_I \wedge X)_\mathfrak{f}\} \to \{(E_n \wedge M_I \wedge X)_\mathfrak{f}\}$ of towers, which yields the weak equivalence

$$E^\vee(X) \cong \mathrm{holim}_I(E_n \wedge M_I \wedge X)_\mathfrak{f} \xleftarrow{\simeq} \mathrm{holim}_I(F_n \wedge M_I \wedge X)_\mathfrak{f}.$$

As in the proof of Theorem 6.6, this implies the following lemma, since the diagram $\{(F_n \wedge M_I \wedge X)_{f,G_n}\}$ is a tower of fibrant spectra.

Lemma 9.1. *Given any spectrum X with trivial G_n-action, the isomorphism*

$$E^\vee(X) \cong \mathrm{holim}_I(F_n \wedge M_I \wedge X)_{f,G_n}$$

makes $E^\vee(X)$ a continuous G_n-spectrum.

Let G be any closed subgroup of G_n. Since $\{(F_n \wedge M_I \wedge X)_{f,G_n}\}$ is a tower of discrete G-spectra that are fibrant in Spt, Lemma 9.1 also shows that $E^\vee(X)$ is a continuous G-spectrum. By Corollary 3.7, the composition

$$(F_n \wedge M_I \wedge X) \to (F_n \wedge M_I \wedge X)_{f,G_n} \to ((F_n \wedge M_I \wedge X)_{f,G_n})_{f,G}$$

is a trivial cofibration in Spt_G, with target fibrant in Spt_G. Therefore, we have $((F_n \wedge M_I \wedge X)_{f,G_n})^{hG} = (F_n \wedge M_I \wedge X)^{hG}$. Then, by Definition 8.1, we obtain the following.

Definition 9.2. Let $G <_c G_n$. Then

$$E_n^{hG} = (\operatorname*{holim}_I (F_n \wedge M_I)_{f,G_n})^{hG} = \operatorname*{holim}_I (F_n \wedge M_I)^{hG}.$$

More generally, for any spectrum X,

$$(E^\vee(X))^{hG} = (\operatorname*{holim}_I (F_n \wedge M_I \wedge X)_{f,G_n})^{hG} = \operatorname*{holim}_I (F_n \wedge M_I \wedge X)^{hG}.$$

Remark 9.3. When X is a finite spectrum, $E_n \wedge X \simeq E^\vee(X)$. Thus, we have $(E_n \wedge X)^{hG} \cong (E^\vee(X))^{hG}$.

Remark 9.4. For any X, $E^\vee(X) \cong \operatorname{holim}_I (F_n \wedge M_I \wedge X)_{f,G}$ also shows that $E^\vee(X)$ is a continuous G-spectrum. By definition, $((F_n \wedge M_I \wedge X)_{f,G})^{hG}$ and $(F_n \wedge M_I \wedge X)^{hG}$ are identical. Thus, as before,

$$(E^\vee(X))^{hG} = \operatorname*{holim}_I ((F_n \wedge M_I \wedge X)_{f,G})^{hG} = \operatorname*{holim}_I (F_n \wedge M_I \wedge X)^{hG}.$$

Note that Definition 9.2 implies the identifications

$$E_n^{hG} = \operatorname*{holim}_I \operatorname*{holim}_\Delta (\Gamma^\bullet_{G_n}(F_n \wedge M_I)_{f,G_n})^G \quad \text{and}$$

$$(E^\vee(X))^{hG} = \operatorname*{holim}_I \operatorname*{holim}_\Delta (\Gamma^\bullet_{G_n}(F_n \wedge M_I \wedge X)_{f,G_n})^G.$$

The first identification, coupled with Theorem 7.14, implies the following.

Theorem 9.5. *There is a functor* $P\colon (\mathcal{O}_{G_n})^{\mathrm{op}} \to \mathrm{Spt}$, *defined by* $P(G_n/G) = E_n^{hG}$, *where* G *is any closed subgroup of* G_n.

In addition to the above identifications, we also have

$$E_n^{hG} = \operatorname*{holim}_I \operatorname*{holim}_\Delta (\Gamma^\bullet_G(F_n \wedge M_I)_{f,G})^G \quad \text{and}$$

$$(E^\vee(X))^{hG} = \operatorname*{holim}_I \operatorname*{holim}_\Delta (\Gamma^\bullet_G(F_n \wedge M_I \wedge X)_{f,G})^G.$$

Below we show that, like E_n^{dhG}, E_n^{hG} is $K(n)$-local.

