RESEARCH STATEMENT

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1. Introduction

Understanding the homotopy groups of the sphere, $\pi_*(S^n)$, remains one of the most interesting and difficult problems in algebraic topology. Currently, many experts in the field are approaching this problem by using ideas and tools from algebraic geometry, commutative algebra, topological group theory, number theory, and homological algebra. This approach is known as chromatic stable homotopy theory and its chief focus is on an object that is referred to as the $K(n)$-local sphere.

My research, which is in chromatic stable homotopy theory, primarily investigates the theory of spectra with a continuous action by a profinite group and their homotopy fixed points. Though my research is shaped by a desire to better understand the $K(n)$-local sphere, my work in this area allows me to interact with several of the branches of math mentioned above. One of the main constructions in this theory is the homotopy fixed point spectrum, which turns out to be very useful in chromatic theory, as illustrated in papers by Behrens, Devinatz, Goerss, Henn, Hopkins, Mahowald, and Rezk (see [2, 1, 18, 19, 20, 23, 24]). This theory is also related to elliptic cohomology and topological modular forms (the work of Hopkins and his collaborators) (see [1] and [26]).

Roughly speaking, a spectrum X is a collection of topological spaces $\{X_i\}_{i\geq 0}$, such that each X_i is related to X_{i+1} in a nice way. Then (oversimplifying somewhat), the spectrum X has a continuous action by a profinite group G (a type of topological group), when there is an action map $G \times X_i \to X_i$ that is continuous, for each i.

In this theory, it is often useful to do the following: (a) replace X with a better-behaved model X' that is equivalent to X, in some sense; (b) take the G-fixed points of X', $(X')^G$; and, finally, (c) construct - and compute, if possible - an algebraic machine (more precisely, a spectral sequence) for calculating $\pi_*((X')^G)$. The object considered in (b) sometimes turns out to be equivalent to another object that is already of considerable interest, so that (c) yields valuable information.

This theory relies upon foundational work by Jardine and Thomason (and, to a lesser extent, Goerss and Mitchell - see [30, 31, 32, 33, 40, 25, 34]), and, for interesting examples, a beautiful paper by Devinatz and Hopkins ([20]) provides critical input. However, it is my thesis [16] and my paper [14] that establish this theory and develop the language required by it.

We note that my work ([16], [14]) has shown itself to be useful: see [1, pp. 2, 26], [3, pg. 89 and the proof of Lemma 11.3.4], [21, pg. 5 and Section 10.3], and [36, pp. 26-27]. My joint work with Mark Behrens of M.I.T. ([4]) was used in a critical way in [2, Theorem 2.3.2] and cited by [1, Proposition 5.2.3]. Also, my paper [13] is cited by [21, Remark 8.29].

2. Doctoral work

For any $n \geq 1$ and any prime p, let E_n be the Lubin-Tate spectrum: the homotopy groups of E_n are

$$\pi_*(E_n) = W(\mathbb{F}_{p^n})[[u_1, ..., u_{n-1}]][u^{\pm 1}],$$

where the complete power series ring over the Witt vectors is in degree zero, and the degree of u is -2. Also, let S_n be the nth Morava stabilizer group - the automorphism group of the Honda formal group law Γ_n of height n over $\mathbb{F}_{p^n}$. Finally, let $G_n = S_n \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$, where $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ is the Galois group.

The objects E_n and G_n play a very important role in chromatic homotopy theory, mainly for the following reason. Work of Goerss, Hopkins, and Miller (see [22, 35]) shows that G_n acts on E_n in a highly structured way. Moreover, this action induces the continuous action of the profinite group G_n on the topological module E_{n*}, and there is a strongly convergent spectral sequence

$$H^s_c(G_n; \pi_t(E_n)) \Rightarrow \pi_{t-s}(L_{K(n)}S^0),$$

where $L_{K(n)}S^0$ is the $K(n)$-local sphere ($K(n)$ is the nth Morava K-theory spectrum) and the E_2-term is the continuous cohomology of G_n (see the paragraph after Theorem 3.2 for the precise definition). Thus, E_n and G_n are closely related to $L_{K(n)}S^0$, so that understanding E_n and G_n helps with understanding the $K(n)$-local sphere.

At a time when the topological ring $E^{\mathsf{s}}_{n*}[\![G_n]\!]$, the completed twisted group ring of E_n-cohomology operations, was not well-understood, I gave a definition of it in [12], a manuscript that is cited by Hovey in [27, pg. 202].

To describe my thesis, I need to introduce some more notation. We often use the symbol $\cong$ to denote an isomorphism in the stable homotopy category. Let $\{M_{I_i}\}$ denote a collection of generalized Moore spectra, such that these spectra form a tower $M_{I_0} \leftarrow M_{I_1} \leftarrow M_{I_2} \leftarrow \cdots$ and $\mathrm{holim}_i\, L_{K(n)}M_{I_i} \cong L_{K(n)}S^0$. Also, for any closed subgroup G of G_n, let E_n^{dhG} be the "ad hoc homotopy fixed point spectrum" constructed by Devinatz and Hopkins in [20].

