THE LUBIN-TATE SPECTRUM AND ITS HOMOTOPY FIXED POINT SPECTRA

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ABSTRACT. This note is a summary of the results of my Ph.D. thesis (plus slight modifications), completed May 9, 2003, under the supervision of Professor Paul Goerss at Northwestern University.

Let E_n be the Lubin-Tate spectrum with

$$E_{n*} = W(\mathbb{F}_{p^n})[[u_1, ..., u_{n-1}]][u^{\pm 1}],$$

where the degree of u is -2 and the complete power series ring over the Witt vectors is in degree zero. Let $G_n = S_n \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$, where S_n is the nth Morava stabilizer group (the automorphism group of the Honda formal group law Γ_n of height n over $\mathbb{F}_{p^n}$), and let G be a closed subgroup of G_n. Note that S_n, G_n and G are all profinite groups. Morava's change of rings theorem yields a spectral sequence

$$H^*_c(G_n; \pi_*(E_n \wedge X)) \Longrightarrow \pi_* L_{K(n)}(X), \tag{1}$$

where the E_2-term is continuous cohomology and X is a finite spectrum (see [7], [1], [5]). Using the G_n-action on E_n by maps of commutative S-algebras (work of Goerss and Hopkins ([3], [4]), and Hopkins and Miller [8]), Devinatz and Hopkins [2] constructed spectra E_n^{hG} with strongly convergent spectral sequences

$$H^*_c(G; \pi_*(E_n \wedge X)) \Longrightarrow \pi_*(E_n^{hG} \wedge X). \tag{2}$$

Also, Devinatz and Hopkins showed that $E_n^{hG_n} \wedge X \simeq L_{K(n)}(X)$.

When K is a discrete group and Y is a K-spectrum, there is a homotopy fixed point spectrum $Y^{hK} = \mathrm{Map}_K(EK_+, Y)$, where EK_+ is a free contractible K-space. Also, there is a conditionally convergent spectral sequence

$$E_2^{s,t} = H^s(K; \pi_t(Y)) \Longrightarrow \pi_{t-s}(Y^{hK}),$$

where the E_2-term is group cohomology [6, §1.1]. Such a spectral sequence is called a descent spectral sequence.

This scenario also occurs in another context. Let K be a profinite group. We say that Y is a discrete K-spectrum, if Y is a K-spectrum of simplicial sets such that each simplicial set Y_k is a simplicial discrete K-set (that is, for each $l \geq 0$, the action map on the l-simplices of Y_k,

$K \times (Y_k)_l \to (Y_k)_l$, is a continuous map, where $(Y_k)_l$ is given the discrete topology). Using work of Jardine, there is a model category $Sp_{_K}$ of discrete K-spectra, and one defines $Y^{hK} = (Y_f)^K$ to be the homotopy fixed point spectrum of Y, where $Y \to Y_f$ is a trivial cofibration and Y_f is fibrant, all in $Sp_{_K}$. Then, if the cohomological dimension of K satisfies an appropriate finiteness hypothesis, there is a conditionally convergent descent spectral sequence

$$H^s_c(K; \pi_t(Y)) \Longrightarrow \pi_{t-s}(Y^{hK}).$$

Upon comparing (2) with the above descent spectral sequences, E_n appears to be a continuous G_n-spectrum with "descent" spectral sequences for "homotopy fixed point" spectra $E_n^{hG} \wedge X$. Indeed, we explain how [2] implies that E_n is a continuous G_n-spectrum - E_n is an inverse limit of discrete G_n-spectra. Using this continuous action, we define homotopy fixed point spectra $(E_n \wedge X)^{hG}$ that are weakly equivalent to $E_n^{hG} \wedge X$, and, for $(E_n \wedge X)^{hG}$, we construct a descent spectral sequence that is isomorphic to (2). We remark that this continuous action is not shown to be by A_∞- or E_∞-maps of ring spectra.

In more detail, the $K(n)_*$-local spectrum E_n has an action by G_n as a commutative S-algebra. The $K(n)_*$-local commutative S-algebra E_n^{hG} has an associated strongly convergent $K(n)_*$-local E_n-Adams spectral sequence

$$E_2^{*,*} \cong H^*_c(G; E_n^*(Z)) \Longrightarrow (E_n^{hG})^*(Z),$$

where Z is any CW-spectrum. Also, [2] proves the remarkable formula

$$E_n \simeq L_{K(n)}(\mathrm{hocolim}_{i \ E_\infty} \, E_n^{hU_i}), \tag{3}$$

where $\{U_i\}$ is a cofinal descending chain of open normal subgroups of G_n, and the homotopy colimit, as the notation indicates, is in the category of commutative S-algebras.

