NORTHWESTERN UNIVERSITY

The Lubin-Tate Spectrum and its Homotopy Fixed Point Spectra

A DISSERTATION

SUBMITTED TO THE GRADUATE SCHOOL

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

for the degree

DOCTOR OF PHILOSOPHY

Field of Mathematics

By

Daniel Glen Davis

EVANSTON, ILLINOIS

December 2003

ABSTRACT

The Lubin-Tate Spectrum and its Homotopy Fixed Point Spectra

Daniel Glen Davis

Using the work of Devinatz and Hopkins, we show that the Lubin-Tate spectrum E_n is a continuous G_n-spectrum, where G_n is the extended Morava stabilizer group. For G closed in G_n and any finite spectrum X, we use this continuous action to define the homotopy fixed point spectrum $(E_n \wedge X)^{hG}$ with the associated descent spectral sequence

$$H_c^s(G; \pi_t(E_n \wedge X)) \Longrightarrow \pi_{t-s}((E_n \wedge X)^{hG}).$$

We show that this spectral sequence is isomorphic to the strongly convergent $K(n)_*$-local E_n-Adams spectral sequence abutting to $\pi_*(E_n^{hG} \wedge X)$. We also have a descent spectral sequence for $\big(L_{K(n)}(E_n \wedge X)\big)^{hG}$, where X satisfies a particular finiteness condition.

Acknowledgements

First of all, many thanks to my thesis advisor Paul Goerss for teaching me so much through my coursework and my research. I had a plethora of helpful conversations with him and received many suggestions that improved the organization and writing of this thesis. His fingerprints are present throughout this work and its proofs.

I want to express my gratitude to Mark Mahowald and Stewart Priddy for the variety of ways that they helped me during the course of my Ph.D. I really appreciate their careful examination of my thesis and their comments about various drafts of it. Most of all, I have been regularly inspired by the wonderful example Professors Goerss, Mahowald and Priddy have set as mathematicians.

I want to point out that [**9**], a paper by Ethan Devinatz and Mike Hopkins, was very inspirational for this work, and I thank Ethan Devinatz for helpful conversations and e-mails.

When I first suspected that $E_n \wedge M_I \simeq F_n \wedge M_I$, I asked Charles Rezk about this, and he saw its validity and sketched a proof. I thank Charles for his insight on this point and related conversations.

I thank Rick Jardine for answering several questions about homotopy fixed points, and he and Jeff Smith for answering questions about the smash product. Also, I had many helpful conversations with Christian Haesemeyer during the course of my research, and I thank him for his time.

I had several helpful conversations with Halvard Fausk, and, in seminars about this work, Sharon Hollander and Hal Sadofsky asked “do I know if?” questions that helped me in formulating Theorems 10.12 and 10.6, respectively.

I thank Mark Pearson for his camaraderie as we worked on our dissertations concurrently, and Paul Pearson for his answers to tricky LaTeX questions.

I am grateful to my family, especially my parents, for their encouragement during my time in graduate school. Thanks Joy, for your friendship and all the fun chats we've had.

Most of all, I thank Lisa, my wonderful wife, for her support, encouragement, sense of humor, and many other things; she made an already pleasureful endeavor so much more satisfying.

Contents

CHAPTER 1

Introduction

Let E_n be the Lubin-Tate spectrum with $E_{n*} = W(\mathbb{F}_{p^n})[[u_1, ..., u_{n-1}]][u^{\pm 1}]$, where the degree of u is -2 and the complete power series ring over the Witt vectors is in degree zero. Let $G_n = S_n \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$, where S_n is the nth Morava stabilizer group, and let G be a closed subgroup of G_n. Morava's change of rings theorem yields a spectral sequence

$$H_c^*(G_n; \pi_*(E_n \wedge X)) \Longrightarrow \pi_* L_{K(n)}(X), \tag{1.1}$$

where the E_2-term is continuous cohomology and X is a finite spectrum. Using the G_n-action on E_n by maps of commutative S-algebras (work of Goerss and Hopkins ([**15**], [**16**]), and Hopkins and Miller [**39**]), Devinatz and Hopkins [**9**] constructed spectra E_n^{hG} with strongly convergent spectral sequences

$$H_c^*(G; \pi_*(E_n \wedge X)) \Longrightarrow \pi_*(E_n^{hG} \wedge X). \tag{1.2}$$

Also, they showed that $E_n^{hG_n} \wedge X \simeq L_{K(n)}(X)$.

When K is a discrete group and Y is a K-spectrum, there is a homotopy fixed point spectrum $Y^{hK} = \mathrm{Map}_K(EK_+, Y)$, where EK_+ is a free contractible K-space. Also, there is a conditionally convergent spectral sequence

$$E_2^{s,t} = H^s(K; \pi_t(Y)) \Longrightarrow \pi_{t-s}(Y^{hK}),$$

where the E_2-term is group cohomology [**34**, §1.1]. Such spectral sequences are called descent spectral sequences.

Thus, by comparison, E_n appears to be a continuous G_n-spectrum with "descent" spectral sequences for "homotopy fixed point" spectra $E_n^{hG} \wedge X$. Indeed, we explain how their work implies that E_n is a continuous G_n-spectrum - E_n is a homotopy inverse limit of discrete G_n-spectra. Using this continuous action, we define homotopy fixed point spectra $(E_n \wedge X)^{hG}$ that are weakly equivalent to $E_n^{hG} \wedge X$, and we construct descent spectral sequences having the form of 1.2.

In more detail, the $K(n)_*$-local spectrum E_n has an action by G_n as a commutative S-algebra. The $K(n)_*$-local commutative S-algebra E_n^{hG} has an associated strongly convergent spectral sequence

$$E_2^{*,*} \cong H_c^*(G; E_n^*(Z)) \Longrightarrow (E_n^{hG})^*(Z),$$

where Z is any CW-spectrum. Also, Devinatz and Hopkins prove the remarkable formula $E_n \simeq L_{K(n)}(\mathrm{hocolim}_{i\ E_\infty}\, E_n^{hU_i})$, where $\{U_i\}$ is a cofinal descending chain of open normal subgroups of G_n, and the homotopy colimit, as the notation indicates, occurs in the category of commutative S-algebras.

Devinatz and Hopkins prove that the homotopy fixed point spectra E_n^{hG} have the expected properties and, when one sets Z equal to the Spanier-Whitehead dual of any finite spectrum X, one obtains a spectral sequence

$$E_2^{*,*} \cong H_c^*(G; \pi_*(E_n \wedge X)) \Longrightarrow \pi_*(E_n^{hG} \wedge X)$$

that has the form of a classical descent spectral sequence. Thus, their constructions strongly suggest that G_n acts on E_n in a continuous sense. However, their highly structured action is not proven to be continuous, their homotopy fixed point spectra are not defined with respect to a continuous action, and their homotopy fixed point spectral sequences are not constructed as descent spectral sequences, but are $K(n)_*$-local E_n-Adams spectral sequences.

Let $F_n = \operatorname{colim}_i E_n^{hU_i}$ and let M_I (when it exists) be the generalized Moore spectrum with $BP_*(M_I) \cong BP_*/I$. The key for getting our work started is knowing that

$$E_n \wedge M_I \simeq F_n \wedge M_I,$$

and thus, $E_n \wedge M_I$ has the homotopy type of a discrete G_n-spectrum, a spectrum that levelwise consists of simplicial discrete G_n-sets. This result is almost immediate due to the work of Devinatz and Hopkins.

Theorem 1.3. *As the homotopy inverse limit of a pro-object in the category of discrete G_n-spectra, $E_n \simeq \operatorname{holim}_I (F_n \wedge M_I)$ is a continuous G_n-spectrum. Also, if the* *tower of abelian groups* $\{\pi_t(E_n \wedge M_I \wedge X)\}_I$ *is Mittag-Leffler for a fixed spectrum* X *and every integer* t *(e.g.* X *is finite), then* $L_{K(n)}(E_n \wedge X) \simeq \operatorname{holim}_I (F_n \wedge M_I \wedge X)$ *is a continuous* G_n*-spectrum.*

Using this continuous action and for X as in Theorem 1.3, we define homotopy fixed point spectra $\big(L_{K(n)}(E_n \wedge X)\big)^{hG}$ and construct their descent spectral sequences. To avoid the pile-up of parentheses, instead of $\big(L_{K(n)}(E_n \wedge X)\big)^{hG}$, we write $L_{K(n)}(E_n \wedge X)^{hG}$.

Theorem 1.4. *Let* G *be a closed subgroup of* G_n *and let* X *satisfy the hypotheses of the previous theorem. Then there is a conditionally convergent descent spectral sequence*

$$H^s_{\text{cts}}(G; \pi_t(L_{K(n)}(E_n \wedge X))) \Longrightarrow \pi_{t-s}(L_{K(n)}(E_n \wedge X)^{hG}),$$

where the E_2*-term is the cohomology of continuous cochains. In particular, if* X *is a finite spectrum, this descent spectral sequence is strongly convergent and has the form*

$$H^s_c(G; \pi_t(E_n \wedge X)) \Longrightarrow \pi_{t-s}((E_n \wedge X)^{hG}), \tag{1.5}$$

where the E_2*-term coincides with the continuous cohomology of* [**9**, Rk. 0.3].

We also show that computing with the descent spectral sequence when X is finite is the same as computing with the Adams spectral sequence of Devinatz and Hopkins: the differentials are identical.

Theorem 1.6. *When X is a finite spectrum, descent spectral sequence* 1.5 *is isomorphic to the $K(n)_*$-local E_n-Adams spectral sequence with abutment $\pi_*(E_n^{hG} \wedge X)$. In particular, in the stable homotopy category, the morphism $E_n^{hG} \wedge X \to (E_n \wedge X)^{hG}$ is an isomorphism.*

Finally, we prove a theorem that chromatic stable homotopy theorists have suspected is true, at least since the advent of Morava's change of rings theorem [**36**] and the Annals work of Miller, Ravenel and Wilson in 1977 [**33**]: the $K(n)_*$-localization of any finite complex is a homotopy fixed points spectrum.

Theorem 1.7. *Let X be a finite complex. Then $L_{K(n)}(X) \cong (E_n \wedge X)^{hG_n}$, in the stable homotopy category. In particular, if X is also of type n, then $L_n X \cong (F_n \wedge X)^{hG_n}$.*

We want to stress at the outset how indebted we are to the beautiful work of Devinatz and Hopkins, and to the work of Paul Goerss and Mike Hopkins upon which [**9**] is reliant ([**15**], [**16**]), and the earlier work of Hopkins and Haynes Miller [**39**].

It is interesting to note that the key weak equivalence $E_n \wedge M_I \simeq F_n \wedge M_I$ is expected, in the following sense. Since $\pi_*(E_n \wedge M_I)$ is a discrete G_n-module, one might hope for a spectrum $E_n/I \simeq E_n \wedge M_I$ that is a discrete G_n-spectrum. Given a profinite group H, the work of Jardine gives a model category structure on the category of discrete H-spectra (see §4.1). Given a discrete H-spectrum Y, one defines $Y^{hH} = (Y_{f,H})^H$, where $Y \xrightarrow{\simeq} Y_{f,H}$ is a fibrant replacement in the category of discrete H-spectra. Also, if K is an open normal subgroup of H, then a fibrant discrete H-spectrum is also fibrant as a discrete K-spectrum [**27**, Rk. 6.26].

Thus,

$$E_n/I \simeq (E_n/I)_{f,G_n} \cong \operatorname{colim}_i (E_n/I)^{U_i}_{f,G_n} \simeq \operatorname{colim}_i (E_n/I)^{U_i}_{f,U_i}$$
$$= \operatorname{colim}_i (E_n/I)^{hU_i} \simeq \operatorname{colim}_i (E_n \wedge M_I)^{hU_i} \simeq \operatorname{colim}_i (E_n^{hU_i} \wedge M_I),$$

where in the last two steps we applied the ideas that (a) $E_n \wedge M_I$ ought to have well-behaved homotopy fixed point spectra $(E_n \wedge M_I)^{hU_i}$; and (b) the work of Devinatz and Hopkins suggests that $E_n^{hU_i} \wedge M_I$ and $(E_n \wedge M_I)^{hU_i}$ ought to have the same homotopy type.

It is important to note that the model given here for E_n as a continuous G_n-spectrum is not completely satisfactory. For example, the continuous action is by morphisms that are just maps of spectra. However, it is known that there are models for E_n where the G_n-action is by A_∞- and E_∞-automorphisms of ring spectra, and therefore we would like to know that such enriched actions are actually continuous.

Another deficiency is that I do not know if $\operatorname{holim}_I (F_n \wedge M_I)$ is a twisted E_n-module spectrum (that is, the module structure map (if it exists), with diagonal action on the domain, is G_n-equivariant). Clearly, one expects any fully functional model to be a twisted E_n-module spectrum, as E_n itself is.

Outline of Contents. In Chapter 2, we give some background material, recall useful facts, and handle several technical issues. In Chapter 3, we show how the Adams-Novikov spectral sequence for $\pi_*(L_{K(n)}X)$, when X is finite, gives 1.1. In Chapter 4, we define "continuous G-spectrum," and discuss the model categories of discrete G-spectra and their towers. Chapter 5 shows that E_n is a continuous G_n-spectrum. Homotopy fixed points for discrete G-spectra are discussed in Chapter 6, while Chapter 7 constructs homotopy fixed point spectra $E_n^{h'G}$ (using the continuous action) that are weakly equivalent to E_n^{hG}. Chapter 8 contains various results about continuous cohomology and towers of abelian groups that are needed for later work. Chapter 9 builds the descent spectral sequence for $L_{K(n)}(E_n \wedge X)^{hG}$, when X is E_n-Mittag-Leffler, and Chapter 10 proves that, when X is a finite spectrum, the descent spectral sequence for $(E_n \wedge X)^{hG}$ is strongly convergent and is isomorphic to the $K(n)_*$-local E_n-Adams spectral sequence for $E_n^{hG} \wedge X$.

Notation and Terminology. Gal is the Galois group $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$. For a profinite group G, we write $G \cong \varprojlim_N G/N$, an inverse limit over the open normal subgroups.

The nth Morava stabilizer group S_n is a profinite group, defined to be the automorphism group of the Honda formal group law Γ_n of height n over $\mathbb{F}_{p^n}$. $G_n = S_n \rtimes \mathrm{Gal}$ is a profinite group with a descending chain of open normal subgroups $G_n = U_0 \gneqq U_1 \gneqq ... \gneqq U_i \gneqq ...$, such that $\bigcap_{i=0}^{\infty} U_i = \{e\}$. The canonical map $G_n \rightarrow \varprojlim_i G_n/U_i$ is a group isomorphism and a homeomorphism. The notation $N <_o G$ means that N is an open subgroup of G; "$<_c$" refers to closed subgroups.

We use G to denote arbitrary profinite groups and, specifically, closed subgroups of G_n, sometimes in the same section. The context should always make it clear which we mean.

Let BP denote the Brown-Peterson spectrum with $BP_* = \mathbb{Z}_{(p)}[v_1, v_2, ...]$, where $|v_i| = 2(p^i - 1)$. The ideal $(p^{i_0}, v_1^{i_1}, ..., v_{n-1}^{i_{n-1}})$ in BP_* is denoted by I; M_I is the corresponding generalized Moore spectrum $M(p^{i_0}, v_1^{i_1}, ..., v_{n-1}^{i_{n-1}})$. Given an ideal I, M_I need not exist; however, enough exist for our constructions (e.g. see 2.3). The spectrum M_I is a finite type n spectrum with the property that $BP_*(M_I) \cong BP_*/I$. (See [**5**, §4], [**23**, §4], and [**30**, Prop. 3.7] for details.) The map $r\colon BP_* \to E_{n*}$ - defined by $r(v_i) = u_i u^{1-p^i}$, where $u_n = 1$ and $u_i = 0$, when $i > n$ - sends I to the ideal $(p^{i_0}, u_1^{i_1}, ..., u_{n-1}^{i_{n-1}})$ in E_{n*}, also denoted by I. Landweber exactness gives $\pi_*(E_n \wedge M_I) \cong E_{n*}/I$. The set $\{i_0, ..., i_{n-1}\}$ of superscripts of positive integers varies so that there is a family of ideals $\{I\}$. Also, let $I_n = (p, u_1, ..., u_{n-1}) \subset E_{n*}$.

Since we will frequently use (homotopy) limits indexed by these ideals, we make the following remarks. Given I_1, I_2 in $\{I\}$, $I_1 \leq I_2$ if and only if $I_2 \subset I_1$. Even though $\{I\}$ is not the inverse system $\{0 \leftarrow 1 \leftarrow 2 \leftarrow ...\}$, since $\{I\}$ is a countable directed poset, it contains a cofinal directed poset $J = \{I_0 \leftarrow I_1 \leftarrow I_2 \leftarrow ...\}$. Therefore, in what follows, the functors $\varprojlim_I (-)$ and $\mathrm{holim}_I (-)$ are always over a cofinal subset with the shape of J. We always write the subscript "I" even though we really mean the cofinal subset.

The homotopy limit of (Bousfield-Friedlander) spectra is constructed levelwise in $\mathcal{S}$, the category of simplicial sets. The homotopy limit in $\mathcal{S}$ of pointed simplicial sets is again a member of $\mathcal{S}_*$, the category of pointed simplicial sets. We use $S^n \in \mathcal{S}$ to denote the n-sphere.

Given a spectrum X, DX is its Spanier-Whitehead dual. Given a spectrum E, $E^j = E \wedge E \wedge \cdots \wedge E$, with j factors. $K(n)$ is the nth Morava K-theory spectrum with $K(n)_* = \mathbb{F}_p[v_n, v_n^{-1}]$. We use L_n to denote the Bousfield localization functor with respect to $E(n)$, where $E(n)_* = \mathbb{Z}_{(p)}[v_1, ..., v_{n-1}][v_n, v_n^{-1}]$. The category of commutative S-algebras is denoted by E_∞. If C is a model category, then $\mathrm{Ho}(\mathsf{C})$ is its homotopy category and $[-,-]_{\mathsf{C}}$ denotes morphisms in $\mathrm{Ho}(\mathsf{C})$. **Spt** is the category of Bousfield-Friedlander spectra, and "the stable category" always refers to Ho(**Spt**).

Given a site $\mathcal{C}$, $\mathcal{C}^\sim$ is the Grothendieck topos of sheaves of sets on the site. $\mathrm{DMod}(G)$ is the category of discrete G-modules. A discrete G-set X is a G-set such that the action map $G \times X \to X$ is continuous when X is given the discrete topology. The map $r_g \colon G_n \to G_n$ denotes right multiplication by $g \in G_n$.

We use $\mathrm{Map}_c(G, A)$ to denote the set of continuous maps out of the profinite group G into the object A, which can be a topological abelian group, a discrete G-spectrum, etc. We often write $\Gamma_G(A)$ for $\mathrm{Map}_c(G, A)$, and $\Gamma_G^k(A) = \mathrm{Map}_c(G^k, A)$, where G^k is the k-fold product of G and $G^0 = *$.

If $M \cong \varprojlim_\alpha M_\alpha$ is an inverse system of discrete G-modules, then we use $H_c^*(G; M)$ to denote $\varprojlim_\alpha H_c^*(G; M_\alpha)$.

Given a topological abelian group $M \cong \varprojlim_\alpha M_\alpha$, where M_α is a discrete abelian group, $\mathrm{Map}_c(G_n^j, M)$ is the topological G_n-module of continuous maps with action defined by $(g' \cdot f)(g_1, ..., g_j) = f(g_1 g', g_2, ..., g_j)$. Also, $\mathrm{Map}_c^\ell(G_n^j, M)$ is the topological G_n-module of continuous maps with action denoted and defined by $(g' \cdot f)^\ell(g_1, ..., g_j) = f((g')^{-1} g_1, g_2, ..., g_j)$. There is a G_n-equivariant isomorphism of topological G_n-modules

$$p\colon \mathrm{Map}_c^\ell(G_n^j, M) \to \mathrm{Map}_c(G_n^j, M), \quad p(f)(g_1, g_2, ..., g_j) = f(g_1^{-1}, g_2, ..., g_j).$$

We use $\bar{k}$ to denote the separable closure of the field k, and Ab is the category of abelian groups.

CHAPTER 2

Preliminaries

In this chapter, we recall some frequently used facts, discuss background material, and handle several technical issues to help get our work started.