My thesis consists of three parts. Part (a) explains the theory of continuous G-spectra Z and their homotopy fixed points Z^{hG}, and is briefly summarized below. Part (b) shows that $L_{K(n)}(E_n \wedge X)$ is a continuous G_n-spectrum, where X is a spectrum such that the tower $\{\pi_t(E_n \wedge X \wedge M_{I_i})\}$ satisfies the Mittag-Leffler condition for every integer t. Also, part (c) shows that, when X is finite and G is any closed subgroup of G_n, $(E_n \wedge X)^{hG} \cong E_n^{dhG} \wedge X$, and the descent spectral sequence for $(E_n \wedge X)^{hG}$ is isomorphic to the strongly convergent $K(n)$-local E_n-Adams spectral sequence for $E_n^{dhG} \wedge X$.

Part (b) was generalized in my paper [14] and part (c) is being written up in [6] (in [6], we use techniques that are different from those of [16]). Also, part (c) is being generalized in work in progress. Thus, we discuss only the generalizations, in Sections 3 and 10.1.

A discrete G-spectrum is a (naive) G-spectrum of simplicial sets, such that each simplicial set is a simplicial object in the category of discrete G-sets.

Theorem 2.1. *The category* Spt_G *of discrete G-spectra is a model category, where a morphism f is a weak equivalence* (*cofibration*) *if and only if f is a weak equivalence* (*cofibration*) *in* Spt*, the model category of spectra. If X is a discrete G-spectrum, then the definition $X^{hG} = (X_f)^G$, where X_f is a fibrant replacement for X in* Spt_G*, generalizes the notion of homotopy fixed points for a finite group.*

If $Z_0 \leftarrow Z_1 \leftarrow Z_2 \leftarrow \cdots$ is a tower of discrete G-spectra that are fibrant as spectra, then $\mathrm{holim}_i\, Z_i$ is a continuous G-spectrum and $(\mathrm{holim}_i\, Z_i)^{hG} = \mathrm{holim}_i (Z_i)^{hG}$ is the homotopy fixed point spectrum of $\mathrm{holim}_i\, Z_i$ with respect to the continuous action of G. This definition of homotopy fixed points is equivalent to the total right derived functor of fixed points, in the appropriate sense, and it agrees with the usual definition of homotopy fixed points, when G is finite. If G satisfies a certain finiteness condition, then we can build the descent spectral sequence.

Theorem 2.2. *If G has finite virtual cohomological dimension, then there is a conditionally convergent descent spectral sequence*

$$E_2^{s,t} \Rightarrow \pi_{t-s}((\operatorname*{holim}_i Z_i)^{hG}).$$

If the tower $\{\pi_t(Z_i)\}$ satisfies the Mittag-Leffler condition for each integer t, then

$$E_2^{s,t} \cong H^s_{\mathrm{cont}}(G; \{\pi_t(Z_i)\}),$$

the continuous cohomology of G, with coefficients in the tower $\{\pi_t(Z_i)\}$ of discrete G-modules.

3. Homotopy fixed points for $L_{K(n)}(E_n \wedge X)$ using the continuous action

Besides containing part (a) of my thesis, my paper [14] generalizes part (b) in that X can be any spectrum. More precisely, in [14], we prove the following results.

Theorem 3.1. *Let $\{U_j\}_{j\geq 0}$ be a descending chain of open normal subgroups of G_n, such that $G_n \cong \lim_j G_n/U_j$, and let $F_n = \mathrm{colim}_j E_n^{dhU_j}$. Then the spectrum $E_n \cong \mathrm{holim}_i (F_n \wedge M_{I_i})_f$ is a continuous G_n-spectrum. More generally, for any spectrum X,*

$$L_{K(n)}(E_n \wedge X) \cong \operatorname*{holim}_i (F_n \wedge M_{I_i} \wedge X)_f$$

is a continuous G_n-spectrum.

Let $\mathcal{O}_{G_n}$ be the orbit category of quotients of G_n by closed subgroups.

Theorem 3.2. *There is a functor $P\colon (\mathcal{O}_{G_n})^{\mathrm{op}} \to \mathrm{Spt}$, defined by $P(G_n/G) = E_n^{hG}$, where G is any closed subgroup of G_n.*

The next theorem is an application of Theorem 2.2. First, we need a definition. If G is a profinite group and $\lim_\alpha M_\alpha$ is a profinite continuous $\mathbb{Z}_p[[G]]$-module, then $H^s_c(G; M)$ will be used to denote $\lim_\alpha H^s_c(G; M_\alpha)$. In all the cases in this research statement where this expression is used, this turns out to be continuous cohomology in a particularly nice sense (see [20, Remark 1.3] and [39]).