Devinatz and Hopkins prove that the homotopy fixed point spectra E_n^{hG} have the expected properties and, when one sets Z equal to the Spanier-Whitehead dual of any finite spectrum X, one obtains a spectral sequence

$$E_2^{*,*} \cong H^*_c(G; \pi_*(E_n \wedge X)) \Longrightarrow \pi_*(E_n^{hG} \wedge X)$$

that has the form of a descent spectral sequence. Thus, their constructions strongly suggest that G_n acts on E_n in a continuous sense. However, their highly structured action is not proven to be continuous and their homotopy fixed point spectra are not defined with respect to a continuous action.

Let $F_n = \mathrm{colim}_i \, E_n^{hU_i}$. Given $I = (p^{i_0}, v_1^{i_1}, ..., v_{n-1}^{i_{n-1}}) \subset BP_*$, let $M_I = M(p^{i_0}, v_1^{i_1}, ..., v_{n-1}^{i_{n-1}})$ (when it exists) be the generalized Moore spectrum with $BP_*(M_I) \cong BP_*/I$. We observe that $F_n \wedge M_I$ is a discrete G_n-spectrum, since it is the colimit of G_n/U_i-spectra. Then the key fact for getting our work started is

$$E_n \wedge M_I \simeq F_n \wedge M_I,$$

obtained by applying (3), since this implies that $E_n \wedge M_I$ has the stable homotopy type of a discrete G_n-spectrum.

Henceforth, X is a finite spectrum only if this is explicitly mentioned.

Theorem 1. *There is a tower*

$$\cdots \to (F_n \wedge M_{I_j})_f \to \cdots \to (F_n \wedge M_{I_1})_f \to (F_n \wedge M_{I_0})_f$$

of fibrations of fibrant spectra in Sp_{G_n}*, such that each* $(F_n \wedge M_{I_j})_f \simeq F_n \wedge M_{I_j}$*, and*

$$E_n \simeq \lim_j (F_n \wedge M_{I_j})_f$$

is an inverse limit of discrete G_n*-spectra. Thus,* E_n *is a continuous* G_n*-spectrum. Similarly, for any spectrum* X*,*

$$L_{K(n)}(E_n \wedge X) \simeq \lim_j (F_n \wedge M_{I_j} \wedge X)_f$$

is a continuous G_n*-spectrum.*

Using this continuous action, we define homotopy fixed point spectra

$$\big(L_{K(n)}(E_n \wedge X)\big)^{hG} = \operatorname{holim}_j (F_n \wedge M_{I_j} \wedge X)^{hG},$$

for any X. When X is a finite spectrum, this definition gives the homotopy fixed point spectrum $(E_n \wedge X)^{hG}$.

Theorem 2. *Let* G *be a closed subgroup of* G_n *and let* X *have the property that the tower of abelian groups* $\{\pi_t(E_n \wedge M_{I_j} \wedge X)\}_j$ *is Mittag-Leffler for every integer* t *(e.g.* X *is finite or* $X = E_n \wedge E_n \wedge \cdots \wedge E_n$*). Then there is a conditionally convergent descent spectral sequence*

$$H^s_{\mathrm{cts}}(G; \pi_t(L_{K(n)}(E_n \wedge X))) \Longrightarrow \pi_{t-s}\big(\big(L_{K(n)}(E_n \wedge X)\big)^{hG}\big),$$

where the E_2*-term is the cohomology of continuous cochains. In particular, if* X *is a finite spectrum, this descent spectral sequence has the form*

$$H^s_c(G; \pi_t(E_n \wedge X)) \Longrightarrow \pi_{t-s}((E_n \wedge X)^{hG}), \tag{4}$$

where $H^s_c(G; \pi_t(E_n \wedge X)) \cong \varprojlim_k H^s_c(G; \pi_t(E_n \wedge X)/I_n^k)$*, where* $I_n = (p, u_1, ..., u_{n-1}) \subset E_{n*}$.

We also show that the descent spectral sequence, when X is finite, is isomorphic to the spectral sequence of Devinatz and Hopkins.

Theorem 3. *When X is a finite spectrum, descent spectral sequence (4) is isomorphic to the strongly convergent $K(n)_*$-local E_n-Adams spectral sequence with abutment $\pi_*(E_n^{hG} \wedge X)$. In particular, in the stable homotopy category, the morphism $E_n^{hG} \wedge X \to (E_n \wedge X)^{hG}$ is an isomorphism.*

Finally, we prove that the $K(n)_*$-localization of any finite complex is a homotopy fixed point spectrum. In particular, $L_{K(n)}(S^0)$ is the G_n-homotopy fixed points of E_n, in a continuous sense.

Theorem 4. *Let X be a finite complex. Then*

$$L_{K(n)}(X) \cong (E_n \wedge X)^{hG_n},$$

in the stable homotopy category. In particular, if X is also of type n, then

$$L_n X \cong (F_n \wedge X)^{hG_n}.$$

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