In [**9**], Devinatz and Hopkins, using work by Goerss and Hopkins ([**15**], [**16**]), and Hopkins and Miller [**39**], show that the action of G_n on E_n is by maps of commutative S-algebras. Previously, Hopkins and Miller showed that G_n acts on E_n by maps of A_∞-ring spectra. However, the continuous action presented here is not so structured. As will be seen later, the starting point for the continuous action is the spectrum $F_n \wedge M_I$ and this spectrum is not known to even be an A_∞-ring spectrum. Thus, we work in an unstructured category of spectra of simplicial sets and the continuous action is not known to have any further structure.

As alluded to in the Introduction, the key to our constructions is the homotopy colimit (see [**9**, Def. 0.5]) $\mathrm{hocolim}_{i\ E_\infty} E_n^{hU_i}$, where the homotopy colimit is taken in the category of commutative S-algebras. As described in [**9**, Rk. 0.2, Lem. 5.2], each $E_n^{hU_i}$ is functorially replaced with a weakly equivalent cofibrant cell commutative S-algebra, also labeled $E_n^{hU_i}$. Our construction of the continuous action requires that this cofibrant $E_n^{hU_i}$ be a G_n/U_i-spectrum. Note that, as will be seen later, since the G_n/U_i-action is given by functoriality, this $E_n^{hU_i}$ is indeed a G_n/U_i-spectrum in the desired sense.

Since $E_n^{hU_i}$ is assumed to be cell commutative, [**9**, Lem. 5.2] shows that

$$\operatorname{hocolim}_{i\ E_\infty} E_n^{hU_i} \simeq \operatorname{hocolim}_i E_n^{hU_i},$$

where the second homotopy colimit is in the category of spectra. Notice that by [**9**, Def. 5.1, Pf. of Lem. 5.2], $\pi_*(\operatorname{hocolim}_{i\ E_\infty} E_n^{hU_i}) \cong \pi_*(\operatorname{colim}_i E_n^{hU_i})$, where the undecorated colim is in the category of spectra. Therefore, the canonical map $\operatorname{hocolim}_{i\ E_\infty} E_n^{hU_i} \to \operatorname{colim}_i E_n^{hU_i}$ is a weak equivalence that is G_n-equivariant.

Since our action does not involve additional structure, but is just by maps of spectra, we work in the technically simpler model category **Spt** of Bousfield-Friedlander spectra. We state precisely how we will work in **Spt** while using [**9**].

We begin with some recollections from [**9**]. Let $R^+_{G_n}$ be the category whose objects are finite discrete left G_n-sets together with the continuous profinite G_n-space G_n itself. The morphisms are continuous G_n-equivariant maps. Devinatz and Hopkins construct a presheaf of spectra $\mathbf{F}\colon \left(R^+_{G_n}\right)^{\mathrm{op}} \to E_\infty$, such that (a) for each $S \in R^+_{G_n}$, $\mathbf{F}(S)$ is $K(n)_*$-local; (b) $\mathbf{F}(G_n) = E_n$; (c) for $U <_o G_n$, $E_n^{hU} := \mathbf{F}(G_n/U)$; and (d) remarkably, $\mathbf{F}(*) = E_n^{hG_n} \simeq L_{K(n)}S^0$.

We need some terminology to simplify our discussion.

Definition 2.1. [**21**, Def. 1.3.1] Suppose $\mathcal{C}$ and $\mathcal{D}$ are model categories. The functor $F\colon \mathcal{C} \to \mathcal{D}$ is a left Quillen functor if F is a left adjoint that preserves cofibrations and trivial cofibrations. The functor $P\colon \mathcal{D} \to \mathcal{C}$ is a right Quillen functor if P is a right adjoint that preserves fibrations and trivial fibrations. In addition, if F and P are adjoint, (F, P) is said to be a Quillen pair for the model categories $(\mathcal{C}, \mathcal{D})$.

Recall [**21**, Lem. 1.3.10] that a Quillen pair (F, P) yields total left and right derived functors $\mathbf{L}F$ and $\mathbf{R}P$, respectively, which give an adjunction between the homotopy categories $\mathrm{Ho}(\mathcal{C})$ and $\mathrm{Ho}(\mathcal{D})$.

Let $Sp^{\Sigma+}_{\mathcal{T}_*}$ be the category of symmetric spectra of topological spaces with the positive stable model category structure [**41**, pg. 529], and let $\mathcal{M}_S$ be the model category of S-modules [**11**]. Then [**41**, pg. 530] defines functors Λ and Φ such that (Λ, Φ) is a Quillen pair for $(Sp^{\Sigma+}_{\mathcal{T}_*}, \mathcal{M}_S)$. We recall that Φ preserves homotopy groups.

Let $Sp^{\Sigma}_{\mathcal{T}_*}$ be the category of symmetric spectra of topological spaces with the stable model category structure. Then the identity functor id goes in both directions between $Sp^{\Sigma+}_{\mathcal{T}_*}$ and $Sp^{\Sigma}_{\mathcal{T}_*}$, and $(\mathrm{id}, \mathrm{id})$ is a Quillen pair for $(Sp^{\Sigma+}_{\mathcal{T}_*}, Sp^{\Sigma}_{\mathcal{T}_*})$ [**31**, Prop. 14.6].

Recall from [**31**, §19] that the singular functor $\mathbb{S}\colon \mathcal{T}_* \to \mathcal{S}_*$ from the category of pointed compactly generated topological spaces extends, by levelwise application, to a functor $\mathbb{S}\colon Sp^{\Sigma}_{\mathcal{T}_*} \to Sp^{\Sigma}$ to the category of symmetric spectra of simplicial sets. Also, it is easy to see that $\mathbb{S}$ preserves homotopy groups. Levelwise application of geometric realization gives a functor $\mathbb{T}\colon Sp^{\Sigma} \to Sp^{\Sigma}_{\mathcal{T}_*}$, and once again, $(\mathbb{T}, \mathbb{S})$ is a Quillen pair for $(Sp^{\Sigma}, Sp^{\Sigma}_{\mathcal{T}_*})$ [**31**, Thm. 19.3].

Finally, let $\mathbf{U}\colon Sp^{\Sigma} \to \mathbf{Spt}$ be the forgetful functor with left adjoint V. Then (V, U) is a Quillen pair for $(\mathbf{Spt}, Sp^{\Sigma})$ [**22**, Prop. 4.2.4]. Let $(-)_{\mathsf{c}}\colon \mathbf{Spt} \to \mathbf{Spt}$ and $(-)_{\mathsf{f}}\colon \mathcal{M}_S \to \mathcal{M}_S$ denote the cofibrant and fibrant replacement functors, respectively, for the stated model categories; e.g. given any S-module M, $M_c \to M$ is a trivial fibration with M_c cofibrant.

Notice that $\pi_*((\mathbf{U}\mathbb{S}\Phi(-)_{\mathsf{f}}\mathbf{F})(S)) \cong \pi_*(\mathbf{F}(S))$. Thus, we can regard $(\mathbf{U}\mathbb{S}\Phi(-)_{\mathsf{f}}\mathbf{F})(S)$ as a model for $\mathbf{F}(S)$, and we **redefine** $\mathbf{F}$ to be the presheaf of spectra $\mathbf{U}\mathbb{S}\Phi(-)_{\mathsf{f}}\mathbf{F} \colon \big(R^+_{G_n}\big)^{\mathrm{op}} \to$ **Spt**. Also, E_n and E_n^{hU} are **redefined** to be $(\mathbf{U}\mathbb{S}\Phi(-)_{\mathsf{f}}\mathbf{F})(G_n)$ and $(\mathbf{U}\mathbb{S}\Phi(-)_{\mathsf{f}}\mathbf{F})(G_n/U)$, respectively.

Since left Quillen functors preserve cofibrant objects and right Quillen functors preserve fibrant objects, we have the following isomorphisms, for an arbitrary Bousfield-Friedlander spectrum X (we use both meanings of $\mathbf{F}$):

$$[\Lambda\mathbb{T}V(X)_{\mathsf{c}}, \mathbf{F}(S)]^t_{\mathcal{M}_S} \cong [\mathbb{T}V(X)_{\mathsf{c}}, (\Phi(-)_{\mathsf{f}}\mathbf{F})(S)]^t_{Sp^{\Sigma+}_{\mathcal{T}_*}} \cong [V(X)_{\mathsf{c}}, (\mathbb{S}\Phi(-)_{\mathsf{f}}\mathbf{F})(S)]^t_{Sp^{\Sigma}}$$
$$\cong [X, (U\mathbb{S}\Phi(-)_{\mathsf{f}}\mathbf{F})(S)]^t_{\mathbf{Spt}} = [X, \mathbf{F}(S)]^t_{\mathbf{Spt}}.$$

This isomorphism will be needed later.

Now we can define a spectrum that is critical for the constructions in this work.

Definition 2.2. Let $F_n = \operatorname{colim}_i E_n^{hU_i}$.

Given a filtered system $\{X_i\}$ of spectra, the canonical map $\operatorname{hocolim}_i X_i \to \operatorname{colim}_i X_i$ is a weak equivalence in **Spt**. Therefore, since Φ preserves homotopy groups, we point out that the four spectra (note the two different meanings of $E_n^{hU_i}$)

$$\operatorname{hocolim}_{i\ E_\infty} E_n^{hU_i} \to \operatorname{colim}_{i\ \mathcal{S}} E_n^{hU_i}, \operatorname{hocolim}_{i\ \mathbf{Spt}} E_n^{hU_i} \leftarrow \operatorname{colim}_{i\ \mathbf{Spt}} E_n^{hU_i}$$

all have the same homotopy groups, and by functoriality, the same G_n-action, and thus, are models for the same G_n-spectrum. For the remainder of this paper, we will use the model $\operatorname{colim}_i E_n^{hU_i}$, where the **Spt** is omitted. Since $\operatorname{colim} X_i \simeq \operatorname{hocolim} X_i$, we can interchange these two as needed, so that, for example, we can assume that $E(n)_*$-localization commutes with colim (see below).

We review some results that will be used frequently in our work. First of all, as implied by the properties of **F** listed earlier, E_n and $E_n^{hU_i}$ are $K(n)_*$-local, and hence, $E(n)_*$-local. Since $E(n)_*$-localization is smashing [**38**, Thm. 7.5.6], for any spectrum X, $L_n X \simeq X \wedge L_n S^0$. Also, $E(n)_*$-localization commutes with homotopy direct limits [**38**, Thm. 8.2.2], so that $L_n(\operatorname{hocolim} X_\alpha) \simeq \operatorname{hocolim} L_n(X_\alpha)$, and if each X_α is $E(n)_*$-local, then so is $\operatorname{hocolim} X_\alpha$. Note that if X is $E(n)_*$-local, then for any spectrum Y, $X \wedge Y$ is also $E(n)_*$-local.

Recall ([**6**, Remark 3.6], [**20**, §2], [**23**, Prop. 7.10]) that if X is any $E(n)_*$-local spectrum, $L_{K(n)}X \simeq \operatorname{holim}_I (X \wedge M_I)$. Since $L_{K(n)}L_n \simeq L_{K(n)}$, for any spectrum X,

$$L_{K(n)}X \simeq \operatorname{holim}_I (L_n X \wedge M_I). \tag{2.3}$$

We also need a result by Hovey and Strickland [**23**, Lem. 7.2]:

Lemma 2.4. *If X is any spectrum, and Y is a finite spectrum of type at least n (so Y is p-local), then $L_{K(n)}(X \wedge Y) \simeq L_{K(n)}(X) \wedge Y \simeq L_n(X) \wedge Y$.*

We make a few remarks about smash products in **Spt**. We require that the smash product be functorial, particularly since our actions are given by functors. Though the smash product is well-known to be cumbersome, [**27**, Chps. 1,2] gives a functorial definition of it, which we now state.

Definition 2.5. Given spectra X and Y, their smash product $X \wedge Y$ is given by

$$(X \wedge Y)_n = \begin{cases} X_k \wedge Y_k, & n = 2k; \\ X_k \wedge Y_{k+1}, & n = 2k+1. \end{cases}$$

Since $K \wedge (-)\colon \mathcal{S}_* \to \mathcal{S}_*$ is a left adjoint, for any K in $\mathcal{S}_*$, smashing with any spectrum in either variable commutes with colimits in **Spt**.

Note that in general, in **Spt**, there are no point-set level maps between X and $X \wedge S^0$.

We recall some useful facts about compact p-adic analytic groups. Our starting point is that S_n is compact p-adic analytic. Since G_n is an extension of S_n by the Galois group, [**42**, Cor. of Thm. 2] implies that G_n is a compact p-adic analytic group. By [**10**, Thm. 10.7], any closed subgroup G of G_n is also compact p-adic analytic. Using (a) the characterization of compact p-adic analytic groups in [**40**, pg. 76]; and (b) the fact that G_n has a finitely generated pro-p subgroup S_n (the unique p-Sylow open normal subgroup of S_n of strict automorphisms) that is strongly complete [**40**, pg. 124], G_n is also strongly complete. Thus, all subgroups in G_n of finite index are open.

Recall that virtual cohomological dimension (vcd) refers to the cohomological dimension of an open subgroup. Since $\mathrm{vcd}(S_n) = n^2$ ([**28**, 2.4.9], [**36**, pg. 30]), if G is a closed subgroup of G_n, then G has finite virtual cohomological dimension. Thus, we write $\mathrm{vcd}(G_n) = m$ so that any closed subgroup G contains an open subgroup H with $cd(H) \leq m$, and hence, $H_c^s(H; M) = 0$ for all discrete H-modules M, whenever $s > m + 1$.

CHAPTER 3

An Adams-Novikov spectral sequence for $L_{K(n)}(X)$

In the Introduction, we referred to the spectral sequence 1.1 motivated by the Morava change of rings theorem. In this chapter we discuss this spectral sequence, since its existence and form is one of the chief impetuses for this work. Since the spectral sequence is rarely written in this form, we show how it is derived from a certain Adams-Novikov spectral sequence.

Definition 3.1. Let $\mathcal{K}_{n,*}(X) = \varprojlim_I \pi_*(E_n \wedge M_I \wedge X)$, where X is a spectrum.

We recall Proposition 7.4 of [**19**]: If $\mathcal{K}_{n,*}(X)$ is finitely generated over E_{n*}, then there is a spectral sequence

$$E_2^{*,*} = H_c^{**}(S_n; \mathcal{K}_{n,*}(X)) \Longrightarrow W(\mathbb{F}_{p^n}) \otimes_{\mathbb{Z}_p} \pi_*(L_{K(n)}X). \tag{3.2}$$

Remark 3.3. The second "$*$" in the superscript of the continuous cohomology above has the following meaning: if N_* is a graded G_n-module, $H_c^{*t}(S_n; N_*) = H_c^*(S_n; N_t)$.

Spectral sequence 3.2, due to Hopkins and Ravenel (see [**18**, §2.2], [**20**, pg. 241], [**44**, pg. 46], and the proof in [**19**, pg. 116]), is constructed by taking an inverse limit over

$\{I\}$ of Adams-Novikov spectral sequences of the form

$$E_2^{*,*} = W(\mathbb{F}_{p^n}) \otimes_{\mathbb{Z}_p} \mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, BP_*(L_nX \wedge M_I)) \Longrightarrow W(\mathbb{F}_{p^n}) \otimes_{\mathbb{Z}_p} \pi_*(L_nX \wedge M_I),$$

and applying the Morava change of rings isomorphism

$$W(\mathbb{F}_{p^n}) \otimes_{\mathbb{Z}_p} \mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, BP_*(L_nX \wedge M_I)) \cong H_c^{**}(S_n; \pi_t(E_n \wedge M_I \wedge X)).$$

Let X be a finite spectrum. Then $\mathcal{K}_{n,*}(X) \cong \pi_*(E_n \wedge X)$ is a finitely generated E_{n*}-module, by 8.10. We proceed to show that 3.2 gives the spectral sequence

$$H_c^*(G_n; \pi_*(E_n \wedge X)) \Longrightarrow \pi_*(L_{K(n)}X).$$

Notice that Gal acts on the abutment of (3.2) by acting only on the Witt vectors. There is also a canonical action of Gal on the E_2-term, which induces the action of Gal on $H_c^s(S_n; \mathcal{K}_{n,*}(X)) = \varprojlim_I H_c^*(S_n; \pi_*(E_n \wedge M_I \wedge X))$ [**4**, pg. 112]. We recall the following definitions from [**4**, Lem. 5.4], [**8**, Def. 3.21, proof of Lem. 3.22].

Definition 3.4. Let M be a $W(\mathbb{F}_{p^n})$-module and a Gal-module. If Gal acts by $\mathbb{Z}_p$-module automorphisms and if $\sigma(cm) = (\sigma c)(\sigma m)$, whenever $\sigma \in$ Gal, $c \in W(\mathbb{F}_{p^n})$, and $m \in M$, then M is called a twisted Gal-$W(\mathbb{F}_{p^n})$ module. A discrete G_n-module N is a discrete twisted G_n-$W(\mathbb{F}_{p^n})$ module if it is also a torsion $W(\mathbb{F}_{p^n})$-module with $\sigma(cn) = (\sigma c)(\sigma n)$, for $n \in N$, and σ and c as above.

We will make use of the following, a version of [**4**, Lem. 5.4].

Lemma 3.5. *If N is a twisted* Gal-$W(\mathbb{F}_{p^n})$ *module, then the inclusion $N^{\mathrm{Gal}} \hookrightarrow N$ extends to a* Gal-*equivariant isomorphism $W(\mathbb{F}_{p^n}) \otimes_{\mathbb{Z}_p} N^{\mathrm{Gal}} \to N$ of $W(\mathbb{F}_{p^n})$-modules.*

Definition 3.6. Given a discrete twisted G_n-$W(\mathbb{F}_{p^n})$ module N, $\mathrm{Map}_c^{\ell}(S_n^k, N)$ denotes the discrete G_n-module of continuous maps into N. Given a continuous map $h\colon S_n^k \to N$ and $(g, \sigma) \in S_n \rtimes \mathrm{Gal} = G_n$, the action is given by $((g, \sigma)h)(s_1, ..., s_k) = \sigma \cdot h(\sigma^{-1}(g^{-1}s_1), \sigma^{-1}s_2, ..., \sigma^{-1}s_k)$. (Note that $\sigma^{-1}s_i$ refers to the action of Gal on S_n, instead of being the product of two elements in G_n.)

Lemma 3.7. *If N is a twisted* Gal-$W(\mathbb{F}_{p^n})$ *module with the discrete topology, then, for any $j \geq 0$, $\mathrm{Map}_c^{\ell}(S_n^j, N)$ is a twisted* Gal-$W(\mathbb{F}_{p^n})$ *module.*

Proof. It is easy to show that the module structure is twisted and, since Gal acts trivially on $\mathbb{Z}_p$, it acts by $\mathbb{Z}_p$-homomorphisms. □

Lemma 3.8. *If N is a finite twisted* Gal-$W(\mathbb{F}_{p^n})$ *module with the discrete topology, then $H^s(\mathrm{Gal}; N) = 0$, for $s > 0$.*

Proof. Since $S_n \lhd_o G_n$, the continuous projection $G_n \to G_n/S_n \cong \mathrm{Gal}$ makes N a discrete G_n-module, and the hypotheses imply that N is a discrete twisted G_n-$W(\mathbb{F}_{p^n})$ module.

Recall [**8**, pg. 23] that given a discrete twisted G_n-$W(\mathbb{F}_{p^n})$ module N and a closed subgroup H of G_n, there is a cochain complex $D_H^*(N) = \mathrm{Map}_c^{\ell}(S_n^{*+1}, N)^H$, such that

$H^*(D_H^*(-))$ is equivalent to $H_c^*(H;-)$ on the category of discrete twisted G_n-$W(\mathbb{F}_{p^n})$ modules. A tedious computation shows that the Gal-action on the cochain complex $\mathrm{Map}_c^\ell(S_n^{*+1}, N)$ is by maps of cochain complexes.