Theorem 3.3. *Let G be a closed subgroup of G_n and let X be any spectrum. Then there is a conditionally convergent descent spectral sequence*

$$E_2^{s,t} \Rightarrow \pi_{t-s}((L_{K(n)}(E_n \wedge X))^{hG}). \tag{3.4}$$

If the tower of abelian groups $\{\pi_t(E_n \wedge M_{I_i} \wedge X)\}$ satisfies the Mittag-Leffler condition for each $t \in \mathbb{Z}$, then

$$E_2^{s,t} \cong H^s_{\mathrm{cont}}(G; \{\pi_t(E_n \wedge M_{I_i} \wedge X)\}).$$

If X is a finite spectrum, then spectral sequence (3.4) has the form

$$H^s_c(G; \pi_t(E_n \wedge X)) \Rightarrow \pi_{t-s}((E_n \wedge X)^{hG}).$$

4. The E_2-term of the descent spectral sequence for a continuous G-spectrum

Let G be a profinite group with finite virtual cohomological dimension, and let $\mathrm{holim}_i Z_i$ be a continuous G-spectrum. The E_2-term $\pi^s\pi_t(\mathrm{holim}_i(\Gamma^\bullet_G(Z_i)_f)^G)$ of the descent spectral sequence for $\pi_*((\mathrm{holim}_i Z_i)^{hG})$, in general, can not be labeled in a nice algebraic way. For example, additional hypotheses are needed for it to be equivalent to some form of continuous cohomology. However, in [13], we show that this E_2-term is always an interesting object.

More precisely, in [13], we show that the E_2-term is always built out of a certain complex of spectra, that, in the context of abelian groups, can be used to compute the continuous cohomology of G with coefficients in $\lim_i M_i$, where $\{M_i\}$ is a tower of discrete G-modules (see [13, Corollary 4.4] for the precise statement and compare this result with [13, Theorem 2.3]).

5. The homotopy orbit spectrum for a profinite group

In this brief section, we summarize the contents of [7]. Let $G = \lim_\alpha G/N_\alpha$ present the profinite group G as the inverse limit of a collection of quotients of G by open normal subgroups. Then, given a diagram $\{X_\alpha\}_\alpha$ of G-spectra and G-equivariant maps, such that each X_α is a fibrant Bousfield-Friedlander spectrum and, for each α, the G-action on X_α factors through G/N_α (so that X_α is a G/N_α-spectrum), we define the homotopy orbit spectrum X_{hG}, where $X = \mathrm{holim}_\alpha X_\alpha$.

If G is a countably based profinite group, then, under reasonable assumptions on the diagram $\{X_\alpha\}$, we construct a homotopy orbit spectral sequence

$$H_p^c(G, \pi_q(X)) \Rightarrow \pi_{p+q}(X_{hG}),$$

where the E_2-term is the continuous homology of G, with coefficients in the profinite $\widehat{\mathbb{Z}}[[G]]$-module $\pi_q(X)$. In particular, we show that one can construct $F(E_n, L_{K(n)}S^0)_{hG_n}$ and that there is a homotopy orbit spectral sequence

$$H_p^c(G_n, \pi_q(F(E_n, L_{K(n)}S^0))) \Rightarrow \pi_{p+q}(F(E_n, L_{K(n)}S^0)_{hG_n}).$$

6. Iterated homotopy fixed points for E_n

In this section, we discuss the main results of [8]. Let H and K be closed subgroups of G_n, with $H \lhd K$. Note that K/H is a profinite group. In [19], Devinatz constructs a strongly convergent Adams-type spectral sequence

$$H_c^s(K/H; \pi_t(E_n^{dhH})) \Rightarrow \pi_{t-s}(E_n^{dhK}).$$

Since $E_n^{dhG} \cong E_n^{hG}$ for G equal to H and K, successively, this spectral sequence looks like a descent spectral sequence for $(E_n^{hH})^{hK/H}$ and it suggests that $(E_n^{hH})^{hK/H} \cong E_n^{hK}$. When K/H is a finite group, [19] and [20], respectively, show that these conclusions are indeed valid.

However, when K/H is not finite, it is not even known how to construct the spectrum $(E_n^{hH})^{hK/H}$. We have remedied this situation with the following result.

Theorem 6.1. *Let H and K be as above (so that K/H need not be finite). Then the spectrum E_n^{hH} is a continuous K/H-spectrum. More generally, if X is any spectrum, $L_{K(n)}(E_n^{hH} \wedge X)$ is a continuous K/H-spectrum.*

By applying the theory discussed in Section 2, Theorem 6.1 implies that $(E_n^{hH})^{hK/H}$ can always be formed. Also, the descent spectral sequence has the desired form.

Theorem 6.2. *There is a descent spectral sequence*

$$H_c^s(K/H; \pi_t(E_n^{hH})) \Rightarrow \pi_{t-s}((E_n^{hH})^{hK/H}).$$

Finally, we are able to show that the iterated homotopy fixed points of E_n behave as desired.