Lemma 9.6. *Let* $G <_c G_n$ *and let* X *be any spectrum. Then* $(F_n \wedge M_I \wedge X)^{hG}$ *and* $(E^\vee(X))^{hG}$ *are* $K(n)$*-local. Also,* $(F_n \wedge X)^{hG}$ *is* $E(n)$*-local.*

Proof. Recall that $(F_n \wedge M_I \wedge X)^{hG} = \operatorname{holim}_\Delta (\Gamma^\bullet_G(F_n \wedge M_I \wedge X)_{f,G})^G$, and

$$(\Gamma^k_G(F_n \wedge M_I \wedge X)_f)^G \simeq \mathrm{Map}_c(G^{k-1}, F_n \wedge X) \wedge M_I,$$

for $k \geq 1$. In the isomorphism

$$\mathrm{Map}_c(G, F_n \wedge X) \cong \operatorname*{colim}_i \textstyle\prod_{G/(U_i \cap G)} (F_n \wedge X),$$

the spectrum $F_n \wedge X$ is $E(n)$-local, the finite product is too, and hence, the direct limit $\mathrm{Map}_c(G, F_n \wedge X)$ is $E(n)$-local. Iterating this argument shows that

$$\mathrm{Map}_c(G^{k-1}, F_n \wedge X) \cong \Gamma_G \Gamma_G \cdots \Gamma_G(F_n \wedge X)$$

is $E(n)$-local. Smashing $\mathrm{Map}_c(G^{k-1}, F_n \wedge X)$ with the spectrum M_I shows that $(\Gamma^k_G(F_n \wedge M_I \wedge X)_f)^G$ is $K(n)$-local. Therefore, since the homotopy limit of an arbitrary diagram of E-local spectra is E-local, $(F_n \wedge M_I \wedge X)^{hG}$ and $(E^\vee(X))^{hG}$ are $K(n)$-local. The same argument shows that $(F_n \wedge X)^{hG}$ is $E(n)$-local. □

The following theorem shows that the homotopy fixed points of $E^\vee(X)$ are obtained by taking the $K(n)$-localization of the homotopy fixed points of the discrete G_n-spectrum $(F_n \wedge X)$.

Theorem 9.7. *For $G <_c G_n$ and any spectrum X with trivial G-action, there is an isomorphism $(E^\vee(X))^{hG} \cong \hat{L}((F_n \wedge X)^{hG})$ in the stable category. In particular, $E_n^{hG} \cong \hat{L}(F_n^{hG})$.*

Proof. After switching M_I and X, $(E^\vee(X))^{hG} \cong \mathrm{holim}_I(F_n \wedge X \wedge M_I)^{hG}$. By Remark 7.16, $(F_n \wedge X \wedge M_I)^{hG} \cong (F_n \wedge X)^{hG} \wedge M_I \simeq ((F_n \wedge X)^{hG} \wedge M_I)_{\mathtt{f}}$, where the isomorphism signifies a zigzag of natural weak equivalences. Thus,

$$(E^\vee(X))^{hG} \cong \operatorname*{holim}_I((F_n \wedge X)^{hG} \wedge M_I)_{\mathtt{f}} \cong \hat{L}((F_n \wedge X)^{hG}),$$

since, by Lemma 9.6, $(F_n \wedge X)^{hG}$ is $E(n)$-local. □

Corollary 9.8. *If X is a finite spectrum of type n, then there is an isomorphism $(F_n \wedge X)^{hG} \cong E_n^{hG} \wedge X$. In particular, $(F_n \wedge M_I)^{hG} \cong E_n^{hG} \wedge M_I$.*

Proof. By Remark 7.16 and the fact that F_n^{hG} is $E(n)$-local (Lemma 9.6),

$$(F_n \wedge X)^{hG} \cong F_n^{hG} \wedge X \cong \hat{L}(F_n^{hG}) \wedge X \cong E_n^{hG} \wedge X.$$

□

We conclude this section by observing that smashing with a finite spectrum commutes with taking the homotopy fixed points of E_n.

Theorem 9.9. *Let G be a closed subgroup of G_n and let X be a finite spectrum. Then there is an isomorphism $(E_n \wedge X)^{hG} \cong E_n^{hG} \wedge X$.*

Proof. Recall that $(E_n \wedge X)^{hG} = \mathrm{holim}_I(F_n \wedge M_I \wedge X)^{hG}$. The zigzag of natural weak equivalences between $(F_n \wedge M_I \wedge X)^{hG}$ and $(F_n \wedge M_I)^{hG} \wedge X$ yields

$$(E_n \wedge X)^{hG} \cong \operatorname*{holim}_I((F_n \wedge M_I)^{hG} \wedge X)_{\mathtt{f}} \simeq (\operatorname*{holim}_I(F_n \wedge M_I)^{hG}) \wedge X,$$

where the weak equivalence is due to Lemma 2.7. □

10. The descent spectral sequence for $(E^\vee(X))^{hG}$

By applying the preceding two sections, it is now an easy matter to build the descent spectral sequence for $(E^\vee(X))^{hG}$.

Definition 10.1. Let X be a spectrum. If the tower $\{\pi_t(E_n \wedge M_I \wedge X)\}_I$ of abelian groups satisfies the Mittag-Leffler condition for all $t \in \mathbb{Z}$, then X is an E_n-*Mittag-Leffler spectrum.* If X is an E_n-Mittag-Leffler spectrum, then, for convenience, we say that X is E_n-ML.