Since $\mathrm{Map}_c^\ell(S_n^j, N)$ is a twisted Gal-$W(\mathbb{F}_{p^n})$ module, there is a Gal-equivariant isomorphism $W(\mathbb{F}_{p^n}) \otimes_{\mathbb{Z}_p} \mathrm{Map}_c^\ell(S_n^j, N)^{\mathrm{Gal}} \to \mathrm{Map}_c^\ell(S_n^j, N)$ of $W(\mathbb{F}_{p^n})$-modules that is induced by the inclusion. As in [**4**, proof of Lem. 5.15], since Gal acts by maps of cochain complexes, there is an isomorphism $W(\mathbb{F}_{p^n}) \otimes_{\mathbb{Z}_p} \mathrm{Map}_c^\ell(S_n^{*+1}, N)^{\mathrm{Gal}} \to \mathrm{Map}_c^\ell(S_n^{*+1}, N)$ of cochain complexes. Since $W(\mathbb{F}_{p^n}) \cong \bigoplus_{i=1}^n \mathbb{Z}_p$ is a free $\mathbb{Z}_p$-module, there is an isomorphism

$$\bigoplus_{i=1}^n H^*(\mathrm{Gal}; N) \cong \bigoplus_{i=1}^n H^*(\mathrm{Map}_c^\ell(S_n^{*+1}, N)^{\mathrm{Gal}}) \cong H^*(\mathrm{Map}_c^\ell(S_n^{*+1}, N)) \cong H^*(\{e\}; N),$$

and the conclusion of the theorem follows. □

Theorem 3.9. *If X is a finite spectrum, then for all s and t,*

$$H_c^s(S_n; \pi_t(E_n \wedge M_I \wedge X))^{\mathrm{Gal}} \cong H_c^s(G_n; \pi_t(E_n \wedge M_I \wedge X)).$$

Proof. Let $M = \pi_t(E_n \wedge M_I \wedge X)$, and consider the restriction map $H_c^s(G_n; M) \to H_c^s(S_n; M)^{\mathrm{Gal}}$ and the Lyndon-Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p(\mathrm{Gal}; H_c^q(S_n; M)) \Longrightarrow H_c^{p+q}(G_n; M).$$

It is enough to prove that $E_2^{p,q} = 0$ for all $p \geq 1, q \geq 0$, since this implies that

$$H_c^q(G_n; M) \cong H^0(\mathrm{Gal}; H_c^q(S_n; M)) \cong H_c^q(S_n; M)^{\mathrm{Gal}}.$$

As a finite G_n-module, M is also an $E_{n,0}$-module with the $E_{n,0}$-module structure map G_n-equivariant (since G_n acts on E_{n*} by ring automorphisms). Then $W(\mathbb{F}_{p^n}) \subset E_{n,0}$ implies that M is a twisted Gal-$W(\mathbb{F}_{p^n})$ module.

By the preceding result, we only have to prove that $H_c^q(S_n; M)$ is a finite discrete twisted Gal-$W(\mathbb{F}_{p^n})$ module, which follows from applying the following: (a) since S_n is a compact p-adic analytic group, the proofs of 8.12 and 8.13 show that $H_c^q(S_n; M)$ is a finite abelian group; and (b) the complex of continuous cochains used to compute $H_c^q(S_n; M)$ consists of the discrete abelian groups $\mathrm{Map}_c(S_n^k, M)$, so that the induced topology on the cohomology is also discrete. □

Since taking fixed points is a limit, we get

Corollary 3.10. *If X is a finite spectrum, then, for all s and t,*

$$H_c^s(S_n; \mathcal{K}_{n,t}(X))^{\mathrm{Gal}} \cong H_c^s(G_n; \pi_t(E_n \wedge X)).$$

Therefore, after taking Gal-fixed points [**44**, pg. 46], the spectral sequence in 3.2 does indeed take on the form of 1.1.

CHAPTER 4

Continuous G-spectra of simplicial sets

In this chapter, we consider the model categories of discrete G-spectra and towers of discrete G-spectra. We give a definition of continuous G-spectrum and see how in certain cases these can be obtained by taking the holim of a tower.

4.1. The model category of discrete G-spectra

Let **Spt** be the category of Bousfield-Friedlander spectra of simplicial sets, and let G be a profinite group. A discrete G-spectrum X is a (naive) G-spectrum of simplicial sets X_k, for $k \geq 0$, such that each simplicial set is a pointed simplicial discrete G-set, and each bonding map $S^1 \wedge X_k \rightarrow X_{k+1}$ is G-equivariant (S^1 has trivial G-action). Thus, a discrete G-spectrum is a continuous G-spectrum in the sense that each set constituting the spectrum is a continuous G-space with the discrete topology and all face and degeneracy maps are (trivially) continuous. We let $\mathbf{Spt}_G$ denote the category of discrete G-spectra.

Example 4.1.1. Let k be a field with separable closure $\bar{k}$. The simplicial set $BGL_\infty(\bar{k})$ is functorially a simplicial discrete G-set, where $G = \mathrm{Gal}(\bar{k}/k)$. Let $K(\bar{k})$ denote the corresponding algebraic K-theory spectrum. Since $\Omega^\infty K(\bar{k}) \simeq \mathbb{Z} \times BGL_\infty(\bar{k})^+$, $K(\bar{k})$ is a discrete G-spectrum [**34**, §1.1].

Let $G-\mathbf{Sets}_{df}$ be the canonical site of finite discrete G-sets. This site has a unique point $u\colon \mathbf{Sets} \to (G-\mathbf{Sets}_{df})^{\sim}$, where $(G-\mathbf{Sets}_{df})^{\sim}$ is the category of sheaves of sets on the site. ("Unique" means that $\{u\}$ is a sufficient set of points for the Grothendieck topos.) The left adjoint of the topos morphism is

$$u^*\colon (G-\mathbf{Sets}_{df})^{\sim} \to \mathbf{Sets}, \quad \mathcal{F} \mapsto \operatorname{colim}_N \mathcal{F}(G/N),$$

with right adjoint

$$u_*\colon \mathbf{Sets} \to (G-\mathbf{Sets}_{df})^{\sim}, \quad X \mapsto \operatorname{Hom}_G(-, \operatorname{Map}_c(G, X)) \cong \mathbf{Sets}(-, X)$$

[**27**, Rk. 6.25]. The action on $\operatorname{Map}_c(G, X)$ is defined by $(g \cdot f)(g') = f(g'g)$, for g, g' in G and f a continuous map $G \to X$.

Let $\mathbf{ShvSpt}(G-\mathbf{Sets}_{df})$ be the category of sheaves of spectra on the site $G-\mathbf{Sets}_{df}$. A sheaf of spectra $\mathcal{F}$ is a presheaf $\mathcal{F}\colon (G\text{–}\mathbf{Sets}_{df})^{\mathrm{op}} \to \mathbf{Spt}$, such that, for each surjective covering family $\{f_i\colon W_i \to U\}$, the diagram of spectra

$$\mathcal{F}(U) \longrightarrow \prod{}_i \, \mathcal{F}(W_i) \rightrightarrows \prod{}_{j,k} \, \mathcal{F}(W_j \times_U W_k)$$

is an equalizer.

Alternatively, a sheaf of spectra $\mathcal{F}$ consists of pointed simplicial sheaves $\mathcal{F}^n$, together with pointed maps of simplicial presheaves $\sigma\colon S^1 \wedge \mathcal{F}^n \to \mathcal{F}^{n+1}$, for $n \geq 0$, where S^1 is the constant simplicial presheaf. A morphism between sheaves of spectra is a natural transformation between the underlying presheaves.

The category $\mathbf{PreSpt}(G{-}\mathbf{Sets}_{df})$ of presheaves of spectra has the following "stable" model category structure ([**26**], [**27**, §2.3]). A map $h\colon \mathcal{F} \to \mathcal{G}$ of presheaves of spectra is a weak equivalence if and only if the canonical map $\operatorname{colim}_N \mathcal{F}(G/N) \to \operatorname{colim}_N \mathcal{G}(G/N)$ is a weak equivalence of spectra. Recall that a map k of simplicial presheaves is a cofibration if it is an objectwise monomorphism in the sense that, for each $S \in G{-}\mathbf{Sets}_{df}$, the map $k(S)$ is a monomorphism of simplicial sets. The map h is a cofibration of presheaves of spectra if the following two conditions hold:

(1) the map $h^0 : \mathcal{F}^0 \to \mathcal{G}^0$ is a cofibration of simplicial presheaves, and

(2) for each $n \geq 0$, the canonical map $(S^1 \wedge \mathcal{G}^n) \cup_{(S^1 \wedge \mathcal{F}^n)} \mathcal{F}^{n+1} \hookrightarrow \mathcal{G}^{n+1}$ is a cofibration of simplicial presheaves.

Fibrations are those maps with the right lifting property with respect to trivial cofibrations.

In this stable model category structure, fibrant presheaves of spectra are often referred to as globally fibrant, and if $\mathcal{F} \to \mathbf{G}\mathcal{F}$ is a weak equivalence of presheaves of spectra, with $\mathbf{G}\mathcal{F}$ globally fibrant, then we say that $\mathbf{G}\mathcal{F}$ is a globally fibrant model for $\mathcal{F}$.

Remark 4.1.2. Henceforth, we use **PreSpt** and **ShvSpt** to denote the categories $\mathbf{PreSpt}(G{-}\mathbf{Sets}_{df})$ and $\mathbf{ShvSpt}(G{-}\mathbf{Sets}_{df})$, respectively.

We recall the following fact, which is especially useful when $S = *$.

Lemma 4.1.3. *Let $S \in G{-}\mathbf{Sets}_{df}$. The S-sections functor* $\mathbf{PreSpt} \to \mathbf{Spt}$*, defined by $\mathcal{F} \mapsto \mathcal{F}(S)$, preserves fibrations, trivial fibrations, and weak equivalences between fibrant objects.*

Proof. The S-sections functor has a left adjoint, obtained by left Kan extension, which preserves cofibrations and weak equivalences. See [**27**, pg. 60] and [**34**, Cor. 3.16] for the details. $\square$

Now we consider sheaves of spectra. Let $i\colon \mathbf{ShvSpt} \to \mathbf{PreSpt}$ denote the inclusion functor, which is right adjoint to $\mathcal{L}^2$, the sheafification functor. In [**13**, Rk. 3.11], **ShvSpt** is given the following model category structure. A map $h\colon \mathcal{F} \to \mathcal{G}$ of sheaves of spectra is a weak equivalence (fibration) if and only if $i(f)$ is a weak equivalence (fibration) of presheaves of spectra. A map k of simplicial sheaves is a cofibration if it is a cofibration of simplicial presheaves. Then h is a cofibration of sheaves of spectra if the following two conditions hold:

(1) the map $h^0 : \mathcal{F}^0 \to \mathcal{G}^0$ is a cofibration of simplicial sheaves, and

(2) for each $n \geq 0$, the canonical map $\mathcal{L}^2\big((S^1 \wedge \mathcal{G}^n) \cup_{(S^1 \wedge \mathcal{F}^n)} \mathcal{F}^{n+1}\big) \hookrightarrow \mathcal{G}^{n+1}$ is a cofibration of simplicial sheaves.

Since i preserves all weak equivalences and fibrations, $\mathcal{L}^2$ preserves cofibrations, trivial cofibrations, and weak equivalences between cofibrant objects, so that $(\mathcal{L}^2, i)$ is a Quillen pair. Given a presheaf of spectra $\mathcal{F}$, $\mathcal{F} \to \mathcal{L}^2\mathcal{F}$ is a weak equivalence (and a strict weak equivalence), so that $\mathbf{L}\mathcal{L}^2$ and $\mathbf{R}i$ give an equivalence of categories

$$\text{Ho}(\mathbf{ShvSpt}) \cong \text{Ho}(\mathbf{PreSpt}).$$

Corresponding to **ShvSpt** is the category $\mathbf{Spt}_G$ of discrete G-spectra. Building on the equivalence between the categories of discrete G-sets and sheaves of sets on G–$\mathbf{Sets}_{df}$, the categories **ShvSpt** and $\mathbf{Spt}_G$ are equivalent to each other, via the functors $L\colon \mathbf{ShvSpt} \to \mathbf{Spt}_G$, where $L(\mathcal{F}) = \text{colim}_N\, \mathcal{F}(G/N)$, and $R\colon \mathbf{Spt}_G \to \mathbf{ShvSpt}$, where $R(X) = \text{Hom}_G(-, X)$.

We make $\mathbf{Spt}_G$ a model category in the following way. Define a map h of discrete G-spectra to be a weak equivalence (fibration) if and only if $\text{Hom}_G(-, h)$ is a weak equivalence (fibration) of sheaves of spectra. Also, define h to be a cofibration if and only if h has the left lifting property with respect to all trivial fibrations. Then it is easy to show that h is a cofibration if and only if $\text{Hom}_G(-, h)$ is a cofibration in **ShvSpt**. Using this, it is easy to show that $\mathbf{Spt}_G$ is a model category.

Note that h is a weak equivalence in $\mathbf{Spt}_G$ if and only if h is a weak equivalence in **Spt**. This setup immediately implies the following.

Theorem 4.1.4. *The Quillen pair (L, R) induces an equivalence of homotopy categories:* $\text{Ho}(\mathbf{ShvSpt}) \cong \text{Ho}(\mathbf{Spt}_G)$.

Lemma 4.1.5. *Let* $\mathrm{e}\colon \mathbf{Spt} \to \mathbf{Spt}_G$ *give a spectrum trivial* G*-action, so that* $\mathrm{e}(X) = X$*. The right adjoint of* e *is the fixed points functor* $(-)^G$*, and* $(\mathrm{e}, (-)^G)$ *is a Quillen pair for* $(\mathbf{Spt}, \mathbf{Spt}_G)$*.*

Proof. It is clear that e preserves weak equivalences; it suffices to show that e preserves cofibrations. Let $X \in \mathbf{Spt}, Y \in \mathbf{Spt}_G$, and let cX denote the constant presheaf on X. In $\mathbf{Spt}_G$, $X \cong \operatorname{colim}_N (\mathcal{L}^2(cX))(G/N)$, due to the adjunction isomorphisms

$$\mathbf{Spt}_G(X,Y) \cong \mathbf{Spt}(X,Y^G) \cong \mathbf{PreSpt}(cX, \mathrm{Hom}_G(-,Y))$$

$$\cong \mathbf{ShvSpt}(\mathcal{L}^2(cX), \mathrm{Hom}_G(-,Y)) \cong \mathbf{Spt}_G(\operatorname{colim}_N (\mathcal{L}^2(cX))(G/N), Y).$$

Let $f\colon X \to Z$ be a cofibration in $\mathbf{Spt}$. By 4.1.3, $c(f)$ is a cofibration in $\mathbf{PreSpt}$, and thus, $\mathcal{L}^2(c(f))$ is a cofibration in $\mathbf{ShvSpt}$. Since $\mathrm{e}(f)$ factors as

$$X \cong \operatorname{colim}_N (\mathcal{L}^2(cX))(G/N) \to \operatorname{colim}_N (\mathcal{L}^2(cZ))(G/N) \cong Z,$$

$\mathrm{e}(f)$ is a cofibration in $\mathbf{Spt}_G$. □

We need the following observation.

Lemma 4.1.6. *If* $h\colon X \to Y$ *is a fibration in* $\mathbf{Spt}_G$*, then it is a fibration in* $\mathbf{Spt}$*. In particular, if* X *is fibrant as a discrete* G*-spectrum, then* X *is fibrant as a spectrum.*

Proof. Since $\mathrm{Hom}_G(-,h)$ is a fibration of presheaves of spectra, $\mathrm{Hom}_G(G/N,h)$ is a fibration of spectra for each open normal subgroup N, by 4.1.3. Thus, the map

$\operatorname{colim}_N \operatorname{Hom}_G(G/N, h)$ is a fibration of spectra. Since h factors as $X \cong \operatorname{colim}_N X^N \to \operatorname{colim}_N Y^N \cong Y$, the result follows. □

4.2. Building discrete G-spectra

In this section, we give several elementary lemmas showing that certain colimits and smash products yield discrete G-spectra.

Lemma 4.2.1. *Given a profinite group $G \cong \varprojlim_N G/N$, let $\{X_N\}$ be a direct system of (discrete) G/N-spectra. Then $\operatorname{colim}_N X_N$ is a discrete G-spectrum.*

Proof. This follows from the fact that given a direct system $\{Z_N\}$ of G/N-sets, the projection $\pi\colon G \to G/N$ makes $\{Z_N\}$ a direct system of discrete G-sets, and the colimit of discrete G-sets is a discrete G-set. We remark that this lemma was used implicitly in defining $L\colon \mathbf{ShvSpt} \to \mathbf{Spt}_G$. (See [**27**, Prop. 6.20], [**29**, III-9, Thm. 1].) □

Lemma 4.2.2. *Given the hypotheses of the previous lemma, let Y be a trivial G-spectrum. Then there is a G-equivariant isomorphism of discrete G-spectra*

$$(\operatorname{colim}_N X_N) \wedge Y \cong \operatorname{colim}_N (X_N \wedge Y).$$

Proof. By the functoriality of $(-) \wedge Y$, $(\operatorname{colim} X_N) \wedge Y$ is a G-spectrum, and $X_N \wedge Y$ is a G/N-spectrum, making $\operatorname{colim}(X_N \wedge Y)$ a discrete G-spectrum. The isomorphism is G-equivariant, since it is so on the level of simplicial sets. The isomorphism makes $(\operatorname{colim} X_N) \wedge Y$ a discrete G-spectrum. □

Lemma 4.2.3. *Let G and Y be as in the two previous lemmas. If X is a discrete G- spectrum, then $X \wedge Y$ is a discrete G-spectrum.*

Proof. We have $X \wedge Y \cong (\operatorname{colim} X^N) \wedge Y \cong \operatorname{colim} (X^N \wedge Y)$, a discrete G-spectrum. $\square$

This lemma is used repeatedly in upcoming chapters.

4.3. Towers of discrete G-spectra

Let $\mathbf{tow}(\mathbf{Spt}_G)$ be the category where a typical object $\{X_i\}$ is a tower $\cdots \to X_i \to X_{i-1} \to \cdots \to X_1 \to X_0$ in $\mathbf{Spt}_G$. The morphisms are natural transformations $\{X_i\} \to \{Y_i\}$ such that each $X_i \to Y_i$ is G-equivariant. Since $\mathbf{Spt}_G$ is a simplicial model category, [**17**, VI, Prop. 1.3] shows that $\mathbf{tow}(\mathbf{Spt}_G)$ is a simplicial model category, where $\{f_i\}$ is a weak equivalence (cofibration) if and only if each f_i is a weak equivalence (cofibration) in $\mathbf{Spt}_G$. By [**17**, Rk. 1.5], if $\{Z_i\}$ is fibrant in $\mathbf{tow}(\mathbf{Spt}_G)$, then each map $Z_i \to Z_{i-1}$ in the tower is a fibration and each Z_i is fibrant in $\mathbf{Spt}_G$.

Example 4.3.1. Let $M(p^i)$ be the mod p^i Moore spectrum. Then the p-adic completion $KU_p^\wedge = L_{M(p)}KU = \operatorname{holim}_{i \geq 1} KU \wedge M(p^i) \simeq E_1$, of the complex K-theory spectrum gives $\{KU \wedge M(p^i)\}$ in $\mathbf{tow}(\mathbf{Spt}_{\mathbf{G_1}})$.

Example 4.3.2. Let ℓ be a prime not equal to $\operatorname{char}(\bar{k})$. Then the ℓ-adic completion $K(\bar{k})_\ell^\wedge$ of $K(\bar{k})$ gives $\{K(\bar{k}) \wedge M(\ell^i)\} \in \mathbf{tow}(\mathbf{Spt}_G)$, where G is the absolute Galois group.

The functor $\varprojlim_i (-)^G \colon \mathbf{tow}(\mathbf{Spt}_G) \to \mathbf{Spt}$, given by $\{X_i\} \mapsto \varprojlim_i X_i^G$, is right adjoint to the functor $\underline{\mathrm{e}} \colon \mathbf{Spt} \to \mathbf{tow}(\mathbf{Spt}_G)$ that sends a spectrum X to the constant diagram $\{X\}$, where X has trivial G-action.

Remark 4.3.3. It is important to note that the inverse limit above is formed in $\mathbf{Spt}$. Limits in $\mathbf{Spt}_G$ are not formed in $\mathbf{Spt}$, and $\lim_{\mathbf{Spt}_G} X_i \cong \operatorname{colim}_N (\lim_{\mathbf{Spt}} X_i)^N$. To prevent any confusion, $\varprojlim$ and holim are always formed in $\mathbf{Spt}$.

Since $\underline{\mathrm{e}}$ preserves all weak equivalences and cofibrations, by 4.1.5, we have:

Lemma 4.3.4. *The pair* $(\underline{\mathrm{e}}, \varprojlim_i (-)^G)$ *is a Quillen pair for* $(\mathbf{Spt}, \mathbf{tow}(\mathbf{Spt}_G))$.