Theorem 6.3. *There is an isomorphism $(E_n^{hH})^{hK/H} \cong E_n^{hK}$ in the stable category.*

The above theorem extends a result of Devinatz and Hopkins (see [20, Theorem 4 and Section 6]), which showed that the statement is valid when H has finite index in K.

7. The category R_G^+ is a Grothendieck site when G is profinite

Given a profinite group G, let R_G^+ be the category that consists of finite discrete G-sets and the profinite G-space G itself, with morphisms being the continuous G-equivariant maps. This category was first considered by Devinatz and Hopkins in [20], in the case of $G = G_n$, because it provided a functorial way of organizing the G_n-action on E_n and the G_n/U-action on E_n^{dhU}, where U is any open normal subgroup of G_n.

In [9], I show that R_G^+ is a Grothendieck site, when it is equipped with the pretopology of epimorphic covers. This result takes some work, since, if G is not finite, then R_G^+ is not closed under pullbacks. Also, we show that the site R_G^+ does not have a canonical topology.

Since R_G^+ is a site, Jardine's machinery yields a model category structure on the category of presheaves of spectra on this site, and we note that such presheaves are useful for organizing the data associated to a continuous G-spectrum Z, when one takes its homotopy fixed points with respect to open subgroups of G.

8. The content of three papers in preparation

8.1. **A universal coefficient spectral sequence for discrete G-spectra.** In [11], a paper in preparation, I am working towards completing a proof of the following theorem.

Theorem 8.1.1. *Let X and F be discrete G-spectra, where G is a profinite group, and let E be a discrete G-A_∞-ring spectrum that is an Adams spectrum. Also, suppose that F is an E-module spectrum in the category of discrete G-spectra. Then there is a spectral sequence*

$$\mathrm{Ext}^s_{\mathcal{C}}(E_*(X), F_{*+t}) \Rightarrow \pi_{t-s}(\mathrm{Map}_G(X, F)),$$

where $\mathcal{C}$ is the category of graded discrete twisted E_*-G-modules, and $\mathrm{Map}_G(X, F)$ is the simplicial mapping space in the category of discrete G-spectra.

I hope that computing with the above spectral sequence will be helpful for learning more about discrete G-spectra. Also, if the above theorem is reworked in the setting of discrete G-A_∞-ring spectra, then the E_2-term of the spectral sequence should be related to the obstructions for realizing a discrete G-module as the homotopy groups of a discrete G-A_∞-ring spectrum.

8.2. **Continuous G-spectra and Brown-Comenetz duality.** Now we describe part of the work in [5], another paper in preparation. Let the profinite group G have a descending chain $\{N_i\}_{i\geq 0}$ of open normal subgroups, such that $\bigcap_i N_i = \{e\}$. Then we obtain the following result.

Theorem 8.2.1. *Let Z be any spectrum and let X be a discrete G-spectrum. Then the function spectrum $F(X, Z)$ is a continuous G-spectrum.*

For example, if X is a discrete G-spectrum, then cX, the Brown-Comenetz dual of X, is a continuous G-spectrum. Also, when certain finiteness hypotheses on G, $\pi_*(X)$, and all the $\pi_*(X^{hN_i})$ are satisfied, there is a descent spectral sequence of the form

$$H^s_c(G; \pi_t(cX)) \Rightarrow \pi_{t-s}((cX)^{hG}).$$

For any $K(n)$-local spectrum X, let $\widehat{I}X$ be the Brown-Comenetz dual of X in the $K(n)$-local category (see [28, Chapter 10]). Thus, $\widehat{I}X = IM_nX$, where IY is the p-local Brown-Comenetz dual of the p-local spectrum Y, and M_nX is the fiber of the map $X \to L_{n-1}X$. Then Theorem 8.2.1 implies that, if F is any finite type n complex, then $\widehat{I}(E_n \wedge F)$ is a continuous G_n-spectrum. Also, $\widehat{I}E_n$ is a continuous G_n-spectrum.

We construct the descent spectral sequences for $(\widehat{I}(E_n \wedge F))^{hG_n}$ and $(\widehat{I}E_n)^{hG_n}$, and we show that they have E_2-terms that can be nicely expressed in terms of continuous cohomology. When $(p-1) \nmid n$, we use these spectral sequences to provide strong evidence that $(\widehat{I}(E_n \wedge F))^{hG_n}$ and $(\widehat{I}E_n)^{hG_n}$ are closely related to $\Sigma^{-n^2}\widehat{I}(L_{K(n)}F)$ and $\Sigma^{-n^2}\widehat{I}$, respectively.

8.3. **Examples of totally hyperfibrant discrete G-spectra.** In [8], I defined the notion of a totally hyperfibrant discrete G-spectrum and showed that if X is such a spectrum, then

$$(X^{hH})^{hK/H} \cong X^{hK}$$

always holds, where H and K are closed subgroups of G, with H normal in K. In [8], I used this notion to show that $(E_n^{hH})^{hK/H} \cong E_n^{hK}$, where H and K are in G_n. In [10], I extend this result to show that

$$((E_n[[G_n/L]])^{hH})^{hK/H} \cong (E_n[[G_n/L]])^{hK},$$

where L is any closed subgroup of G_n.