Any finite spectrum X is E_n-ML, since $\{\pi_t(E_n \wedge M_I \wedge X)\}_I$ is a tower of finite abelian groups, by Lemma 2.13. However, an E_n-ML spectrum need not be finite. For example, for $j \geq 1$, let $X = E_n^{(j)}$. Then $\pi_t(E_n \wedge M_I \wedge X) \cong \mathrm{Map}_c^{\ell}(G_n^j, \pi_t(E_n)/I)$. Since $\{\pi_t(E_n)/I\}$ is a tower of epimorphisms, the tower $\{\mathrm{Map}_c^{\ell}(G_n^j, \pi_t(E_n)/I)\}$ is also, and $E_n^{(j)}$ is E_n-ML.

Theorem 10.2. *Let G be a closed subgroup of G_n and let X be any spectrum with trivial G-action. Let $E_2^{s,t} = \pi^s\pi_t(\mathrm{holim}_I(\Gamma^\bullet_G(F_n \wedge M_I \wedge X)_{f,G})^G)$. Then there is a conditionally convergent descent spectral sequence*

$$E_2^{s,t} \Rightarrow \pi_{t-s}((E^\vee(X))^{hG}). \tag{10.3}$$

If X is E_n-ML, then $E_2^{s,t} \cong H^s_{\mathrm{cont}}(G; \{\pi_t(E_n \wedge M_I \wedge X)\})$. In particular, if X is a finite spectrum, then descent spectral sequence (10.3) has the form

$$H^s_c(G; \pi_t(E_n \wedge X)) \Rightarrow \pi_{t-s}((E_n \wedge X)^{hG}).$$

Proof. As in Remark 9.4, $E^\vee(X) \cong \mathrm{holim}_I(F_n \wedge M_I \wedge X)_{f,G}$ is a continuous G-spectrum. Then (10.3) follows from Theorem 8.8 by considering the tower of spectra $\{(F_n \wedge M_I \wedge X)_f\}_I$. When X is E_n-ML, $\{(F_n \wedge M_I \wedge X)_f\}$ is a Mittag-Leffler tower of spectra, and thus, the simplification of the E_2-term in this case follows from Theorem 8.8. By Definition 2.17, when X is finite, there is an isomorphism $H^s_{\mathrm{cont}}(G; \{\pi_t(E_n \wedge M_I \wedge X)\}) \cong H^s_c(G; \pi_t(E_n \wedge X))$. □

As discussed in Remark 8.10, Theorem 8.5 gives a spectral sequence with abutment $\pi_*((E^\vee(X))^{hG})$, the same as the abutment of (10.3), but with E_2-term given by $H^s_{\mathrm{cont}}(G; \{\pi_t(E_n \wedge M_I \wedge X)\})$, which is, in general, different from the E_2-term of (10.3). We are interested in the descent spectral sequence of Theorem 10.2, not just because it is a second descent spectral sequence with an interesting E_2-term, but, as mentioned in §8, it can be compared with the $K(n)$-local E_n-Adams spectral sequence. (This comparison is work in progress.)

We conclude this paper with a computation that uses spectral sequence (10.3) to compute $\pi_*((\hat{L}(E_n \wedge E_n^{(j)}))^{hG_n})$, where $j \geq 1$ and G_n acts only on the leftmost factor. By Theorem 2.4,

$$\begin{aligned}\pi_t(\hat{L}(E_n^{(j+1)})) &\cong \mathrm{Map}_c^{\ell}(G_n^j, \pi_t(E_n)) \cong \lim_I \mathrm{Map}_c^{\ell}(G_n, \mathrm{Map}_c(G_n^{j-1}, \pi_t(E_n \wedge M_I))) \\ &\cong \lim_I \mathrm{Map}_c(G_n, \mathrm{Map}_c(G_n^{j-1}, \pi_t(E_n \wedge M_I))),\end{aligned}$$

where the tower $\{\mathrm{Map}_c(G_n, \mathrm{Map}_c(G_n^{j-1}, \pi_t(E_n \wedge M_I)))\}$ satisfies the Mittag-Leffler condition, by Remark 2.11. Thus, in spectral sequence (10.3), Theorem 2.16 implies that $E_2^{s,t} \cong \lim_I H^s_c(G_n; \mathrm{Map}_c(G_n, \mathrm{Map}_c(G_n^{j-1}, \pi_t(E_n \wedge M_I))))$, which vanishes for $s > 0$, and equals $\mathrm{Map}_c(G_n^{j-1}, \pi_t(E_n))$, when $s = 0$. Thus,

$$\pi_*((\hat{L}(E_n^{(j+1)}))^{hG_n}) \cong \mathrm{Map}_c(G_n^{j-1}, \pi_*(E_n)) \cong \pi_*(\hat{L}(E_n^{(j)})),$$

as abelian groups. Therefore, for $j \geq 1$, there is an isomorphism

$$(\hat{L}(E_n^{(j+1)}))^{hG_n} \cong \hat{L}(E_n^{(j)}).$$

The techniques described in this paper do not allow us to handle the $j = 0$ case, which would say that $E_n^{hG_n}$ and $\hat{L}(S^0) \simeq E_n^{dhG_n}$ are isomorphic.

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Purdue University, Department of Mathematics, 150 N. University Street, West Lafayette, IN 47907-2067, USA
E-mail address: dgdavis@math.purdue.edu