This implies the existence of the total right derived functor

$$\mathbf{R}\big(\varprojlim_i (-)^G\big) \colon \mathrm{Ho}(\mathbf{tow}(\mathbf{Spt}_G)) \to \mathrm{Ho}(\mathbf{Spt}), \quad \{X_i\} \mapsto \varprojlim_i (X_i')^G,$$

where $\{X_i\} \to \{X_i'\}$ is a trivial cofibration with $\{X_i'\}$ fibrant in $\mathbf{tow}(\mathbf{Spt}_G)$.

4.4. Continuous G-spectra

Definition 4.4.1. Let $\{X_i\}$ be in $\mathbf{tow}(\mathbf{Spt}_G)$. Then the inverse limit $\varprojlim_i X_i$, formed in $\mathbf{Spt}$, is a continuous G-spectrum. Let $\{Y_i\}$ be a tower in $\mathbf{tow}(\mathbf{Spt}_G)$. If $X \to \operatorname{holim}_i Y_i$ or $\operatorname{holim}_i Y_i \to X$ is a G-equivariant weak equivalence in $\mathbf{Spt}$, where X is a continuous G-spectrum, then $\operatorname{holim}_i Y_i$ is a continuous G-spectrum.

Remark 4.4.2. Definition 4.4.1 can be generalized to more complicated diagrams of discrete G-spectra, but our definition is sufficient for the applications considered here. Also, we will use the term "continuous G-spectrum" more loosely. Suppose that X is a continuous G-spectrum, Z is a G-spectrum, and $X \to Z$ or $Z \to X$ is a weak equivalence of spectra that is G-equivariant. Then X is a model for Z as a continuous G-spectrum and we call Z a continuous G-spectrum. Also, if a G-spectrum Z is isomorphic to a continuous G-spectrum in the stable category, then Z is a continuous G-spectrum.

Let $\{X_i\}$ be in $\mathbf{tow}(\mathbf{Spt}_G)$. Given X in $\mathbf{Spt}_G$, let $X \to X_f$ be a trivial cofibration with X_f fibrant in $\mathbf{Spt}_G$, so that there is a weak equivalence $\{X_i\} \to \{(X_i)_f\}$ of towers. Let $\{(X_i)_f\} \to \{(X_i)'_f\}$ be a trivial cofibration with $\{(X_i)'_f\}$ fibrant in $\mathbf{tow}(\mathbf{Spt}_G)$. Then there is a commutative diagram, all of whose maps are G-equivariant:

(4.4.3)
$$\begin{array}{ccc} \varprojlim_i X_i & \longrightarrow & \operatorname{holim}_i X_i \\ \downarrow & & \downarrow{\scriptstyle a} \\ \varprojlim_i (X_i)_f & \xrightarrow{b} & \operatorname{holim}_i (X_i)_f \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle d} \\ \varprojlim_i (X_i)'_f & \xrightarrow{e} & \operatorname{holim}_i (X_i)'_f. \end{array}$$

Remark 4.4.4. We recall the definition of the Mittag-Leffler condition. Let $\{A_i\}$ be a tower $\cdots \to A_2 \to A_1 \to A_0$ of abelian groups. If for each k there exists a $j \geq k$ such that the image of $A_i \to A_k$ equals the image of $A_j \to A_k$ for all $i \geq j$, then the tower

$\{A_i\}$ is said to be Mittag-Leffler. For example, if all the maps $A_i \to A_{i-1}$ are onto, then $\{A_i\}$ is Mittag-Leffler. Also, a tower of finite abelian groups is Mittag-Leffler.

Lemma 4.4.5. *In diagram 4.4.3,* $\operatorname{holim}_i (X_i)_f$ *and* $\operatorname{holim}_i (X_i)'_f$ *are continuous G-spectra. If the tower* $\{\pi_t(X_i)\}$ *is Mittag-Leffler for every integer t, then* $\operatorname{holim}_i X_i$ *is a continuous G-spectrum.*

Proof. Since d is the homotopy limit of a diagram that consists of weak equivalences between fibrant spectra, it is a weak equivalence. If $\{\pi_t(X_i)\}$ is always Mittag-Leffler, then $\pi_*(\operatorname{holim}_i X_i) \cong \varprojlim_i \pi_*((X_i)_f) \cong \pi_*(\operatorname{holim}_i (X_i)_f)$, and a is a weak equivalence.

Each $(X_i)'_f$ is a fibrant spectrum and, as a fibration of discrete G-spectra, the map $(X_i)'_f \to (X_{i-1})'_f$ is a fibration of spectra, for $i \geq 1$. Thus, for any $j \geq 0$, $((X_i)'_f)_j \to ((X_{i-1})'_f)_j$ is a fibration between Kan complexes, and $\varprojlim_i ((X_i)'_f)_j \to \operatorname{holim}_i ((X_i)'_f)_j$ is a weak equivalence in $\mathcal{S}$. This implies that e is a weak equivalence of spectra. $\square$

Lemma 4.4.5 will be used frequently in later chapters to obtain continuous G-spectra.

CHAPTER 5

A model for E_n as a continuous G_n-spectrum

5.1. The spectrum F_n

We consider the spectrum $F_n = \operatorname{colim}_i E_n^{hU_i}$ and its relationship to E_n in more detail. Note that F_n is the stalk of the presheaf of spectra $\mathbf{F}|_{(G_n\text{–}\mathbf{Sets}_{df})^{\mathrm{op}}}$, and $L_{K(n)}S^0 \simeq \mathbf{F}(*)$ is the "spectrum of global sections" $\Gamma_*\mathbf{F} = \varprojlim_{S \in R^+_{G_n}} \mathbf{F}(S)$ of the presheaf $\mathbf{F}$.

After the author realized that $\operatorname{holim}_I (F_n \wedge M_I)$ might provide a model of E_n with a continuous action, Devinatz pointed out to him that by applying the results of [**9**], $E_n \simeq L_{K(n)}F_n$ is immediate. We pause to explain this and temporarily return to the notation of [**9**].

Theorem 5.1.1 (Devinatz and Hopkins)**.** *There is a weak equivalence $E_n \simeq L_{K(n)}F_n$.*

Proof. As in [**9**], let $E_n^{h'G}$ denote the ordinary homotopy fixed point spectrum for the action of G on E_n, when G is a finite subgroup of G_n. By applying [**9**, Thm. 3, Def. 0.5, Rk. 0.6], we have: $E_n \simeq E_n^{h'\{e\}} \simeq E_n^{h\{e\}} = L_{K(n)}\big((\operatorname{hocolim}_{i\ E_\infty} E_n^{hU_i})_{cc}\big)$, where $\operatorname{hocolim}_{i\ E_\infty} E_n^{hU_i}$ is functorially replaced with $(\operatorname{hocolim}_{i\ E_\infty} E_n^{hU_i})_{cc}$, a weakly equivalent cell commutative S-algebra. Since $(\operatorname{hocolim}_{i\ E_\infty} E_n^{hU_i})_{cc} \simeq F_n$, we obtain the result. □

We return to our earlier notation and F_n, as defined in Definition 2.2. Without relying on the results of [**9**] used in proving Theorem 5.1.1, we give a different proof of the key weak equivalence that underlies this work. The proof follows an outline given by Charles Rezk.

Theorem 5.1.2. *There is a weak equivalence* $E_n \simeq L_{K(n)}F_n$.

Proof. Let $f\colon F_n \to E_n$ be the canonical map out of the colimit, determined by the obvious maps $\mathbf{F}(G_n/U_i) \to \mathbf{F}(G_n)$. Let K_n be the two-periodic version of $K(n)$ satisfying $K_{n*} \cong \mathbb{F}_{p^n}[u^{\pm 1}]$, where $|u| = -2$. Let $\eta\colon F_n \to L_{K_n}F_n$ be the K_{n*}-localization of F_n. Since E_n is $K(n)_*$-local, it is K_{n*}-local, so if f is a K_{n*}-equivalence, then f and η are canonically equivalent, implying that $E_n \simeq L_{K_n}F_n \simeq L_{K(n)}F_n$, since K_n and $K(n)$ have weakly equivalent localization functors. Thus, it suffices to prove that $K_{n*}(f)$ is an isomorphism.

Observe that $K_{n*}(F_n) \cong \operatorname{colim}_i K_{n*}(E_n^{hU_i})$. Let E be the version of "Morava E-theory" considered in [**23**]. Since E_n is a finite wedge of suspended copies of E [**45**, pg. 1022], E_{n*} is the direct sum of various shifted copies of E_*, with the consequence that the results of Proposition 8.4 in [**23**] are valid with E replaced with E_n and $K = K(n)$ replaced with K_n. By [**9**, Theorem 2], as E_{n*}-modules, $\pi_*(L_{K(n)}(E_n \wedge E_n^{hU_i})) \cong \operatorname{Map}_c^{\ell}(G_n, E_{n*})^{U_i} \cong \prod_{G_n/U_i} E_{n*}$ is a pro-free E_{n*}-module, by [**23**, Corollary A.11], since E_{n*} is pro-free (because E_{n*} is a complete ring). Therefore, Proposition 8.4(e) implies that $K_{n*}(E_n^{hU_i}) \cong \big(\prod_{G_n/U_i} E_{n*}\big)/I_n \cong (\bigoplus_{G_n/U_i} E_{n*}) \otimes_{E_{n*}} E_{n*}/I_n \cong \mathrm{Map}_c(G_n/U_i, K_{n*})$. Thus, $K_{n*}(F_n) \cong \mathrm{Map}_c(G_n, K_{n*})$, since K_{n*} is a discrete G_n-module.

By [**9**, pg. 8], $\pi_*(L_{K(n)}(E_n \wedge E_n)) \cong \mathrm{Map}_c(G_n, E_{n*})$ is a pro-free E_{n*}-module. Thus, Proposition 8.4(e) implies that $K_{n*}(E_n) \cong \mathrm{Map}_c(G_n, E_{n*})/I_n$. This isomorphism indicates that $K_{n*}(E_n)$ is two-periodic, since E_{n*} is two-periodic. As in [**45**, Thm. 12], $(K_n)_0(E_n) = \mathrm{Map}_c(G_n, E_{n,0})/I_n = \mathrm{Map}_c(G_n, E_{n,0}/I_n)$, so that two-periodicity gives $K_{n*}(E_n) \cong \mathrm{Map}_c(G_n, E_{n*}/I_n) \cong \mathrm{Map}_c(G_n, K_{n*})$.

Since $K_{n*}(f) = K_{n*}(\mathrm{colim}_i\,(E_n^{hU_i} \to E_n)) = \mathrm{colim}_i\,\big(K_{n*}(E_n^{hU_i}) \to K_{n*}(E_n)\big)$, it is clear that $K_{n*}(f)$ is an isomorphism. □

Corollary 5.1.3. *There is a weak equivalence of spectra*

$$E_n \simeq \mathrm{holim}_I\,(F_n \wedge M_I) \simeq \mathrm{holim}_I\,(E_n \wedge M_I).$$

Proof. We have $E_n \simeq \mathrm{holim}_I\, L_n(F_n) \wedge M_I$. Since each $E_n^{hU_i}$ is $K(n)_*$-local and hence, $E(n)_*$-local, the colimit F_n is also $E(n)_*$-local so that $L_nF_n \simeq F_n$. □

The following result, whose proof was suggested by Rezk, shows that $E_n \wedge M_I \simeq F_n \wedge M_I$. As will be seen later, this weak equivalence is the main fact that makes it possible to construct the homotopy fixed point spectra of E_n.

Corollary 5.1.4. *If Y is a finite spectrum of type n, then the G_n-equivariant map $F_n \wedge Y \to E_n \wedge Y$ is a weak equivalence. In particular, $E_n \wedge M_I \simeq F_n \wedge M_I$.*

Proof. We have $E_n \wedge Y \simeq L_{K(n)}(F_n) \wedge Y \simeq L_n(F_n) \wedge Y \simeq F_n \wedge Y$. $\square$

We point out some simple facts that are necessary for building the model in the next section.

Lemma 5.1.5. *Let N be an open normal subgroup of G_n. The homotopy fixed point spectrum E_n^{hN} is a G_n/N-spectrum and the projection $\pi\colon G_n \to G_n/N$ gives a G_n-equivariant map $\mathbf{F}(\pi)\colon E_n^{hN} \to E_n$. Thus, the canonical map $f\colon F_n \to E_n$ is G_n-equivariant.*

Proof. Since $\mathrm{Hom}_{G_n}(G_n/N, G_n/N) \cong G_n/N$, $\mathbf{F}$ gives a G_n/N-action on $\mathbf{F}(G_n/N)$. $\square$

The following is an example of a useful manipulation that can be performed with F_n.

Lemma 5.1.6. *The maps $L_{K(n)}(F_n \wedge F_n) \to L_{K(n)}(E_n \wedge F_n) \to L_{K(n)}(E_n \wedge E_n)$ are weak equivalences.*

Proof. Since $F_n \wedge M_I \simeq E_n \wedge M_I$, $F_n \wedge F_n \wedge M_I \simeq E_n \wedge F_n \wedge M_I \simeq E_n \wedge E_n \wedge M_I$. Since $F_n \wedge F_n$, $E_n \wedge F_n$ and $E_n \wedge E_n$ are $E(n)_*$-local, the result follows. $\square$

We conclude this section with the following observation.

Theorem 5.1.7. *The spectrum F_n is not $K(n)_*$-local, and F_n and E_n are not weakly equivalent.*

Proof. Suppose F_n and E_n are weakly equivalent. Then F_n is $K(n)_*$-local and since $f\colon F_n \to E_n$ is a K_{n*}-equivalence, the G_n-equivariant map f is a weak equivalence.

Then $\pi_*(f)\colon \pi_*(F_n) \to \pi_*(E_n)$ is a G_n-equivariant isomorphism from a discrete module to a graded profinite module, which is impossible, so that F_n and E_n are not weakly equivalent. If F_n were $K(n)_*$-local, we obtain the same contradiction. $\square$

5.2. Constructing the continuous action

We explain how the continuous action is constructed. Recall that $E_n = \mathbf{F}(G_n)$. Also, in **Spt**, there is no map between $E_n \wedge S^0$ and E_n, but only an isomorphism in the stable category. There is a G_n-equivariant map $\operatorname{holim}_I (F_n \wedge M_I) \to \operatorname{holim}_I (E_n \wedge M_I)$, where M_I has trivial G_n-action. By the unit map $S \to M_I$ [**23**, Prop. 4.22], there is a map $E_n \wedge S^0 \to \lim_I (E_n \wedge M_I) \to \operatorname{holim}_I (E_n \wedge M_I)$. Notice that there is no map between $\operatorname{holim}_I (F_n \wedge M_I)$ and $E_n \wedge S^0$, since, for example, there is no map $E_n \to F_n$.

Consider the following zig-zag of maps:

$$\operatorname{holim}_I (F_n \wedge M_I) \xrightarrow{\alpha} \operatorname{holim}_I (E_n \wedge M_I) \xleftarrow{\beta} E_n \wedge S^0 \overset{\gamma}{\cong} E_n.$$

The maps α and β are G_n-equivariant and weak equivalences, and γ is an isomorphism in the stable category. Since $E_n \wedge S^0$ is a model for E_n as a G_n-spectrum, the zig-zag presents $\operatorname{holim}_I (F_n \wedge M_I)$ as a model for E_n as a G_n-spectrum.

Theorem 5.2.1. *The spectrum $\mathbb{E}_n = \operatorname{holim}_I (F_n \wedge M_I)$ is a continuous G_n-spectrum, and thus, E_n is a continuous G_n-spectrum.*

Proof. Since $E_n^{hU_i}$ is a (discrete) G_n/U_i-spectrum, F_n and $F_n \wedge M_I$ are discrete G_n-spectra. Since $\{\pi_*(F_n \wedge M_I)\}$ is a Mittag-Leffler tower of epimorphisms, $\mathbb{E}_n$ is a continuous G_n-spectrum, by 4.4.5. □

Now that we have a model for E_n as a continuous G_n-spectrum, one might wonder if there is a simpler model. We believe that there is not a simpler model, and we explain why one natural proposal does not work. Notice that $\pi_*(E_n \wedge M_I) = E_{n*}/I$ is two-periodic and finite in degree zero. This implies that the G_n-action on $E_{n,0}/I$ factors through some finite quotient G_n/U_I, so that $\pi_0(E_n \wedge M_I)$ is a G_n/U_I-module. Thus, one might conjecture that $E_n \wedge M_I$ is a G_n/U_I spectrum and hence, a particularly simple discrete G_n-spectrum, providing E_n with a simpler continuous action. However, we will show that this is false: $E_n \wedge M_I$ is not a G_n/U_i-spectrum for any i. As far as the author knows, Mike Hopkins first made this observation and the author learned the justification given below from Hal Sadofsky.

First of all, note that it is morally clear that $E_n \wedge M_I$ should not be a G_n/U_I-spectrum since it has the "same action" as the *weakly* equivalent $F_n \wedge M_I$, which by construction "is not" a G_n/U_i-spectrum for any U_i. However, the author does not know rigorous proofs of these assertions without using the kind of argument given below.

Theorem 5.2.2. *There is no open normal subgroup U of G_n such that the G_n-action on $E_n \wedge M_I$ factors through G_n/U; $E_n \wedge M_I$ is not a G_n/U-spectrum for any $U \triangleleft_o G_n$.*

Proof. Suppose the G_n-spectrum $E_n \wedge M_I$ is a G_n/U-spectrum. This implies that the G_n-action on the middle factor of $E_n \wedge E_n \wedge M_I$ factors through G_n/U, so that $\pi_*(E_n \wedge E_n \wedge M_I)$ is naturally a G_n/U-module. Also, $E_n \wedge E_n \wedge M_I \simeq L_n(E_n \wedge E_n \wedge M_I) \simeq L_{K(n)}(E_n \wedge E_n) \wedge M_I$, so that $\pi_*(E_n \wedge E_n \wedge M_I) \cong \mathrm{Map}_c^\ell(G_n, E_{n*}/I)$ by [**9**, pg. 10].

Since the G_n-module $\mathrm{Map}_c^\ell(G_n, E_{n*}/I)$ is a G_n/U-module, there is an isomorphism of sets $\mathrm{Map}_c^\ell(G_n, E_{n,0}/I) = \mathrm{Map}_c^\ell(G_n, E_{n,0}/I)^U \cong \mathrm{Map}_c(G_n/U, E_{n,0}/I)$. But the first set is infinite and the last is finite, a contradiction. □

As mentioned in the Introduction, one of the deficiencies of the model given in this paper for E_n as a continuous G_n-spectrum is that even though E_n is a twisted E_n-module spectrum ("twisted" means that the module structure map is G_n-equivariant, where $E_n \wedge E_n$ has the diagonal action), we can not prove that $\mathrm{holim}_I\, (F_n \wedge M_I)$ is an E_n-module spectrum. Clearly, any satisfactory model of E_n as a continuous G_n-spectrum would also be a twisted E_n-module continuous G_n-spectrum. Similarly, since $E_n \wedge M_I$ is an E_n-module, one would hope that $F_n \wedge M_I$ is an E_n-module. However, the primary cause of this deficiency in our model is that it appears that F_n is not an E_n-module spectrum.

The apparent failure of F_n to be an E_n-module is a topological version of the fact that if R is a non-discrete continuous G-ring, then the discrete G-module $\mathrm{colim}_N\, R^N$ need not be an R-module.

CHAPTER 6

Homotopy fixed point spectra for discrete G-spectra

Let G be a closed subgroup of G_n. Then G is a profinite group and E_n is a continuous G-spectrum by the inclusion of G in G_n. Recall that [**9**, Def. 0.5] defines homotopy fixed point spectra

$$E_n^{hG} = L_{K(n)}\big(\mathrm{hocolim}_{i\ E_\infty}\, E_n^{hU_iG}\big),$$

where U_iG is an open subgroup of G_n, with associated spectral sequences

$$H_c^s(G; \pi_t(E_n)) \Longrightarrow \pi_{t-s}(E_n^{hG}).$$

The spectra E_n^{hG} are called homotopy fixed point spectra because these associated spectral sequences have the form of descent spectral sequences. However, E_n^{hG} is not defined with respect to any continuous action, and the associated spectral sequences are not actually descent spectral sequences, but are instead $K(n)_*$-local E_n-Adams spectral sequences.