9. Joint work, in preparation

I briefly discuss the joint work [4], in preparation, with Mark Behrens of M.I.T. As mentioned earlier, this work was used in [2, Theorem 2.3.2] to help prove an interesting result.

Let A be an E-local E_∞-ring spectrum, where E is a spectrum such that its localization functor L_E behaves like $L_{K(n)}$. For a profinite group G, we study faithful E-local profinite G-Galois extensions of A, which, basically, are defined by Rognes in [36]. Our main theorem is the following.

Theorem 9.1. *Let R be a faithful E-local profinite G-Galois extension of A such that $A \cong L_E(R^{hG})$. If G has finite virtual cohomological dimension, and H and K are closed subgroups of G, then*

$$F_A(L_E(R^{hH}), L_E(R^{hK})) \cong L_E((R[[G/H]])^{hK}),$$

where $R[[G/H]]$ has the diagonal K-action.

10. Other work

10.1. **Generalizing part of my thesis.** In work in progress, I will prove the following theorem. This generalizes part (c) of my thesis.

Theorem 10.1.1. *Let G be any closed subgroup of G_n and let X be any spectrum. Then*

$$(L_{K(n)}(E_n \wedge X))^{hG} \cong L_{K(n)}(E_n^{dhG} \wedge X).$$

Also, spectral sequence (3.4), the descent spectral sequence for the spectrum $(L_{K(n)}(E_n \wedge X))^{hG}$, is isomorphic to the strongly convergent $K(n)$-local E_n-Adams spectral sequence for $L_{K(n)}(E_n^{dhG} \wedge X)$. In particular, letting $G = G_n$ implies that

$$L_{K(n)}(X) \cong (L_F(E_n \wedge X))^{hG_n}, \tag{10.1.2}$$

where F is any finite type n complex.

The isomorphism in (10.1.2) gives an interesting model for $K(n)$-localization. Also, experts believe that the category of $K(n)$-local spectra is equivalent to some category $\mathcal{C}$ of continuous G_n-E_n-module spectra, and I believe that (10.1.2) is a first step in proving such an equivalence. For example, (10.1.2) suggests that $\mathcal{C}$ should be equivalent to the category of Morava modules, that is, all spectra of the form $L_{K(n)}(E_n \wedge X)$.

10.2. **Other versions of Morava E-theory.** Let k be any finite field that contains $\mathbb{F}_{p^n}$, so that $k = \mathbb{F}_{p^{mn}}$, for some $m \geq 1$. Let Γ be any height n formal group law over k. Let $E(k,\Gamma)$ be the version of Morava E-theory associated to the pair (k,Γ) by the Goerss-Hopkins theorem ([22, Section 7]): $E(k,\Gamma)$ is an E_∞-ring spectrum with

$$E(k,\Gamma)_* = W(\mathbb{F}_{p^{mn}})[[u_1, ..., u_{n-1}]][u^{\pm 1}],$$

where, as before, u has degree -2. Note that $E(k,\Gamma)_*$ is a profinite abelian group in each degree.

Let $G(m) = S_n \rtimes \mathrm{Gal}(k/\mathbb{F}_p)$. By functoriality of the Goerss-Hopkins construction, $G(m)$ acts on $E(k,\Gamma)$ through maps of E_∞-ring spectra. There is a recent extension of [20] by Devinatz ([17]), which shows that the results of [20] carry over with E_n replaced by $E(k,\Gamma)$ and G_n replaced by $G(m)$. Then, using this work, together with the techniques in [14], we obtain the following results.

Theorem 10.2.1. *The spectrum $E(k,\Gamma)$ is a continuous $G(m)$-spectrum. More generally, if X is any spectrum, then $L_{K(n)}(E(k,\Gamma) \wedge X)$ is a continuous $G(m)$-spectrum.*

Theorem 10.2.2. *Let H be any closed subgroup of $G(m)$. There is a conditionally convergent descent spectral sequence*

$$E_2^{s,t} \Rightarrow \pi_{t-s}((L_{K(n)}(E(k,\Gamma) \wedge X))^{hH}).$$

When $X = S^0$, there is an isomorphism $E_2^{s,t} \cong H_c^s(H; \pi_t(E(k,\Gamma)))$.

10.3. **The content of an archived preprint.** In [36], when the group G is infinite and profinite, Rognes does not make use of the notion of a continuous G-spectrum in his definition of Galois extension. In my preprint [15], available at the Hopf archive, I study Galois extensions from the perspective of continuous G-spectra. Also, assuming my conjecture that $\pi_*(E_n \wedge M_{I_i})$ can be realized by a discrete G_n-symmetric ring spectrum (see Section 11.1 below), I give examples of Galois extensions.

11. Future work

In this section, I describe various problems and projects that I am interested in working on.

11.1. $L_nM_{I_i}$ **as an** A_∞**-ring spectrum.** Currently, it is known that G_n acts on E_n by maps of E_∞-ring spectra. However, the continuous G_n-action is only known to act by unstructured maps of spectra. Thus, I am interested in showing that the continuous G_n-action is actually structured. As a first step, I want to show that G_n acts continuously on E_n through maps of A_∞-ring spectra, by showing that, in the presentation $E_n \cong \mathrm{holim}_i(E_n \wedge M_{I_i})$, each $E_n \wedge M_{I_i}$ is equivalent to a discrete G_n-symmetric ring spectrum E_n/I_i (that is, E_n/I_i is an A_∞-ring spectrum such that the discrete G_n-action is by maps of A_∞-ring spectra).

Let Spt_G be the category of discrete G-symmetric spectra, Spt_G^a the category of discrete G-symmetric ring spectra, and Spt^a the category of symmetric ring spectra. Then the first two categories should be model categories such that the forgetful functor $U\colon \mathrm{Spt}_G^a \to \mathrm{Spt}_G$ and the fixed points functor $(-)^G\colon \mathrm{Spt}_G^a \to \mathrm{Spt}^a$ preserve all weak equivalences and fibrations. Thus, if X is a discrete G-symmetric ring spectrum, then X^{hG} is a symmetric ring spectrum.

One exciting consequence of the above is that it would immediately imply that $(E_n/I_i)^{hG_n} \cong L_nM_{I_i}$ is an A_∞-ring spectrum. This would be a new and unexpected example of an A_∞-ring spectrum.

To solve this problem, as a part of my doctoral work, I partially developed a version of André-Quillen cohomology for twisted discrete associative R-G-algebras, where R is a twisted discrete commutative G-ring. Then, to prove the existence of E_n/I_i, one must show that certain obstruction groups (defined using this version of André-Quillen cohomology) vanish.

From Mark Behrens, I have learned that Mike Hopkins has an argument that might show that the tower $\{L_nM_{I_i}\}_i$ is an H_∞-object in the category of pro-spectra, and there is the possibility that H_∞ can be replaced with E_∞. Note that $F_n \wedge L_nM_{I_i}$ is a discrete G_n-spectrum and $E_n \wedge M_{I_i} \cong F_n \wedge L_nM_{I_i}$. Thus, perhaps one could argue as follows: (a) $\{L_nM_{I_i}\}$ is an E_∞-object in pro-spectra; (b) hence, for each j, $\{E_n^{dhU_j} \wedge L_nM_{I_i}\}$ is an E_∞-object in the pro-category of discrete G_n/U_j-spectra ($E_n^{dhU_j}$ is a commutative S-algebra by [20]); and (c) thus, $\{\mathrm{colim}_j(E_n^{dhU_j} \wedge L_nM_{I_i})\} \cong \{F_n \wedge L_nM_{I_i}\}$ is an E_∞-object in the pro-category of discrete G_n-spectra. Then, using the perspective of pro-spectra, one could say that E_n has a continuous G_n-E_∞-ring spectrum action. Thus, there is work to be done in understanding exactly what Hopkins's argument proves.

11.2. **The** G_n**-homotopy orbit spectrum of** $F(E_n, L_{K(n)}S^0)$**.** By using the homotopy orbit spectral sequence that I constructed in [7], it appears that, if $(p-1) \nmid n$, then $F(E_n, L_{K(n)}S^0)_{hG_n}$ can be identified with $L_{K(n)}S^0$. By using a different argument, it seems that this identification actually holds for all n and p. In future work, I would like to make these arguments precise and thereby give a proof of the identification.

11.3. **Dualizing spectra for profinite groups and the equivalence** $F(E_n, L_{K(n)}S^0) \simeq \Sigma^{-n^2}E_n$**.** In [7, Appendix], I explain an insight of Mark Behrens: in the equivalence

$$F(E_n, L_{K(n)}S^0) \simeq \Sigma^{-n^2}E_n \tag{11.3.1}$$

([38, Proposition 16]), the obvious point-set level G_n-actions are not necessarily compatible (for example, they are not compatible when $n = 2$ and $p = 3$), even though the induced actions by G_n, in the stable homotopy category, are compatible. I would like to understand this subtlety further.

By [24, (2.10), Proposition 2.6 (1)], there is a weak equivalence $\psi_i \colon E_n[G_n/U_i] \to F(E_n^{dhU_i}, E_n)$ that is G_n-equivariant, with G_n acting (a) diagonally on the source of this map, and (b) only on E_n in the target of the map. This yields a G_n-equivariant equivalence $\psi \colon E_n[[G_n]] \simeq F(E_n, E_n)$. By understanding the equivariance of ψ_i when G_n acts only on $E_n^{dhU_i}$ in the target of ψ_i, I should be able to show that, after taking the G_n-homotopy fixed points of ψ, there is a weak equivalence $(E_n[[G_n]])^{hG_n} \simeq F(E_n, L_{K(n)}S^0)$ that is G_n-equivariant. I want to show that this weak equivalence, when pushed down to the homotopy category, is exactly the equivalence (11.3.1), with its compatible G_n-actions (in the homotopy category). Thus, I want to understand the G_n-equivariant homotopy type of $(E_n[[G_n]])^{hG_n}$ and any possible relationship to the expression $L_{K(n)}(C \wedge E_n)$, where C is a spectrum that is non-equivariantly equivalent to S^{-n^2} and has a G_n-action so that $L_{K(n)}(C \wedge E_n)$ has a diagonal G_n-action.