In this chapter, we define homotopy fixed point spectra for discrete G-spectra. Also, when G has finite virtual cohomological dimension, we develop a specific model of the homotopy fixed points that is better suited for building the descent spectral sequence. In subsequent chapters, we use this model to define homotopy fixed point spectra $E_n^{h'G}$

and construct descent spectral sequences, using the continuous action. Furthermore, we partially recover the constructions of [**9**], by using the continuous action, in the following sense: our homotopy fixed point spectra are weakly equivalent to theirs, and our descent spectral sequences are isomorphic to theirs.

Definition 6.1. Given a discrete G-spectrum X, X^{hG} denotes the homotopy fixed point spectrum of X with respect to the continuous action of G. We define $X^{hG} = (X_f)^G$, where $X \to X_{f,G} \equiv X_f$ is a trivial cofibration, and X_f is a fibrant discrete G-spectrum. (This definition has been developed in other categories: see [**14**] for simplicial discrete G-sets, [**25**] for simplicial (pre)sheaves, and, for example, [**27**, Chp. 6] for presheaves of spectra.)

We make explicit several properties of homotopy fixed points.

Lemma 6.2. *The homotopy fixed points functor* $(-)^{hG} \colon \mathrm{Ho}(\mathbf{Spt}_G) \to \mathrm{Ho}(\mathbf{Spt})$ *is the total right derived functor of the fixed points functor* $(-)^G \colon \mathbf{Spt}_G \to \mathbf{Spt}$. *In particular, if* $X \to Y$ *is a weak equivalence of discrete* G*-spectra, then* $X^{hG} \to Y^{hG}$ *is a weak equivalence.*

Proof. Lemma 4.1.5 implies $\mathbf{R}(-)^G = (-)^{hG}$. The map $\mathrm{Hom}_G(-, X_f) \to \mathrm{Hom}_G(-, Y_f)$ is a weak equivalence between globally fibrant presheaves, and evaluation at $*$ gives a weak equivalence, by 4.1.3. $\square$

Remark 6.3. As expected, the definition of homotopy fixed points for a profinite group generalizes the definition for a finite group. Let G be a finite group, X a G-spectrum, and let $X \to X'$ be a weak equivalence that is G-equivariant, with X' a fibrant spectrum. Then the definition of the homotopy fixed point spectrum $X^{h'G}$ for a finite group G is $\operatorname{holim}_G X'$. Note that by Remark 6.9 and [**3**, Ch. XI, 6.3], there is a descent spectral sequence

$$E_2^{s,t} = \lim{}^s{}_G\, \pi_t(X) \cong H^s(G; \pi_t(X)) \Longrightarrow \pi_{t-s}(X^{hG}).$$

Since G is profinite, X is a discrete G-spectrum, and $X_{f,G}$ is a fibrant spectrum, so that $X^{h'G} = \operatorname{holim}_G X_f$. Then the canonical map $X^{hG} = \lim_G X_f \to \operatorname{holim}_G X_f = X^{h'G}$ is a weak equivalence, as desired, by [**27**, Prop. 6.39].

Unlike the simplicial presheaves setting, not every presheaf of spectra is cofibrant. Thus, given two globally fibrant models $\mathbf{G}\mathcal{F}$ and $\mathbf{G}'\mathcal{F}$ for the presheaf of spectra $\mathcal{F}$, there need not be a weak equivalence $\mathbf{G}\mathcal{F} \to \mathbf{G}'\mathcal{F}$. Also, a cofibration of sheaves of spectra can fail to be a cofibration of presheaves. Given these facts, we state the following relationship between homotopy fixed points and the global sections of globally fibrant models.

Lemma 6.4. *Let X be a discrete G-spectrum. Then*

$$X^{hG} = (X_f)^G \xleftarrow{\simeq} \mathcal{F}(*) \xrightarrow{\simeq} \mathbf{G}\mathrm{Hom}_G(-, X)(*),$$

where $\mathrm{Hom}_G(-,X) \to \mathcal{F}$ *is a trivial cofibration of presheaves of spectra and* $\mathcal{F}$ *is globally fibrant. Thus,* X^{hG} *and* $\mathbf{G}\mathrm{Hom}_G(-,X)(*)$ *have the same stable homotopy type.*

Proof. Weak equivalences from $\mathcal{F}$ to the globally fibrant models $\mathrm{Hom}_G(-,X_f)$ and $\mathbf{G}\mathrm{Hom}_G(-,X)$ exist by the left lifting property of trivial cofibrations with respect to fibrations. Evaluation at the point gives the desired weak equivalences. □

We ease our construction of the descent spectral sequence by using a different model for X^{hG}. We do this by discussing some tools that utilize the fact that the closed subgroups of G_n have uniformly bounded finite virtual cohomological dimension m.

Definition 6.5. The functor $\Gamma_G \colon \mathbf{Spt}_G \to \mathbf{Spt}_G$ is given by $\Gamma_G(X) = \mathrm{Map}_c(G,X)$, where $\Gamma_G(X)$ has the G-action defined in §4.1. We write Γ instead of Γ_G, when G is understood from the context. There is a G-equivariant monomorphism $\iota \colon X \to \Gamma X$ defined by $\iota(x)(g) = g \cdot x$. Since Γ forms a triple, there is a cosimplicial discrete G-spectrum $\Gamma^{\cdot} X$.

We recall the construction of Thomason's hypercohomology spectrum (see [**48**, 1.31-1.33] and [**34**, §3.2] for more details). Let $\mathcal{F}$ be a sheaf of fibrant spectra so that for each $S \in G{-}\mathbf{Sets}_{df}$, $\mathcal{F}(S)$ is a fibrant spectrum. Then the functor

$$T = u_* u^* \colon \mathbf{ShvSpt} \to \mathbf{ShvSpt}, \quad \mathcal{F} \mapsto \mathrm{Hom}_G(-, \mathrm{Map}_c(G, \mathrm{colim}_N\, \mathcal{F}(G/N))),$$

gives a cosimplicial sheaf $T^\bullet\mathcal{F}$ of fibrant spectra. This last fact is verified in the following lemma.

Lemma 6.6. *If $\mathcal{F}$ is a sheaf of fibrant spectra, then $T^\bullet\mathcal{F}$ is a cosimplicial sheaf of fibrant spectra.*

Proof. It suffices to show that $T\mathcal{F}$ is a sheaf of fibrant spectra. Any $S \in G{-}\mathbf{Sets}_{df}$ is isomorphic to a finite disjoint union $\coprod_i G/H_i$, where each H_i is an open subgroup of G. Thus,

$$(T\mathcal{F})(S) \cong \prod_i \mathrm{Hom}_G(G/H_i, \mathrm{Map}_c(G, \mathrm{colim}_N\, \mathcal{F}(G/N))) \cong \prod_i \prod_{G/H_i} \mathrm{colim}_N\, \mathcal{F}(G/N).$$

Since filtered colimits and products of fibrant spectra are fibrant, $(T\mathcal{F})(S)$ is a fibrant spectrum. □

Definition 6.7. The presheaf of hypercohomology spectra of $\Phi = G{-}\mathbf{Sets}_{df}$ with coefficients in $\mathcal{F}$ is the presheaf of spectra

$$\mathbb{H}^\bullet_\Phi(-;\mathcal{F})\colon (G{-}\mathbf{Sets}_{df})^{\mathrm{op}} \to \mathbf{Spt}, \quad U \mapsto \mathbb{H}^\bullet_\Phi(U;\mathcal{F}) = \mathrm{holim}_\Delta\, (T^\bullet\mathcal{F})(U),$$

and $\mathbb{H}^\bullet_\Phi(U;\mathcal{F})$ is the hypercohomology spectrum of U with coefficients in $\mathcal{F}$.

Lemma 6.8. *Let $X \to X_f$ in $\mathbf{Spt}_G$ be a trivial cofibration with X_f a fibrant discrete G-spectrum. Then $T^\bullet \mathrm{Hom}_G(-, X_f) \cong \mathrm{Hom}_G(-, \Gamma^\bullet X_f)$.*

Proof. Note that the fibrant presheaf $\mathrm{Hom}_G(-, X_f)$ is a sheaf of fibrant spectra, as required. The result is then obtained by induction. By definition,

$$T\big(\mathrm{Hom}_G(-, X_f)\big) \cong \mathrm{Hom}_G(-, \mathrm{Map}_c(G, \mathrm{colim}_N\,(X_f)^N)) \cong \mathrm{Hom}_G(-, \mathrm{Map}_c(G, X_f)).$$

Similarly, the inductive hypothesis gives

$$\begin{aligned} T^{k+1}\big(\mathrm{Hom}_G(-, X_f)\big) &\cong T\big(\mathrm{Hom}_G(-, \mathrm{Map}_c(G^k, X_f))\big) \\ &\cong \mathrm{Hom}_G(-, \mathrm{Map}_c(G, \mathrm{colim}_N\,\mathrm{Map}_c(G^k, X_f)^N)) \\ &\cong \mathrm{Hom}_G(-, \mathrm{Map}_c(G^{k+1}, X_f)). \end{aligned}$$

□

The map ι induces the map $X_f \to \Gamma^\bullet X_f$ out of the constant cosimplicial diagram, which induces the canonical map

$$\mathrm{Hom}_G(-, X_f) \to \lim \mathrm{Hom}_G(-, X_f) \to \mathrm{holim}\,\mathrm{Hom}_G(-, X_f) \to \mathrm{holim}\,\mathrm{Hom}_G(-, \Gamma^\bullet X_f),$$

where each (ho)lim is over Δ. Then the preceding lemma gives a map

$$\mathrm{Hom}_G(-, X_f) \longrightarrow \mathrm{holim}_\Delta\,\mathrm{Hom}_G(-, \Gamma^\bullet X_f) \xrightarrow{\cong} \mathbb{H}^\bullet_\Phi\,(-; \mathrm{Hom}_G(-, X_f)).$$

Remark 6.9. We recall the Bousfield-Kan spectral sequence, an important tool for our work. [**3**, Chp. XI, §7] constructs a homotopy spectral sequence

$$\pi^s\pi_t(F^\bullet) \Longrightarrow \pi_{t-s}(\mathrm{Tot}F^\bullet),$$

where $F^\bullet$ is a fibrant cosimplicial pointed simplicial set. This spectral sequence extends to cosimplicial spectra: if $F^\bullet$ is a cosimplicial diagram of fibrant spectra (a "cosimplicial fibrant spectrum"), then there is a spectral sequence

$$E_2^{s,t} = \pi^s\pi_t(F^\bullet) \Longrightarrow \pi_{t-s}(\operatorname{holim}_\Delta F^\bullet) \tag{6.10}$$

[**48**, Prop. 5.13, Lem. 5.31].

Here, as elsewhere, the holim of spectra is constructed levelwise in $\mathcal{S}$ and is defined as in [**3**]. Thus, associated to $F^\bullet$ is the cosimplicial spectrum $\prod^* F^\bullet$, with $\prod^n F^\bullet = \prod_{(B\Delta)_n} F^{j_n}$, where the n-simplices of the classifying space $B\Delta$ consist of all strings $[j_0] \to \cdots \to [j_n]$ of n morphisms in Δ. For any $k \geq 0$, $(\prod^* F^\bullet)_k$ is a fibrant cosimplicial pointed simplicial set, and $\operatorname{holim}_\Delta F^\bullet = \mathrm{Tot}(\prod^* F^\bullet)$.

More generally, if J is a small category and $F\colon J \to \mathbf{Spt}$ is a diagram of fibrant spectra, then there is a spectral sequence

$$E_2^{s,t} = \lim{}^s{}_J\, [Z, F]_t \Longrightarrow [Z, \operatorname{holim}_J F]_{t-s}, \tag{6.11}$$

where Z is a finite spectrum equivalent to a CW-spectrum, and $\lim^s$ is the sth right derived functor of $\lim_J\colon \mathrm{Ab}^J \to \mathrm{Ab}$. All of the above spectral sequences are conditionally convergent [**34**, pg. 246].

Now we prove the result that gives us a more useful model of X^{hG}. This result is not original: a version of it is in [**25**, Prop. 3.3], and more general forms of it appear in [**34**, Prop. 3.20] and [**35**, Prop. 6.1].

Theorem 6.12. *Let G be a profinite group with $\mathrm{vcd}(G) \leq m$, and let X be a discrete G-spectrum. Then there are weak equivalences*

$$\mathrm{Hom}_G(-, X) \xrightarrow{\simeq} \mathrm{Hom}_G(-, X_f) \xrightarrow{\simeq} \mathbb{H}^{\bullet}{}_{\Phi}\,(-; \mathrm{Hom}_G(-, X_f)),$$

where $\mathbb{H}^{\bullet}{}_{\Phi}\,(-; \mathrm{Hom}_G(-, X_f))$ is a globally fibrant model for $\mathrm{Hom}_G(-, X)$. Then the canonical map

$$X^{hG} = \mathrm{Hom}_G(*, X_f) \longrightarrow \operatorname{holim}_{\Delta}\,(\Gamma^{\bullet}_G X_f)^G \cong \mathbb{H}^{\bullet}{}_{\Phi}\,(*; \mathrm{Hom}_G(-, X_f))$$

is a weak equivalence.

Remark 6.13. Since $(X_f)^G$ is already X^{hG} and given Lemma 6.17 below, the reader might wonder if X_f can be replaced by a weaker model for X. This can be done, but after trying several alternatives, X_f was decided upon as the best choice, in part because of the above canonical map.

Proof. We refer to the references above, especially [**25**, Prop. 3.3], for the proof that the hypercohomology presheaf $\mathbb{H}^{\bullet}_{\Phi}(-;\mathrm{Hom}_G(-,X_f))$ is globally fibrant. It only remains to show that $\eta\colon \mathrm{Hom}_G(-,X) \to \mathbb{H}^{\bullet}_{\Phi}(-;\mathrm{Hom}_G(-,X_f))$ is a weak equivalence.

By hypothesis, G contains an open subgroup H with $\mathrm{cd}(H) \leq m$. Then by [**50**, Lem. 0.3.2], H contains a subgroup K that is an open normal subgroup of G. Let $\{N\}$ be the collection of open normal subgroups of G. Set $N' = N \cap K$. Observe that $\{N'\}$ is a cofinal subcollection of open normal subgroups of G so that $G \cong \varprojlim_{N'} G/N'$. Since $N' <_c H$, $\mathrm{cd}(N') \leq \mathrm{cd}(H)$. Thus, $H^s_c(N';M) = 0$, for all $s > m+1$, whenever M is a discrete N'-module.

Henceforth, we drop the $'$ from N' to ease the notation: N is really $N \cap K$.

Any sheaf of sets $\mathcal{F}$ has stalk $\mathrm{colim}_N\, \mathcal{F}(G/N)$ and so η is a weak equivalence of presheaves of spectra if $\hat{\eta}\colon X \cong \mathrm{colim}_N\, X^N \to \mathrm{colim}_N\, \mathbb{H}^{\bullet}_{\Phi}(G/N;\mathrm{Hom}_G(-,X_f))$ is a weak equivalence, where $\mathrm{colim}_N\, \mathbb{H}^{\bullet}_{\Phi}(G/N;\mathrm{Hom}_G(-,X_f)) \cong \mathrm{colim}_N\, \mathrm{holim}_{\Delta} \left(\Gamma^{\bullet}_G X_f\right)^N$. For each N, apply 6.10 to obtain the conditionally convergent spectral sequence

$$E_2^{s,t}(N) = \pi^s\pi_t\big((\Gamma^{\bullet}_G X_f)^N\big) \Longrightarrow \pi_{t-s}\big(\mathrm{holim}_{\Delta} \left(\Gamma^{\bullet}_G X_f\right)^N\big). \tag{6.14}$$

Since $T^{\bullet}\,\mathrm{Hom}_G(-,X_f) \cong \mathrm{Hom}_G(-,\Gamma^{\bullet} X_f)$ is a cosimplicial sheaf of fibrant spectra, the diagram $(\Gamma^{\bullet} X_f)^N$ is a cosimplicial fibrant spectrum, as required. Note that

$$\pi_t(\mathrm{Map}_c(G,X_f)^N) \cong \pi_t(\mathrm{Map}_c(G/N,X_f) \cong \textstyle\prod_{G/N} \pi_t(X) \cong \mathrm{Map}_c(G,\pi_t(X))^N.$$

Thus, $\pi^s\pi_t\big((\Gamma^\bullet_G X_f)^N\big) \cong H^s\big((\Gamma^*\pi_t(X))^N\big)$. The cochain complex $0 \to \pi_t(X) \to \Gamma^*\pi_t(X)$ is exact and $H^s_c(N;\Gamma^k\pi_t(X)) \cong H^s_c(N;\mathrm{Map}_c(G,\Gamma^{k-1}\pi_t(X))) = 0$, for $s > 0$, by 8.16. Thus, $\Gamma^*\pi_t(X)$ is a resolution of $\pi_t(X)$ by $(-)^N$-acyclics and thus,

$$E_2^{s,t}(N) \cong H^s_c(N;\pi_t(X)).$$

Taking a colimit over $\{N\}$ of 6.14 gives the spectral sequence

$$\text{(6.15)} \qquad \mathrm{colim}_N\, H^s_c(N;\pi_t(X)) \Longrightarrow \pi_{t-s}\big(\mathrm{colim}_N\, \mathrm{holim}_\Delta \left(\Gamma^\bullet X_f\right)^N\big).$$

Since $E_2^{s,*}(N) = 0$ whenever $s > m+1$, the E_2-terms $E_2(N)$ are uniformly bounded on the right. Therefore, by [**34**, Prop. 3.3], the colimit of the spectral sequences does converge to the colimit of the abutments, as stated in 6.15. Finally,

$$\mathrm{colim}_N\, H^*_c(N;\pi_t(X)) \cong H^*_c(\varprojlim N;\pi_t(X)) \cong H^*(\{e\};\pi_t(X)),$$

which is isomorphic to $\pi_t(X)$, concentrated in degree 0. Thus, 6.15 collapses and for all t, $\pi_t\big(\mathrm{colim}_N\, \mathrm{holim}_\Delta \left(\Gamma^\bullet_G X_f\right)^N\big) \cong \pi_t(X)$, showing that $\hat{\eta}$ is a weak equivalence. □

Remark 6.16. Because of Theorem 6.12, whenever the profinite group G has finite virtual cohomological dimension, and X is a discrete G-spectrum, we will always make the identification $X^{hG} = \mathrm{holim}_\Delta\, (\Gamma^\bullet_G(X_f))^G$. (There are a few exceptions to this, where it is clear from the context.)

The following lemma is obtained from 6.6 and 6.8 by evaluating at $* \in G-\mathbf{Sets}_{df}$. It is used repeatedly so we make it explicit.

Lemma 6.17. *If X is fibrant in $\mathbf{Spt}_G$, then $(\Gamma^\bullet_G X)^G$ is a cosimplicial fibrant spectrum.*

Given a profinite group G of finite virtual cohomological dimension and a discrete G-spectrum X, the following theorem shows that there is always a descent spectral sequence.

Remark 6.18. Given a discrete G-module M, there are canonical maps

$$\eta(M)\colon M \to \Gamma_G(M) = \mathrm{Map}_c(G, M) \text{ and } \varepsilon\colon \mathrm{Map}_c(G, \mathrm{Map}_c(G, M)) \to \mathrm{Map}_c(G, M),$$

so that Γ gives a triple and there is an associated cosimplicial discrete G-module $\Gamma^\bullet_G(M)$. Recall that $H^s_c(G, M)$ can be computed by taking the cohomology of the cochain complex

$$M \to \mathrm{Map}_c(G, M) \to \mathrm{Map}_c(G^2, M) \to \cdots .$$

As pointed out in [**14**, Lem. 5.4 and its proof], the maps in this cochain complex are identical to the maps in the cochain complex associated to the cosimplicial abelian group $(\Gamma^\bullet(M))^G$. Thus, $H^s_c(G; M) \cong \pi^s\big((\Gamma^\bullet(M))^G\big)$.

Theorem 6.19. *If the profinite group G has finite virtual cohomological dimension and X is a discrete G-spectrum, then there is a conditionally convergent descent spectral* *sequence*

$$E_2^{s,t} = H_c^s(G; \pi_t(X)) \Longrightarrow \pi_{t-s}(X^{hG}).$$

Proof. The preceding lemma and 6.10 give a spectral sequence

$$\pi^s \pi_t\big((\Gamma^{\bullet} X_f)^G\big) \Longrightarrow \pi_{t-s}(X^{hG}).$$

By the above remark, $E_2^{s,t} \cong H_c^s(G; \pi_t(X))$, since $\pi_t(X_f) \cong \pi_t(X)$ is an isomorphism of discrete G-modules. □

CHAPTER 7

Homotopy fixed point spectra for E_n

Using the machinery of the previous chapter, it is easy to define homotopy fixed point spectra $E_n^{h'G}$ for $G <_c G_n$, using the continuous action. In this chapter, we also show that $E_n^{h'G}$ matches the homotopy fixed point spectrum E_n^{hG} of Devinatz and Hopkins.