Thus far, I have approached the last goal by studying $\mathrm{holim}_i(E_n \wedge M_{I_i} \wedge ((L_{K(n)}S^0)[G_n/U_i]))^{hG_n}$, where the homotopy fixed point spectrum still has a G_n-action. This naturally leads one to consider if some role is played by $((L_{K(n)}S^0)[[G_n]])^{hG_n} \cong \mathrm{holim}_i(L_{K(n)}S^0)^{hU_i}$. Also, one is led to wonder if, in analogy with [37, Sections 2.5 and 5.2], it might be helpful to formulate a norm map

$$N \colon (L_{K(n)}(E_n \wedge S^{adG_n}))_{hG_n} \to E_n^{hG_n},$$

which would require the development of a notion of the dualizing spectrum for profinite groups. The construction of N would involve using [7] and, perhaps, the work described in Section 11.2.

11.4. **Homotopy fixed points for pro-objects in discrete G-spectra.** My work in [14] only defines homotopy fixed points for towers of discrete G-spectra. However, in my work [4] with Mark Behrens, for an arbitrary pro-object $\{X_\alpha\}_\alpha$ in the category of discrete G-spectra, we define $\mathrm{holim}_\alpha(X_\alpha)^{hG}$ to be the homotopy fixed point spectrum $(\mathrm{holim}_\alpha X_\alpha)^{hG}$ of the continuous G-spectrum $\mathrm{holim}_\alpha X_\alpha$. However, we did not show that this definition makes the homotopy fixed points a total right derived functor of fixed points, in the appropriate sense. Thus, I would like to obtain a model category structure on $\mathrm{pro}-\mathrm{Spt}_G$, so that these homotopy fixed points are the correct total right derived functor. It should be possible to do this by applying work of Isaksen (see [29]).

References

[1] Mark Behrens. Buildings, elliptic curves, and the $K(2)$-local sphere. *To appear in the American Journal of Mathematics, available at www-math.mit.edu/~mbehrens/papers/index.html.*

[2] Mark Behrens. A modular description of the $K(2)$-local sphere at the prime 3. *Topology*, 45(2):343–402, 2006.

[3] Mark Behrens and Tyler Lawson. Topological automorphic forms. *In preparation, 152 pp., October 26, 2006, available at www-math.mit.edu/~mbehrens/preprints.*

[4] Mark J. Behrens and Daniel G. Davis. The homotopy fixed point spectra of profinite Galois extensions. *In preparation, 33 pp., current version available at http://dgdavis.web.wesleyan.edu.*

[5] Daniel G. Davis. Continuous G-spectra and Brown-Comenetz duality. *In preparation, 15 pp.*

[6] Daniel G. Davis. The homotopy fixed points of E_n and the Devinatz-Hopkins construction. *In preparation, 12 pp.*

[7] Daniel G. Davis. The homotopy orbit spectrum for profinite groups. *Submitted to Homology, Homotopy and Applications, received favorable referee's report, under revision, 13 pp., available online as arXiv:math.AT/0608262.*

[8] Daniel G. Davis. Iterated homotopy fixed points for the Lubin-Tate spectrum. *Submitted to Algebraic and Geometric Topology, 15 pp., available online as arXiv:math.AT/0610907.*

[9] Daniel G. Davis. The site R_G^+ for a profinite group G. *Submitted to Theory and Applications of Categories, 14 pp., available online as arXiv:math.AT/0610772.*

[10] Daniel G. Davis. Some examples of totally hyperfibrant discrete G-spectra. *In preparation, 4 pp.*

[11] Daniel G. Davis. A universal coefficient spectral sequence for discrete G-spectra. *In preparation, 28 pp.*

[12] Daniel G. Davis. The topological ring $E_{n*}^{\mathfrak{s}}[\![G_n]\!]$ and results on topological modules for Lubin-Tate theory. *Manuscript, Northwestern University*, 2002.

[13] Daniel G. Davis. The E_2-term of the descent spectral sequence for continuous G-spectra. *New York J. Math.*, 12:183–191 (electronic), published at http://nyjm.albany.edu/j/2006/12–11.html, 2006.

[14] Daniel G. Davis. Homotopy fixed points for $L_{K(n)}(E_n \wedge X)$ using the continuous action. *J. Pure Appl. Algebra*, 206(3):322–354, 2006.