Definition 7.1. Let $E_n^{h'G} = \operatorname{holim}_I (F_n \wedge M_I)^{hG}$ be the homotopy fixed point spectrum of E_n with respect to the continuous action of G. Since $F_n \wedge M_I$ is a discrete G-spectrum and $\mathrm{vcd}(G) < \infty$, $(F_n \wedge M_I)^{hG}$ is defined as in Remark 6.16.

Remark 7.2. Note that the map $((F_n \wedge M_I)_f)^G \to \operatorname{holim}_\Delta (\Gamma^{\cdot}_G(F_n \wedge M_I)_f)^G$ is a weak equivalence between fibrant spectra, since the map is obtained by taking the global sections of a weak equivalence between fibrant presheaves of spectra. Thus, there is a weak equivalence $\operatorname{holim}_I ((F_n \wedge M_I)_f)^G \to \operatorname{holim}_I \operatorname{holim}_\Delta (\Gamma^{\cdot}_G(F_n \wedge M_I)_f)^G$ between the "strict" definition of $E_n^{h'G}$ and our definition.

Remark 7.3. We make explicit the following elementary fact, which will be used several times later. If $\{X_j\}$ is a finite set of spectra and Y is any spectrum, then there is a weak equivalence $(\prod X_j) \wedge Y \to \prod (X_j \wedge Y)$.

It is not hard to see that $E_n^{h'G}$ is $K(n)_*$-local, as expected, since E_n^{hG} is.

Theorem 7.4. *The homotopy fixed point spectra* $(F_n \wedge M_I)^{hG}$ *and* $E_n^{h'G}$ *are* $K(n)_*$*-local. Also,* F_n^{hG} *is* $E(n)_*$*-local.*

Proof. Note that

$$(\Gamma^k(F_n \wedge M_I)_f)^G \cong \mathrm{Map}_c(G^{k-1}, (F_n \wedge M_I)_f) \simeq \mathrm{colim}_N \prod_{(G/N)^{k-1}} (F_n \wedge M_I)$$

$$\simeq (\mathrm{colim}_N \prod_{(G/N)^{k-1}} F_n) \wedge M_I.$$

Since F_n is $E(n)_*$-local, the finite product is also. Since L_n commutes with direct limits, $\mathrm{colim}_N \prod_{(G/N)^{k-1}} F_n$ is $E(n)_*$-local, and thus, $(\Gamma^k(F_n \wedge M_I)_f)^G$ is $K(n)_*$-local. Given a CW-spectrum E, the homotopy limit of an arbitrary diagram of E_*-local spectra is E_*-local [**2**, pg. 259], making $(F_n \wedge M_I)^{hG} = \mathrm{holim}_\Delta (\Gamma^\bullet(F_n \wedge M_I)_f)^G$ $K(n)_*$-local. Hence, the homotopy limit $E_n^{h'G}$ is $K(n)_*$-local. The same arguments show that F_n^{hG} is $E(n)_*$-local. □

After seeing how $E_n^{h'G}$ is defined, we now examine the construction of E_n^{hU} by Devinatz and Hopkins, for $U <_o G_n$. For this discussion, we use the notation of [**9**] and work, as needed, in the category of S-modules. At times, we will not be completely rigorous, since we are trying to understand from the perspective of a continuous action how the construction of Devinatz and Hopkins works.

Construction 3.11 of [**9**] defines a diagram

$$\mathbf{C}\colon (R^+_{G_n})^{\mathrm{op}} \times \Delta \to \mathrm{Ho}(E_\infty), \quad \mathbf{C}(S, [j]) = L_{K(n)}(\mathbf{X}(S) \wedge E_n^j),$$

where $\mathbf{X}\colon (R^+_{G_n})^{\mathrm{op}} \to \mathrm{Ho}(E_\infty)$ is a certain diagram with $\mathbf{X}(G_n/U) = \prod_{G_n/U} E_n$. Then [**9**] shows that $\mathbf{C}$ lifts to a diagram $\overline{\mathbf{C}}\colon (R^+_{G_n})^{\mathrm{op}} \times \Delta \to E_\infty$, such that there is a natural transformation $\overline{\mathbf{C}} \to \mathbf{C}$ in $\mathrm{Ho}(E_\infty)$, with $\overline{\mathbf{C}}(S,[j]) \to \mathbf{C}(S,[j])$ a weak equivalence for every S and $[j]$. Henceforth, we write $C^\bullet_{G_n/U}$ instead of $\overline{\mathbf{C}}(G_n/U,-)$ for the cosimplicial diagram $\Delta \to E_\infty$, so that $C^j_{G_n/U} \equiv \overline{\mathbf{C}}(G_n/U,[j])$. Then, by [**9**, Def. 3.12],

$$E_n^{hU} = \mathrm{Tot}(\textstyle\prod^* C^\bullet_{G_n/U}) = \mathrm{holim}_\Delta\, C^\bullet_{G_n/U}.$$

Since $\mathbf{C}^\bullet_{G_n/U} \in \big(\mathrm{Ho}(E_\infty)\big)^\Delta$, one cannot form $\mathrm{holim}_\Delta\, \mathbf{C}^\bullet_{G_n/U}$. However, since $C^\bullet_{G_n/U}$ is a strict diagram of spectra, $\mathrm{holim}_\Delta\, C^\bullet_{G_n/U}$ is a model for the undefined $\mathrm{holim}_\Delta\, \mathbf{C}^\bullet_{G_n/U}$.

We work to understand $\mathbf{C}^\bullet_{G_n/U}$ better. We ignore the fact that it is only a diagram in the homotopy category; we assume that $E_n^j \wedge M_I$ is a discrete G_n-spectrum (as is its weakly equivalent model $F_n^j \wedge M_I$); we assume that holim preserves natural transformations consisting of weak equivalences; and we ignore the fact that the G_n-action we should have below is that of $\mathrm{Map}^\ell_c(G_n^j,-)$ instead of $\mathrm{Map}_c(G_n^j,-)$ [**9**, 3.19]. Applying [**9**, pg. 10], we have

$$\begin{aligned} C^j_{G_n/U} &\simeq \mathbf{C}^j_{G_n/U} = L_{K(n)}\big((\textstyle\prod_{G_n/U} E_n) \wedge E_n^j\big) \simeq \mathrm{holim}_I \big(\textstyle\prod_{G_n/U} (E_n^{j+1} \wedge M_I)\big) \\ &\simeq \mathrm{holim}_I\, \mathrm{Map}_c(G_n, E_n^{j+1} \wedge M_I)^U \simeq \mathrm{holim}_I\, \mathrm{Map}_c(G_n, \mathrm{Map}_c(G_n^j, E_n \wedge M_I))^U \\ &\simeq \mathrm{holim}_I\, \mathrm{Map}_c(G_n^{j+1}, E_n \wedge M_I)^U. \end{aligned}$$

Thus, $\operatorname{holim}_\Delta C^\bullet_{G_n/U} \simeq \operatorname{holim}_I \operatorname{holim}_\Delta (\Gamma^\bullet_{G_n}(E_n \wedge M_I))^U$. We examine this last expression further.

Consider the functor $(U - \mathbf{Sets}_{df})^{\mathrm{op}} \to (G_n - \mathbf{Sets}_{df})^{\mathrm{op}}$, defined by $S \mapsto G_n/U \times S$, where S has trivial G_n-action. Composition with this functor gives the restriction functor res: $\mathbf{PreSpt}(G_n - \mathbf{Sets}_{df}) \to \mathbf{PreSpt}(U - \mathbf{Sets}_{df})$, which sends the cosimplicial sheaf $\mathrm{Hom}_{G_n}(-, \Gamma^\bullet_{G_n} X_f)$ to the cosimplicial sheaf

$$\mathrm{Hom}_{G_n}(G_n/U \times (-), \Gamma^\bullet_{G_n} X_f) \cong \mathrm{Hom}_{G_n}(G_n/U, \mathbf{Sets}(-, \Gamma^\bullet_{G_n} X_f)) \cong \mathbf{Sets}(-, \Gamma^\bullet_{G_n} X_f)^U$$
$$\cong \mathrm{Hom}_U(-, \Gamma^\bullet_{G_n} X_f).$$

Since U is an open subgroup of G_n, the restriction of the globally fibrant presheaf $\operatorname{holim}_\Delta \mathrm{Hom}_{G_n}(-, \Gamma^\bullet_{G_n} X_f)$ is the globally fibrant presheaf $\operatorname{holim}_\Delta \mathrm{Hom}_U(-, \Gamma^\bullet_{G_n} X_f)$. (When U is a normal subgroup, [**27**, Rk. 6.26] shows that res preserves globally fibrant presheaves.)

Note that $\mathrm{Hom}_U(S, \Gamma^\bullet_{G_n} X_f) \cong \mathrm{Hom}_{G_n}(G_n/U \times S, \Gamma^\bullet_{G_n} X_f)$ is a cosimplicial fibrant spectrum since $\mathrm{Hom}_{G_n}(-, \Gamma^\bullet_{G_n} X_f)$ is a cosimplicial sheaf of fibrant spectra. Any subgroup H that is open in U, is open in G_n. These two observations imply that the proof of Theorem 6.12 goes through to show that $\mathrm{Hom}_U(-, X) \to \operatorname{holim}_\Delta \mathrm{Hom}_U(-, \Gamma^\bullet_{G_n} X_f)$ is a weak equivalence of presheaves.

Since $\mathrm{Hom}_{G_n}(-, X_{f,G_n})$ is globally fibrant on $(G_n - \mathbf{Sets}_{df})^{\mathrm{op}}$, $\mathrm{Hom}_U(-, X_{f,G_n})$ is globally fibrant on $(U - \mathbf{Sets}_{df})^{\mathrm{op}}$, and X_{f,G_n} is a fibrant discrete U-spectrum. Thus, there is a weak equivalence $\mathrm{Hom}_U(-, X_{f,G_n}) \to \operatorname{holim}_\Delta \mathrm{Hom}_U(-, \Gamma^\bullet_{G_n} X_{f,G_n})$ between globally fibrant presheaves, which gives a weak equivalence

$$X^{hU} \cong \mathrm{Hom}_U(*, X_{f,G_n}) \to \operatorname{holim}_\Delta \mathrm{Hom}_U(*, \Gamma^\bullet_{G_n} X_{f,G_n}) = \operatorname{holim}_\Delta (\Gamma^\bullet_{G_n} X_{f,G_n})^U,$$

where the isomorphism is in Ho(**Spt**). Therefore, given any discrete G_n-spectrum X, $\operatorname{holim}_\Delta (\Gamma^\bullet_{G_n}(X_{f,G_n}))^U$ is a model for X^{hU}.

Let us return to our assumptions about $\mathbf{C}^\bullet_{G_n/U}$ and allow ourselves to move freely between S-modules and Bousfield-Friedlander spectra. Observe that

$$(\Gamma^\bullet_{G_n}(E_n \wedge M_I))^U \to (\Gamma^\bullet_{G_n}(E_n \wedge M_I)_{f,G_n})^U$$

is a weak equivalence in each cosimplicial degree. Thus,

$$\begin{aligned} \operatorname{holim}_\Delta C^\bullet_{G_n/U} &\simeq \operatorname{holim}_I \operatorname{holim}_\Delta (\Gamma^\bullet_{G_n}(E_n \wedge M_I))^U \\ &\simeq \operatorname{holim}_I \operatorname{holim}_\Delta (\Gamma^\bullet_{G_n}(E_n \wedge M_I)_{f,G_n})^U. \end{aligned}$$

We highlight our conclusion in

Remark 7.5. The above discussion shows that, in some sense,

$$E_n^{hU} = \mathrm{Tot}(\textstyle\prod^* C^\bullet_{G_n/U}) \simeq \operatorname{holim}_I (E_n \wedge M_I)^{hU},$$

and thus, E_n^{hU} is the continuous homotopy fixed point spectrum. Also, conceptually, we see that E_n^{hU} and $E_n^{h'U}$ are the same.

We make this remark precise by showing in what sense the continuous homotopy fixed point spectra $E_n^{h'G}$ match the homotopy fixed point spectra E_n^{hG} of Devinatz and Hopkins.

Lemma 7.6 ([**7**, proof of Lem. 3.5])**.** *For any $G <_c G_n$ and any integer t, the homotopy group $\pi_t(E_n^{hG} \wedge M_I)$ of the S-module $E_n^{hG} \wedge M_I$ is a finite abelian group.*

Remark 7.7. By the discussion about homotopy colimits in Chapter 2, in **Spt**, $L_{K(n)}(\operatorname{colim}_i E_n^{hU_iG})$ is a model for the S-module E_n^{hG} defined by Devinatz and Hopkins. Thus, in the context of **Spt**, we set $E_n^{hG} \equiv L_{K(n)}(\operatorname{colim}_i E_n^{hU_iG})$.

However, clarification is needed if $G = U$ is an open subgroup. In S-modules, the spectra E_n^{hU} and $L_{K(n)}(\operatorname{colim}_i E_n^{hU_iU})$ have the same homotopy type. But it is not obvious that this is the case in **Spt**, where $E_n^{hU} = (U\mathbb{S}\Phi(-)_{\mathsf{f}}\mathbf{F})(G_n/U)$. Let $X_{\mathcal{C}}$ and $X \wedge_{\mathcal{C}} Y$ refer to the spectra X and $X \wedge Y$, respectively, where X and Y are in $\mathcal{C}$, a category of spectra. As in the proof of 7.9, there are strongly convergent spectral sequences

$$H_c^s(U; \pi_t(E_n \wedge_{\mathbf{Spt}} M_I)) \Longrightarrow \pi_{t-s}(E_n^{hU} \wedge_{\mathbf{Spt}} M_I),$$

and

$$H_c^s(U; \pi_t(E_n \wedge_{\mathcal{M}_S} M_I)) \Longrightarrow \pi_{t-s}(E_n^{hU} \wedge_{\mathcal{M}_S} M_I).$$

By Landweber exactness, $\pi_*(E_n \wedge_{\mathbf{Spt}} M_I) \cong E_{n*}/I \cong \pi_*(E_n \wedge_{\mathcal{M}_S} M_I)$. Thus, the spectral sequences have isomorphic E_2-terms and $\pi_*(E_n^{hU} \wedge_{\mathbf{Spt}} M_I)$ and $\pi_*(E_n^{hU} \wedge_{\mathcal{M}_S} M_I)$ are the same up to group extensions. Furthermore, since $(E_n^{hU})_{\mathbf{Spt}}$ is a model for $(E_n^{hU})_{\mathcal{M}_S}$ and M_I has the same characterization in either category of spectra, we conclude that $\pi_*(E_n^{hU} \wedge_{\mathbf{Spt}} M_I) \cong \pi_*(E_n^{hU} \wedge_{\mathcal{M}_S} M_I)$. Thus, since $E_n^{hU} \wedge_{\mathcal{M}_S} M_I \simeq \operatorname{colim}_i (E_n^{hU_iU} \wedge_{\mathcal{M}_S} M_I)$ and $\pi_*(E_n^{hU} \wedge M_I)$ is degree-wise finite,

$$\begin{aligned}\pi_*((E_n^{hU})_{\mathbf{Spt}}) &\cong \pi_*((E_n^{hU})_{\mathcal{M}_S}) \cong \pi_*(L_{K(n)}(\operatorname{colim}_i (E_n^{hU_iU})_{\mathcal{M}_S}))\\ &\cong \pi_*(\operatorname{holim}_I \operatorname{colim}_i (E_n^{hU_iU} \wedge_{\mathcal{M}_S} M_I)) \cong \varprojlim_I \operatorname{colim}_i \pi_*(E_n^{hU_iU} \wedge_{\mathcal{M}_S} M_I)\\ &\cong \varprojlim_I \operatorname{colim}_i \pi_*(E_n^{hU_iU} \wedge_{\mathbf{Spt}} M_I) \cong \pi_*(\operatorname{holim}_I \operatorname{colim}_i (E_n^{hU_iU} \wedge_{\mathbf{Spt}} M_I))\\ &\cong \pi_*(L_{K(n)}(\operatorname{colim}_i (E_n^{hU_iU})_{\mathbf{Spt}})).\end{aligned}$$

Therefore, the canonical map $L_{K(n)}(\operatorname{colim}_i (E_n^{hU_iU})_{\mathbf{Spt}}) \to (E_n^{hU})_{\mathbf{Spt}}$ in $\operatorname{Ho}(\mathbf{Spt})$ is an isomorphism. Thus, in $\mathbf{Spt}$, just as in $\mathcal{M}_S$, we can move freely between the two different meanings of E_n^{hU}, $L_{K(n)}(\operatorname{colim}_i (E_n^{hU_iU})_{\mathbf{Spt}})$ and $(U\mathbb{S}\Phi(-)_{\mathsf{f}}\mathbf{F})(G_n/U)$ in the stable category. In any particular instance, the context will make it clear which meaning we intend.

Given a spectrum X, though there is no point-set level map between X and $X \wedge S^0$, there is an isomorphism $E_n^{hG} = L_{K(n)}(\operatorname{colim}_i E_n^{hU_iG}) \cong L_{K(n)}((\operatorname{colim}_i E_n^{hU_iG}) \wedge S^0)$ in $\operatorname{Ho}(\mathbf{Spt})$. Also, since there is no point-set level map between $L_{K(n)}(X \wedge S^0)$ and $\operatorname{holim}_I (X \wedge M_I)$, for an $E(n)_*$-local spectrum X, the best we can do in our case is the following zigzag:

$$\begin{array}{ccc} L_{K(n)}((\operatorname{colim}_i E_n^{hU_iG}) \wedge S^0) & & \operatorname{holim}_I((\operatorname{colim}_i E_n^{hU_iG}) \wedge M_I) \\ \downarrow \simeq & \overset{\simeq}{\swarrow} & \\ L_{K(n)}(\operatorname{holim}_I((\operatorname{colim}_i E_n^{hU_iG}) \wedge M_I)). & & \end{array}$$

Now we specify the map $\psi_G\colon \operatorname{holim}_I((\operatorname{colim}_i E_n^{hU_iG}) \wedge M_I) \to E_n^{h'G}$. When G is finite, [**9**, pg. 5] shows that there is a canonical map into the homotopy fixed points $E_n^{hG} \to \operatorname{holim}_G E_n$, defined by the composition $E_n^{hG} \to \lim_G E_n \to \operatorname{holim}_G E_n$. Similarly, we would like to construct ψ by composing with a map

$$\phi_G\colon \lim_G(\mathbf{F}(G_n) \wedge S^0) \to E_n^{h'G} = \operatorname{holim}_I(F_n \wedge M_I)^{hG}.$$

However, $\phi_{\{e\}}$ forces the existence of a map $\mathbf{F}(G_n) \wedge S^0 \to \operatorname{holim}_I(F_n \wedge M_I)$. But we saw earlier that there is no such canonical G_n-map, and thus, there is no $\phi_{\{e\}}$. We conclude that there is no canonical map ϕ_G, and we proceed to construct ψ another way.