[15] Daniel G. Davis. Rognes's theory of Galois extensions and the continuous action of G_n on E_n. *Preprint, 14 pp., available at the Hopf archive at http://hopf.math.purdue.edu*, May, 2004.

[16] Daniel G. Davis. The Lubin-Tate spectrum and its homotopy fixed point spectra. *Thesis, 106 pp., Northwestern University*, May 9, 2003, available at http://dgdavis.web.wesleyan.edu.

[17] Ethan S. Devinatz. E-mail correspondence. *May 8, 2005*.

[18] Ethan S. Devinatz. Homotopy groups of homotopy fixed point spectra associated to E_n. *Available at the Hopf topology archive at http://hopf.math.purdue.edu, 14 pp.*

[19] Ethan S. Devinatz. A Lyndon-Hochschild-Serre spectral sequence for certain homotopy fixed point spectra. *Trans. Amer. Math. Soc.*, 357(1):129–150 (electronic), 2005.

[20] Ethan S. Devinatz and Michael J. Hopkins. Homotopy fixed point spectra for closed subgroups of the Morava stabilizer groups. *Topology*, 43(1):1–47, 2004.

[21] Halvard Fausk. Equivariant homotopy theory for pro-spectra. *57 pp., available online as arXiv:math.AT/0609635*.

[22] P. G. Goerss and M. J. Hopkins. Moduli spaces of commutative ring spectra. In *Structured ring spectra*, volume 315 of *London Math. Soc. Lecture Note Ser.*, pages 151–200. Cambridge Univ. Press, Cambridge, 2004.

[23] Paul Goerss, Hans-Werner Henn, and Mark Mahowald. The homotopy of $L_2V(1)$ for the prime 3. In *Categorical decomposition techniques in algebraic topology (Isle of Skye, 2001)*, volume 215 of *Progr. Math.*, pages 125–151. Birkhäuser, Basel, 2004.

[24] Paul Goerss, Hans-Werner Henn, Mark Mahowald, and Charles Rezk. A resolution of the K(2)-local sphere at the prime 3. *Annals of Math.*, 162(2):777–822, 2005.

[25] Paul G. Goerss. Homotopy fixed points for Galois groups. In *The Čech centennial (Boston, MA, 1993)*, pages 187–224. Amer. Math. Soc., Providence, RI, 1995.

[26] M. J. Hopkins. Algebraic topology and modular forms. In *Proceedings of the International Congress of Mathematicians, Vol. I (Beijing, 2002)*, pages 291–317, Beijing, 2002. Higher Ed. Press.

[27] Mark Hovey. Operations and co-operations in Morava E-theory. *Homology Homotopy Appl.*, 6(1):201–236 (electronic), 2004.

[28] Mark Hovey and Neil P. Strickland. Morava K-theories and localisation. *Mem. Amer. Math. Soc.*, 139(666):viii+100, 1999.

[29] Daniel C. Isaksen. Strict model structures for pro-categories. In *Categorical decomposition techniques in algebraic topology (Isle of Skye, 2001)*, volume 215 of *Progr. Math.*, pages 179–198. Birkhäuser, Basel, 2004.

[30] J. F. Jardine. Simplicial presheaves. *J. Pure Appl. Algebra*, 47(1):35–87, 1987.

[31] J. F. Jardine. Stable homotopy theory of simplicial presheaves. *Canad. J. Math.*, 39(3):733–747, 1987.

[32] J. F. Jardine. *Generalized étale cohomology theories*. Birkhäuser Verlag, Basel, 1997.

[33] J. F. Jardine. Generalised sheaf cohomology theories. In *Axiomatic, enriched and motivic homotopy theory*, volume 131 of *NATO Sci. Ser. II Math. Phys. Chem.*, pages 29–68. Kluwer Acad. Publ., Dordrecht, 2004.

[34] Stephen A. Mitchell. Hypercohomology spectra and Thomason's descent theorem. In *Algebraic K-theory (Toronto, ON, 1996)*, pages 221–277. Amer. Math. Soc., Providence, RI, 1997.

[35] Charles Rezk. Notes on the Hopkins-Miller theorem. In *Homotopy theory via algebraic geometry and group representations (Evanston, IL, 1997)*, pages 313–366. Amer. Math. Soc., Providence, RI, 1998.

[36] John Rognes. Galois extensions of structured ring spectra. *To appear in Memoirs of the A.M.S., August 31, 2005, 93 pp.*

[37] John Rognes. Stably dualizable groups. *To appear in Memoirs of the A.M.S., 31 pp.*

[38] N. P. Strickland. Gross-Hopkins duality. *Topology*, 39(5):1021–1033, 2000.

[39] Peter Symonds and Thomas Weigel. Cohomology of p-adic analytic groups. In *New horizons in pro-p groups*, pages 349–410. Birkhäuser Boston, Boston, MA, 2000.

[40] R. W. Thomason. Algebraic K-theory and étale cohomology. *Ann. Sci. École Norm. Sup. (4)*, 18(3):437–552, 1985.