Since U_iG/U_i is a subgroup of G_n/U_i, the canonical map $E_n^{hU_iG} \to E_n^{hU_i}$ gives a map $E_n^{hU_iG} \to \lim_{U_iG/U_i} E_n^{hU_i} \xrightarrow{\cong} (E_n^{hU_i})^{U_iG/U_i}$. Letting $(E_n^{hU_i})^{U_iG/U_i}$ also denote the constant cosimplicial spectrum, there are canonical maps

$$(E_n^{hU_i})^{U_iG/U_i} \to \lim(E_n^{hU_i})^{U_iG/U_i} \to \operatorname{holim}(E_n^{hU_i})^{U_iG/U_i} \to \operatorname{holim}(\Gamma^\bullet_{U_iG/U_i}(E_n^{hU_i}))^{U_iG/U_i},$$

where all (ho)limits are over Δ. Also, the projection $U_iG \to U_iG/U_i$ induces canonical maps

$$\operatorname{holim}_\Delta (\Gamma^\bullet_{U_iG/U_i}(E_n^{hU_i}))^{U_iG/U_i} \to \operatorname{holim}_\Delta (\Gamma^\bullet_{U_iG} E_n^{hU_i})^{U_iG/U_i} \to \operatorname{holim}_\Delta (\Gamma^\bullet_{U_iG} E_n^{hU_i})^{U_iG}.$$

The inclusion $G \to U_iG$ gives maps

$$\operatorname{holim}_\Delta (\Gamma^\bullet_{U_iG} E_n^{hU_i})^{U_iG} \to \operatorname{holim}_\Delta (\Gamma^\bullet_{U_iG} E_n^{hU_i})^{G} \to \operatorname{holim}_\Delta (\Gamma^\bullet_{G} E_n^{hU_i})^{G}.$$

Putting all these maps together gives a canonical map $E_n^{hU_iG} \to \operatorname{holim}_\Delta (\Gamma^\bullet_G E_n^{hU_i})^G$. The G_n-equivariant map $E_n^{hU_i} \to F_n$ induces the map

$$\operatorname{holim}_\Delta (\Gamma^\bullet_G E_n^{hU_i})^G \to \operatorname{holim}_\Delta (\Gamma^\bullet_G F_n)^G,$$

and thus, there are maps

$$\operatorname{colim}_i E_n^{hU_iG} \to \operatorname{colim}_i \operatorname{holim}_\Delta (\Gamma^\bullet_G E_n^{hU_i})^G \to \operatorname{holim}_\Delta (\Gamma^\bullet_G F_n)^G.$$

This composition defines the map $\operatorname{colim}_i E_n^{hU_iG} \to \operatorname{holim}_\Delta (\Gamma^\bullet_G F_n)^G$.

Definition 7.8. Since there is a map

$$(\operatorname{holim}_\Delta (\Gamma^\bullet F_n)^G) \wedge M_I \to \operatorname{holim}_\Delta (\Gamma^\bullet (F_n \wedge M_I))^G,$$

we define $\phi_{I,G} \colon (\operatorname{colim}_i E_n^{hU_iG}) \wedge M_I \to (F_n \wedge M_I)^{hG}$ to be the composition

$$(\operatorname{colim}_i E_n^{hU_iG}) \wedge M_I \to \operatorname{holim}_\Delta (\Gamma^\bullet_G(F_n \wedge M_I))^G \to \operatorname{holim}_\Delta (\Gamma^\bullet_G(F_n \wedge M_I)_f)^G$$

Thus, the map $\psi_G \colon \operatorname{holim}_I ((\operatorname{colim}_i E_n^{hU_iG}) \wedge M_I) \to E_n^{h'G}$ is defined to be $\operatorname{holim}_I \phi_{I,G}$.

The following sequence of results shows that ψ_G is a weak equivalence for all $G <_c G_n$. We first do this for G open, then for G closed. Though we will prove this again in Chapter 10 with a stronger theorem, we include these proofs because the technique is simple and useful.

Lemma 7.9. *For any open subgroup U of G_n, the map*

$$\phi_{I,U} \colon (\operatorname{colim}_i E_n^{hU_iU}) \wedge M_I \to (F_n \wedge M_I)^{hU}$$

is a weak equivalence. Also, there is an isomorphism in the stable category:

$$E_n^{hU} \wedge M_I \cong (F_n \wedge M_I)^{hU}.$$

Proof. By [**9**, (3.13), Prop. 4.16], there is a conditionally convergent (actually, strongly convergent) spectral sequence with

$$E_2^{s,t} \cong H_c^s(U; E_n^{-t}(\Lambda\mathbb{T}V(DM_I)_{\mathsf{c}})) \cong \pi^s([\Lambda\mathbb{T}V(DM_I)_{\mathsf{c}}, C^*_{G_n/U}]^{-t}_{\mathcal{M}_S}$$

and abutment

$$[\Lambda\mathbb{T}V(DM_I)_{\mathsf{c}}, \mathrm{Tot}\big(\textstyle\prod^* C^{\bullet}_{G_n/U}\big)]^{-t+s}_{\mathcal{M}_S} \cong [DM_I, E_n^{hU}]^{-t+s}_{\mathbf{Spt}} \cong \pi_{t-s}(E_n^{hU} \wedge M_I)$$

$$\cong \pi_{t-s}((\mathrm{colim}_i\, E_n^{hU_iU}) \wedge M_I),$$

by Remark 7.7.

Also, by 6.19, there is a conditionally convergent spectral sequence

$$H^s_c(U; \pi_t(F_n \wedge M_I)) \cong \pi^s\big(\pi_t((\Gamma^*_U(F_n \wedge M_I)_f)^U)\big) \Longrightarrow \pi_{t-s}((F_n \wedge M_I)^{hU}).$$

Since

$$E_n^{-t}(\Lambda\mathbb{T}V(DM_I)_{\mathsf{c}}) = [\Lambda\mathbb{T}V(DM_I)_{\mathsf{c}}, \mathbf{F}(G_n)]^{-t}_{\mathcal{M}_S} \cong [DM_I, E_n]^{-t}_{\mathbf{Spt}} \cong \pi_t(E_n \wedge M_I),$$

there is an isomorphism between the E_2-terms of the spectral sequences that is compatible with the map $\phi_{I,U}$ between the abutments. Thus, by [**1**, Thm 7.2], comparison of spectral sequences gives an isomorphism of abutments. □

Corollary 7.10. *For any open subgroup U of G_n, there is a weak equivalence*

$$\psi_U \colon \mathrm{holim}_I\, ((\mathrm{colim}_i\, E_n^{hU_iU}) \wedge M_I) \to E_n^{h'U},$$

and an isomorphism $E_n^{hU} \cong E_n^{h'U}$ in the stable category.

Proof. This follows from the preceding lemma by changing $\pi_*(\text{holim}\,(-))$ into $\varprojlim \pi_*(-)$ by a $\varprojlim^1$ argument. $\square$

Theorem 7.11. *Let G be closed in G_n. Then*

$$\phi_{I,G} \colon (\text{colim}_i\, E_n^{hU_iG}) \wedge M_I \longrightarrow (F_n \wedge M_I)^{hG}$$

and $\psi_G \colon \text{holim}_I\,((\text{colim}_i\, E_n^{hU_iG}) \wedge M_I) \longrightarrow E_n^{h'G}$ are weak equivalences. Also, there are isomorphisms $E_n^{hG} \wedge M_I \cong (F_n \wedge M_I)^{hG}$ and $E_n^{hG} \cong E_n^{h'G}$ in the stable category.

Proof. As above, it suffices to verify the first weak equivalence, which we do by a comparison of spectral sequences. There is a conditionally convergent descent spectral sequence

$$H_c^s(G; \pi_t(F_n \wedge M_I)) \Longrightarrow \pi_{t-s}((F_n \wedge M_I)^{hG}).$$

Let $C^{\bullet}_{G_n/G}$ be the cosimplicial S-module with $C^j_{G_n/G} = L_{K(n)}(E_n^{hG} \wedge_{\mathcal{M}_S} E_n^{j+1})$ [**9**, Rk. AI.9]. Then, by remarks from the proof of Lemma 4.11, pg. 35, and Prop. AI.5 in [**9**], there is a conditionally convergent spectral sequence with E_2-term

$$H_c^s(G; \pi_t(E_n \wedge_{\mathcal{M}_S} M_I)) \cong \pi^s([DM_I, C^*_{G_n/G}]^{-t}_{\mathcal{M}_S}$$

and abutment

$$[DM_I, \mathrm{Tot}(\textstyle\prod^* C^\bullet_{G_n/G})]^{-t+s}_{\mathcal{M}_S} \cong \pi_{t-s}(E_n^{hG} \wedge_{\mathcal{M}_S} M_I) \cong \operatorname{colim}_i \pi_{t-s}(E_n^{hU_iG} \wedge_{\mathcal{M}_S} M_I)$$
$$\cong \pi_{t-s}((\operatorname{colim}_i E_n^{hU_iG}) \wedge_{\mathbf{Spt}} M_I).$$

The proof is finished by the same argument used for Lemma 7.9. □

We want to show that $E_n^{h'G}$ can be viewed as the $K(n)_*$-localization of F_n^{hG}. To do this, we give some results about commuting finite spectra with holim that will also be helpful later.

If J is a small category and $\{X_j\}$ is a diagram in $(\mathcal{S}_*)^J$, then for any $i \in J$, the functor $\{i\} \to J$ induces a canonical map $\operatorname{holim}_J X_j \to \operatorname{holim}_{\{i\}} X_i \cong X_i$, where the holim is taken in $\mathcal{S}$. If K is any pointed simplicial set, then there is a canonical map

$$(\operatorname{holim}_J X_j) \wedge K \to \lim_J (X_j \wedge K) \to \operatorname{holim}_J (X_j \wedge K).$$

Thus, if $\{X_j\}$ is a diagram in $\mathbf{Spt}^J$ and Y is any spectrum, there is a canonical map

$$(\operatorname{holim}_J X_j) \wedge Y \to \operatorname{holim}_J (X_j \wedge Y).$$

The following result gives conditions for when smashing with a finite spectrum commutes with homotopy limits (see also [**34**, pg. 251], [**49**, pg. 96]).

Lemma 7.12. *Let J be a small category and let $\{X_j\}$ be a J-shaped diagram of spectra. Let Y be any finite spectrum, and given a spectrum Z, let $Z \to Z_{\mathtt{f}}$ be a weak equivalence with $Z_{\mathtt{f}}$ a fibrant spectrum. Then there is a weak equivalence*

$$(\operatorname{holim}(X_j)_{\mathtt{f}}) \wedge Y \to \operatorname{holim}((X_j)_{\mathtt{f}} \wedge Y)_{\mathtt{f}}.$$

Furthermore, if J is the category $\{0 \leftarrow 1 \leftarrow 2 \leftarrow \cdots\}$ and if $\varprojlim^1 \pi_*(X_j) = 0 = \varprojlim^1 \pi_*(X_j \wedge Y)$, then the canonical map $(\operatorname{holim} X_j) \wedge Y \to \operatorname{holim}(X_j \wedge Y)$ is a weak equivalence.

Remark 7.13. The last statement of the lemma is used several times in Chapter 10 and Corollary 8.10. In Lemma 8.18, we verify the $\varprojlim^1$ hypotheses for the cases we need.

Proof. Setting Z equal to DY and S^0 in 6.11 successively gives the spectral sequences

$$\varprojlim{}^s{}_j\, \pi_t(X_j \wedge Y) \Longrightarrow \pi_{t-s}((\operatorname{holim}(X_j)_{\mathtt{f}}) \wedge Y)$$

and

$$\varprojlim{}^s{}_j\, \pi_t(X_j \wedge Y) \Longrightarrow \pi_{t-s}(\operatorname{holim}((X_j)_{\mathtt{f}} \wedge Y)_{\mathtt{f}}),$$

respectively.

Comparison of these two spectral sequences, using the map

$$(\operatorname{holim}(X_j)_{\mathtt{f}}) \wedge Y \to \operatorname{holim}((X_j)_{\mathtt{f}} \wedge Y) \to \operatorname{holim}((X_j)_{\mathtt{f}} \wedge Y)_{\mathtt{f}},$$

implies that the desired map is a weak equivalence ([**1**, Thm. 7.2], [**34**, pf. of Prop. 3.3]).

The last statement of the theorem, given the hypotheses, follows from

$$\pi_t(\operatorname{holim}(X_j \wedge Y)) \cong \varprojlim \pi_t(X_j \wedge Y) \cong \varprojlim \pi_t(((X_j)_{\mathtt{f}} \wedge Y)_{\mathtt{f}}) \cong \pi_t(\operatorname{holim}((X_j)_{\mathtt{f}} \wedge Y)_{\mathtt{f}}),$$

and the fact that $\pi_t(\operatorname{holim} X_j) \cong \varprojlim \pi_t((X_j)_{\mathtt{f}}) \cong \pi_t(\operatorname{holim}(X_j)_{\mathtt{f}})$. □

Let X be a discrete G-spectrum and Y a finite spectrum with trivial G-action. Then there is a map

$$X^{hG} \wedge Y = (\operatorname{holim}_\Delta (\Gamma^\bullet_G X_f)^G) \wedge Y \to \operatorname{holim}_\Delta ((\Gamma^\bullet_G X_f)^G \wedge Y) \to \operatorname{holim}_\Delta ((\Gamma^\bullet_G X_f) \wedge Y)^G.$$

Consider the isomorphism $((\Gamma^\bullet_G X_f) \wedge Y)^k \cong \operatorname{colim}_N ((\prod_{(G/N)^{k+1}} X_f) \wedge Y)$ and the canonical weak equivalence

$$\operatorname{colim}_N ((\prod_{(G/N)^{k+1}} X_f) \wedge Y) \to \operatorname{colim}_N \prod_{(G/N)^{k+1}} (X_f \wedge Y) \cong (\Gamma^\bullet_G (X_f \wedge Y))^k.$$

Putting these maps together gives a G-equivariant map of cosimplicial spectra

$$(\Gamma^\bullet X_f) \wedge Y \to \Gamma^\bullet (X_f \wedge Y).$$

Thus, there is a canonical map $X^{hG} \wedge Y \to (X_f \wedge Y)^{hG}$ defined by

$$X^{hG} \wedge Y \to \operatorname{holim}_\Delta ((\Gamma^\bullet X_f) \wedge Y)^G \to \operatorname{holim}_\Delta (\Gamma^\bullet (X_f \wedge Y))^G \to \operatorname{holim}_\Delta (\Gamma^\bullet (X_f \wedge Y)_f)^G.$$

A version of the following result appears in [**34**, Prop. 3.10].

Lemma 7.14. *Suppose G has finite virtual cohomological dimension, X is a discrete G-spectrum, and Y is a finite spectrum with trivial G-action. Then the canonical maps*

$$X^{hG} \wedge Y \longrightarrow (X_f \wedge Y)^{hG} \longleftarrow (X \wedge Y)^{hG}$$

are weak equivalences, where $X \to X_f$, in $\mathbf{Spt}_G$, is a trivial cofibration with X_f fibrant.

Remark 7.15. Because there is no canonical G_n-equivariant map $(F_n)_f \wedge M_I \to (F_n \wedge M_I)_f$, as explained in Remark 9.2.7, there is no map $F_n^{hG} \wedge M_I \to (F_n \wedge M_I)^{hG}$ that can be used to define a map $\operatorname{holim}_I (F_n^{hG} \wedge M_I) \to \operatorname{holim}_I (F_n \wedge M_I)^{hG}$. Thus, we do not concern ourselves in the theorem with the existence of a map $X^{hG} \wedge Y \to (X \wedge Y)^{hG}$.

Proof. By 6.19, there is a conditionally convergent spectral sequence

$$H_c^s(G; \pi_t(X \wedge Y)) \Longrightarrow \pi_{t-s}((X_f \wedge Y)^{hG}).$$

Also, by 6.11 with $Z = DY$, there is a conditionally convergent spectral sequence

$$H_c^s(G; \pi_t(X \wedge Y)) \cong H^s\big[\pi_t\big((\Gamma_G^*(X_f \wedge Y))^G\big)\big] \cong H^s\big([DY, (\Gamma_G^* X_f)^G]^{-t}\big) \Longrightarrow \pi_{t-s}(X^{hG} \wedge Y).$$

The map $X^{hG} \wedge Y \to (X_f \wedge Y)^{hG}$, comparison of spectral sequences, and [**1**, Thm. 7.2] give the desired conclusion. $\square$

Corollary 7.16. *For any $G <_c G_n$, there is an isomorphism $E_n^{h'G} \cong L_{K(n)}(F_n^{hG})$ in the stable category.*

Proof. This follows from a $\varprojlim^1$ argument applied to $\pi_*(\text{holim}\,(-))$ using the fact that $\pi_*(F_n^{hG} \wedge M_I) \cong \pi_*((F_n \wedge M_I)^{hG}) \cong \pi_*(E_n^{hG} \wedge M_I)$ is finite. $\square$

The next result is an example of homotopy fixed points commuting with a non-finite spectrum, which is not true in general.

Lemma 7.17. *For any $G <_c G_n$, the spectra $F_n^{hG} \wedge F_n \wedge M_I$ and $(F_n \wedge F_n \wedge M_I)^{hG}$ have the same homotopy type. (The second F_n in each expression is regarded as a trivial G_n-spectrum.)*

Proof. By [**9**, proof of Prop. 5.3], $\pi_*(F_n^{hG} \wedge F_n \wedge M_I) \cong \pi_*(E_n^{hG} \wedge E_n \wedge M_I) \cong \text{Map}_c^{\ell}(G_n, E_{n*}/I)^G$. Also, there is the descent spectral sequence

$$H_c^s(G; \pi_t(F_n \wedge F_n \wedge M_I)) \Longrightarrow \pi_{t-s}\big((F_n \wedge F_n \wedge M_I)^{hG}\big),$$

and because $\pi_*(F_n \wedge F_n \wedge M_I) \cong \text{Map}_c^{\ell}(G_n, E_{n*}/I) \cong \text{Map}_c(G_n, E_{n*}/I)$ is G-acyclic, by 8.16, $\pi_*\big((F_n \wedge F_n \wedge M_I)^{hG}\big) \cong \text{Map}_c^{\ell}(G_n, E_{n*}/I)^G$. $\square$

Theorem 7.18. *If $G <_c G_n$ and X is any finite spectrum of type n, then there are isomorphisms $(F_n \wedge X)^{hG} \cong F_n^{hG} \wedge X \cong E_n^{h'G} \wedge X \cong E_n^{hG} \wedge X$ in* $\text{Ho}(\mathbf{Spt})$.

Proof. We have $F_n^{hG} \wedge X \cong L_n(F_n^{hG}) \wedge X \cong L_{K(n)}(F_n^{hG}) \wedge X \cong E_n^{h'G} \wedge X$. $\square$

Corollary 7.19. *If $G <_c G_n$, then $(F_n \wedge M_I)^{hG} \cong \operatorname{colim}_i (F_n \wedge M_I)^{hU_iG}$ in the stable category.*

Proof. This follows from $(F_n \wedge M_I)^{hG} \cong \operatorname{colim}_i (E_n^{hU_iG} \wedge M_I) \cong \operatorname{colim}_i (F_n \wedge M_I)^{hU_iG}$. $\square$

Recall that if $G \lhd_c G_n$ does not have finite index, then $G_n/G \cong \varprojlim_i G_n/U_iG$ is profinite and uncountable.

Corollary 7.20. *If the normal subgroup G is closed in G_n, then $(F_n \wedge M_I)^{hG}$ is a discrete G_n/G-spectrum. Thus, E_n^{hG} is a continuous G_n/G-spectrum.*

Proof. Note that $(F_n \wedge M_I)^{hG} \simeq \operatorname{colim}_i (E_n^{hU_iG} \wedge M_I)$. Since U_iG is normal and has finite index in G_n, $E_n^{hU_iG} \wedge M_I$ is automatically a discrete G_n/U_iG-spectrum. The canonical continuous map $G_n/G \to G_n/U_iG$ makes $E_n^{hU_iG} \wedge M_I$ a discrete G_n/G-spectrum. Since the category of discrete G_n/G-spectra is equivalent to the category of sheaves of spectra on the site $G_n/G-\mathbf{Sets}_{df}$, it is easy to see that $\mathbf{Spt}_{G_n/G}$ is closed under colimits, so that $\operatorname{colim}_i (E_n^{hU_iG} \wedge M_I)$ is a discrete G_n/G-spectrum. $\square$

Example 7.21. At the prime 2, $G_1 = \mathbb{Z}_2 \times \mathbb{Z}/2$ and $E_1^{h\mathbb{Z}/2} \simeq KO_2^{\wedge}$ is a continuous $\mathbb{Z}_2$-spectrum [**19**, pg. 101].

CHAPTER 8

Auxiliary results for building the descent spectral sequence

This chapter collects together various results about continuous cohomology, towers of abelian groups, and inverse limits that are required for the work of later chapters.

Definition 8.1. Let $\mathrm{DMod}(G)^{\mathbb{N}}$ denote the category of diagrams in discrete G-modules of the form $\cdots \to M_2 \to M_1 \to M_0$. Define $H^s_{\mathrm{cont}}(G; \{M_i\})$, the continuous cohomology of G with coefficients in the tower $\{M_i\}$, to be the sth right derived functor of the left exact functor

$$\varprojlim_i (-)^G \colon \mathrm{DMod}(G)^{\mathbb{N}} \to \mathrm{Ab}, \quad \{M_i\} \mapsto \varprojlim_i M_i^G.$$

(This version of continuous cohomology is developed in [**24**].)

The next two results are special cases of statements in [**12**, Lem. 3.1.3] and its proof, for the site $G-\mathbf{Sets}_{df}$. Recall that $\lim^s{}_{\Delta\times\mathbb{N}}$ is the sth right derived functor of $\lim_{\Delta\times\mathbb{N}} \colon \mathrm{Ab}^{\Delta\times\mathbb{N}} \to \mathrm{Ab}$.

Lemma 8.2. *If* $I = \{I_i\}$ *is an injective object in* $\mathrm{DMod}(G)^{\mathbb{N}}$*, then, for* $s > 0$*,* $\lim^s{}_{\Delta\times\mathbb{N}} (\Gamma^\bullet_G I_i)^G = 0$.

Proof. Let (j, i) be a typical element of $\Delta \times \mathbb{N}$. The functors $\varprojlim_i \colon \mathrm{Ab}^{\mathbb{N}} \to \mathrm{Ab}$ and

$$\lim{}_{\Delta} \colon \mathrm{Ab}^{\Delta \times \mathbb{N}} \to \mathrm{Ab}^{\mathbb{N}}, \quad \{A_{j,i}\}_{j,i} \mapsto \{\lim{}_{\Delta} A_{j,i}\}_i$$

are left exact functors. Also, $\lim_{\Delta}$ is right adjoint to the diagonal functor $c \colon \mathrm{Ab}^{\mathbb{N}} \to \mathrm{Ab}^{\Delta \times \mathrm{N}}$ that sends $\{A_i\}$ in $\mathrm{Ab}^{\mathbb{N}}$ to the diagram with $A_{j,i} = A_i$, for every $j \in \Delta$. Since c is exact, $\lim_{\Delta}$ preserves injectives. Thus, there is a Grothendieck spectral sequence

$$E_2^{s,t} = \varprojlim{}^s{}_i \left(\lim{}^t{}_{\Delta} \, (\Gamma^{\bullet}_G I_i)^G\right) \Longrightarrow \lim{}^{s+t}{}_{\Delta \times \mathbb{N}} \, (\Gamma^{\bullet}_G I_i)^G.$$

Since I is an injective object, each I_i is injective in $\mathrm{DMod}(G)$, and every map $d_i \colon I_{i+1} \to I_i$ is split surjective, with section r_i [**24**, Prop. 1.1]. This implies that $d_i \colon I_{i+1}^G \to I_i^G$ is surjective for each i: if $m \in I_i^G$, then $m = d_i(r_i(m))$, and $g \cdot r_i(m) = r_i(g \cdot m) = r_i(m)$, for all $g \in G$, so that $r_i(m) \in I_{i+1}^G$. Thus, $\{I_i^G\}$ is a Mittag-Leffler tower of epimorphisms.

Then $\lim^t{}_{\Delta} \, (\Gamma^{\bullet}_G I_i)^G \cong H^t((\Gamma^{*}_G I_i)^G) \cong H^t_c(G; I_i)$, which equals 0 whenever $t > 0$, and is I_i^G when $t = 0$. Thus, for $t > 0$, $E_2^{s,t} = 0$, so that, for $s > 0$, $\lim^s{}_{\Delta \times \mathbb{N}} \, (\Gamma^{\bullet}_G I_i)^G \cong E_2^{s,0} = \varprojlim^s{}_i \, I_i^G = 0$. □

Theorem 8.3. *Let* $\{M_i\}$ *be in* $\mathrm{DMod}(G)^{\mathbb{N}}$. *Then for all* $s \geq 0$,

$$\lim{}^s{}_{\Delta \times \mathbb{N}} \, (\Gamma^{\bullet}_G M_i)^G \cong H^s_{\mathrm{cont}}(G; \{M_i\}).$$

Proof. By definition, $H^s_{\text{cont}}(G;\{M_i\}) \cong H^s\big(\varprojlim_i (-)^G(I^0 \to I^1 \to \cdots)\big)$, where $0 \to \{M_i\} \to I^0 \to I^1 \to \cdots$ is an injective resolution in $\text{DMod}(G)^{\mathbb{N}}$. If $\{N_i\}$ is in $\text{DMod}(G)^{\mathbb{N}}$, then

$$\lim{}_{\Delta\times\mathbb{N}} (\Gamma^\bullet_G N_i)^G \cong \varprojlim{}_i \lim{}_\Delta (\Gamma^\bullet_G N_i)^G \cong \varprojlim{}_i H^0((\Gamma^*_G N_i)^G)$$
$$\cong \varprojlim{}_i H^0_c(G;N_i) \cong \varprojlim{}_i N_i^G.$$

Thus, $H^s_{\text{cont}}(G;\{M_i\}) \cong H^s\big(\lim_{\Delta\times\mathbb{N}}\big((\Gamma^\bullet_G I^0)^G \to (\Gamma^\bullet_G I^1)^G \to \cdots\big)\big)$, which is associated to the sequence

$$0 \to (\Gamma^\bullet\{M_i\})^G \to (\Gamma^\bullet I^0)^G \to (\Gamma^\bullet I^1)^G \to \cdots, \tag{8.4}$$

in $\text{Ab}^{\Delta\times\mathbb{N}}$. Since $\text{Ab}^{\Delta\times\mathbb{N}}$ is an abelian category with enough injectives, and the functor $\lim_{\Delta\times\mathbb{N}}\colon \text{Ab}^{\Delta\times\mathbb{N}} \to \text{Ab}$ is left exact, if 8.4 is a $(\lim_{\Delta\times\mathbb{N}})$-acyclic resolution of $(\Gamma^\bullet\{M_i\})^G$, then $H^s_{\text{cont}}(G;\{M_i\}) \cong \lim^s{}_{\Delta\times\mathbb{N}} (\Gamma^\bullet M_i)^G$.

Fix any $([j],k) \in \Delta\times\mathbb{N}$ and consider the sequence

$$0 \to (\Gamma^{j+1}M_k)^G \to (\Gamma^{j+1}I^0_k)^G \to (\Gamma^{j+1}I^1_k)^G \to \cdots. \tag{8.5}$$

Since $0 \to M_k \to I^0_k \to I^1_k \to \cdots$ is exact in $\text{DMod}(G)$,

$$0 \to \Gamma^{j+1}M_k \to \Gamma^{j+1}I^0_k \to \Gamma^{j+1}I^1_k \to \cdots \tag{8.6}$$

is exact. Since $H_c^s(G;\Gamma^{j+1}I_k^m) = 0$, for any $m \geq 0$ and $s > 0$, 8.6 is a resolution of $\Gamma^{j+1}M_k$ by $(-)^G$-acyclics, and thus, $H^s\big((\Gamma^{j+1}I_k^0)^G \to (\Gamma^{j+1}I_k^1)^G \to \cdots\big) \cong H_c^s(G;\Gamma^{j+1}M_k) = 0$, for $s > 0$. Since $(-)^G$ is left exact, it follows that 8.5 is an exact sequence. Thus, 8.4 is an exact sequence in $\mathrm{Ab}^{\Delta\times\mathbb{N}}$.

The proof is finished by applying Lemma 8.2. $\square$

The following lemma is used, for example, with the exact functors $\Gamma_G^k\colon \mathrm{DMod}(G) \to \mathrm{DMod}(G)$, for any $k \geq 1$.

Lemma 8.7. *Let* $\cdots \xrightarrow{f_2} A_2 \xrightarrow{f_1} A_1 \xrightarrow{f_0} A_0$ *be a tower of abelian groups, and let* $F\colon \mathrm{Ab} \to \mathrm{Ab}$ *be an additive exact functor. If* $\{A_i\}$ *satisfies the Mittag-Leffler condition, then so does* $\{F(A_i)\}$.

Proof. For each $i \geq 0$, there exists some $j > i$, such that, for all $k \geq j$, $\mathrm{im}(f_k^i) = \mathrm{im}(f_j^i)$, where $f_k^i\colon A_k \to A_i$. (The definition of Mittag-Leffler provides for "$j \geq i$," but this implies our statement, and "$j > i$" simplifies the argument.) We use $\bar{f}_k^i$ to denote the obvious map $A_k \to \mathrm{im}(f_k^i)$. Since F is exact, there is an isomorphism $h\colon \mathrm{im}(F(f_k^i)) \to \mathrm{im}(F(f_j^i))$ given by the composite $\mathrm{im}(F(f_k^i)) \xrightarrow{\cong} F(\mathrm{im}(f_k^i)) = F(\mathrm{im}(f_j^i)) \xrightarrow{\cong} \mathrm{im}(F(f_j^i))$. If $f \in F(A_k)$, h is defined by $F(f_k^i)(f) \mapsto F(\bar{f}_k^i)(f) = F(\bar{f}_j^i)(f') \mapsto F(f_j^i)(f')$, where $F(\bar{f}_k^i)(f) = F(\bar{f}_j^i)(f')$ for some $f' \in F(A_j)$. Note that $\mathrm{im}(F(f_k^i)) \subset \mathrm{im}(F(f_j^i))$. Consider the commutative diagram

$$\begin{tikzcd} A_k \arrow[r, "f_k^j"'] \arrow[rr, bend left, "\bar{f}_k^i"] & A_j \arrow[r, "p"] \arrow[rr, bend right, "\bar{f}_j^i"'] & \operatorname{im}(f_k^i) \arrow[r, "i_1"] & \operatorname{im}(f_j^i) \arrow[r, "i_2"] & A_i, \end{tikzcd}$$

where i_1 and i_2 are inclusions, i_1 is the identity map in this case, and $i_2 i_1 p f_k^j = f_k^i$. Functoriality gives the commutative diagram

$$\begin{tikzcd} F(A_k) \arrow[r, "F(f_k^j)"'] \arrow[rr, bend left, "F(\bar{f}_k^i)"] & F(A_j) \arrow[r, "F(p)"] \arrow[rr, bend right, "F(\bar{f}_j^i)"'] & F(\operatorname{im}(f_k^i)) \arrow[r, "F(i_1)"] & F(\operatorname{im}(f_j^i)) \arrow[r, "F(i_2)"] & F(A_i), \end{tikzcd}$$

where $F(i_1)$ is the identity. Therefore,

$$h(F(f_k^i)(f)) = F(f_j^i)(f') = F(i_2)(F(\bar{f}_j^i)(f')) = F(i_2)(F(i_1)(F(\bar{f}_k^i)(f))) = F(f_k^i)(f).$$

Thus, h is the inclusion map and $\operatorname{im}(F(f_k^i)) = \operatorname{im}(F(f_j^i))$. This proves that $\{F(A_i)\}$ is Mittag-Leffler. $\square$

Lemma 8.8. *Let $M = \varprojlim_\alpha M_\alpha$ be an inverse limit of topological abelian groups. Let H be any profinite group. Then the canonical map $\kappa \colon \operatorname{Map}_c(H, M) \to \varprojlim_\alpha \operatorname{Map}_c(H, M_\alpha)$ is an isomorphism of groups.*

Proof. Inverse limits in Ab and in topological spaces are created in **Sets**: $\operatorname{Map}_c(H, M) = \operatorname{Top}(H, \varprojlim_\alpha M_\alpha) \cong \varprojlim_{\mathbf{Sets}\,\alpha} \operatorname{Top}(H, M_\alpha) \cong \varprojlim_{Ab\,\alpha} \operatorname{Map}_c(H, M_\alpha)$. $\square$

Lemma 8.9. *If X is a finite spectrum, then for any $G <_c G_n$ and any integer t, the abelian group $\pi_t(E_n^{hG} \wedge M_I \wedge X)$ is finite. In particular, $\pi_t(E_n \wedge M_I \wedge X)$ is finite.*

Proof. We use the fact that $\pi_t(E_n^{hG} \wedge M_I)$ is finite, and for convenience, we work in the stable homotopy category of CW-spectra. Since X is a finite spectrum, there exists some m such that $X_n = \Sigma^{n-m} X_m$ whenever $n \geq m$, and X_m is a finite complex. Since $\pi_t(E_n \wedge M_I \wedge X) \cong \pi_{t-m}(E_n \wedge M_I \wedge X_m)$ and X_m can be built out of a finite number of cofiber sequences, the result follows. □

Corollary 8.10 ([**19**, pg. 116]). *If X is a finite spectrum, then $\pi_t(E_n \wedge X) \cong \varprojlim_I \pi_t(E_n \wedge M_I \wedge X)$.*

Proof. Since $\pi_t(E_n \wedge X) \cong \pi_t((\operatorname{holim}_I E_n \wedge M_I) \wedge X) \cong \pi_t(\operatorname{holim}_I (E_n \wedge M_I \wedge X))$, the result follows from the preceding lemma. □

We need to be precise about the definition of continuous cohomology that we use. We let $H^s_{\mathrm{cts}}(G; M)$ denote the cohomology of continuous cochains for a profinite group G with coefficients in the topological G-module M, in the sense of [**47**, §2]. If M is a discrete G-module, this is the usual continuous (Galois) cohomology $H^s_c(G; M)$ of [**43**, §2.2]. Also, for an inverse system of discrete G-modules, there is the following result.

Theorem 8.11 ([**24**, (2.1), Thm. 2.2]). *Let $\{M_n\}_{n \geq 0}$ be an inverse system of discrete G-modules satisfying the Mittag-Leffler condition and let $M = \varprojlim_n M_n$ as a topological*

G-module. Then, for each $s \geq 0$, there is a short exact sequence

$$0 \to \varprojlim^1{}_n\, H_c^{s-1}(G; M_n) \to H^s_{\mathrm{cts}}(G; M) \to \varprojlim{}_n\, H_c^s(G; M_n) \to 0,$$

where $H_c^{-1}(G; -) = 0$.

Corollary 8.12. *If the profinite group G is a compact p-adic analytic group and $\{M_n\}_{n \geq 0}$ is an inverse system of finite discrete $\mathbb{Z}_p[\![G]\!]$-modules, with $M = \varprojlim_n M_n$, a topological G-module, then $H^s_{\mathrm{cts}}(G; M) \cong \varprojlim_n H_c^s(G; M_n)$, for $s \geq 0$.*

Proof. When $s = 0$, the statement is immediate. For $s \geq 1$, it suffices to prove that for $t \geq 0$, $H_c^t(G, N)$ is a finite abelian group whenever N is a finite discrete $\mathbb{Z}_p[\![G]\!]$-module.

Since G is a compact p-adic analytic group, [**46**, Thm. 5.1.2] implies that the trivial $\mathbb{Z}_p[\![G]\!]$-module $\mathbb{Z}_p$ (that is, $\mathbb{Z}_p$ has trivial action by G) has a resolution by free objects of the form $\cdots \to P_2 \to P_1 \to P_0 \to \mathbb{Z}_p \to 0$, in the category $\mathfrak{C}_p(G)$, whose objects are topological $\mathbb{Z}_p[\![G]\!]$-modules that are inverse limits of finite discrete left $\mathbb{Z}_p[\![G]\!]$-modules, and whose morphisms are continuous $\mathbb{Z}_p[\![G]\!]$-module homomorphisms. Furthermore, we can assume that each P_i is topologically finitely generated as a $\mathbb{Z}_p[\![G]\!]$-module. Then [**46**, (3.2.4)] implies that

$$H_c^s(G; N) = \mathbf{Ext}^s_{\mathbb{Z}_p[\![G]\!]}(\mathbb{Z}_p, N) = H^s\big(\mathrm{Hom}^c_{\mathbb{Z}_p[\![G]\!]}(P_0, N) \to \mathrm{Hom}^c_{\mathbb{Z}_p[\![G]\!]}(P_1, N) \to ...\big),$$

where $\mathrm{Hom}^c_{\mathbb{Z}_p[\![G]\!]}(-, -)$ denotes continuous $\mathbb{Z}_p[\![G]\!]$-module homomorphisms.

Let P_i be topologically generated by the finite subset $\{p_{i1}, ..., p_{ik}\}$. This set generates the submodule $\bigoplus_j \mathbb{Z}_p[\![G]\!] \cdot p_{ij}$. Since this submodule is closed in P_i, by [**50**, Lem. 7.2.2], $P_i = \bigoplus_j \mathbb{Z}_p[\![G]\!] \cdot p_{ij}$. Thus, each P_i is isomorphic to a finite number of copies of $\mathbb{Z}_p[\![G]\!]$. Then any $\mathbb{Z}_p[\![G]\!]$-module homomorphism from P_i to N is determined by a finite number of values and since N is finite, $\mathrm{Hom}^c_{\mathbb{Z}_p[\![G]\!]}(P_i, N)$ is a finite abelian group, so that $H^s_c(G; N)$ is itself finite. □

Corollary 8.13. *If G is closed in G_n and X is a finite spectrum, then*

$$H^s_{\mathrm{cts}}(G; \pi_t(E_n \wedge X)) \cong \varprojlim_k H^s_c(G; E_{n,t}(X)/I^k_n), \quad s \geq 0,$$

where I^k_n is the submodule $I^k_n E_{n,t}(X)$.

Proof. Note that G is a compact p-adic analytic group, and since X is a finite spectrum, $E_{n,t}(X) \cong \varprojlim_k E_{n,t}(X)/I^k_n$ [**5**, pg. 767]. Also, $\pi_t(E_n \wedge X) \cong \varprojlim_I \pi_t(E_n \wedge M_I \wedge X)$, and the profinite and I_n-adic topologies on $\pi_t(E_n \wedge X)$ are identical [**23**, pf. of Prop. 11.9]. Since I^k_n is an open subgroup, $E_{n,t}(X)/I^k_n$ is a finite discrete G-module.

By the previous result, our proof is finished by showing that $E_{n,t}(X)/I^k_n$ is a discrete $\mathbb{Z}_p[\![G]\!]$-module. By [**4**, Def. 5.8], $E_{n,t}(X)/I^k_n$ is a discrete twisted $E_{n,0}$-G_n-module such that $g \cdot (rm) = (g \cdot r)(g \cdot m)$, for $g \in G_n, r \in E_{n,0}$, and $m \in E_{n,t}(X)/I^k_n$. Since $\mathbb{Z}_p \subset E_{n,0}$, if the G_n-action on $\mathbb{Z}_p$ is trivial, then $E_{n,t}(X)/I^k_n$ is a discrete $\mathbb{Z}_p[\![G]\!]$-module [**40**, Prop. 5.3.6(d)].

Gal acts trivially on $\mathbb{Z}_p$, and since S_n acts on $E_{n,t}(X)$ by $W(\mathbb{F}_{p^n})$-module homomorphisms, the S_n-action on $\mathbb{Z}_p \subset W(\mathbb{F}_{p^n})$ is trivial. Thus, G_n acts trivially on $\mathbb{Z}_p$, and $E_{n,t}(X)/I_n^k$ is a finite discrete $\mathbb{Z}_p[\![G]\!]$-module. $\square$

Remark 8.14. This result shows that $H^s_{\mathrm{cts}}(G; \pi_t(E_n \wedge X))$ agrees with the definition of continuous cohomology used in [**9**, Remark 0.3].

The next result extends the conclusion of Remark 6.18 to pro-discrete G-modules, and is useful in the proof of Theorem 10.6.

Theorem 8.15. *If $M = \varprojlim_\alpha M_\alpha$ is, as a topological G-module, an inverse limit of discrete G-modules, then $H^s_{\mathrm{cts}}(G; M) \cong \pi^s\big((\Gamma^\bullet_G M)^G\big)$, for $s \geq 0$.*

Proof. We have:

$$H^s_{\mathrm{cts}}(G; M) = H^s(\varprojlim M_\alpha \to \mathrm{Map}_c(G, \varprojlim M_\alpha) \to \mathrm{Map}_c(G^2, \varprojlim M_\alpha) \to \cdots)$$

$$\cong H^s(\varprojlim (M_\alpha \to \mathrm{Map}_c(G, M_\alpha) \to \mathrm{Map}_c(G^2, M_\alpha) \to \cdots))$$

$$\cong H^s\big(\varprojlim (\Gamma^* M_\alpha)^G\big) \cong \pi^s\big(\varprojlim (\Gamma^\bullet M_\alpha)^G\big) \cong \pi^s\big((\Gamma^\bullet M)^G\big).$$

$\square$

The proof of the following result closely follows [**37**, pp. 30-1].

Theorem 8.16. *Let K be a closed subgroup of a profinite group H. Let $A = \varprojlim_\alpha A_\alpha$, as a topological abelian group, be an inverse limit of discrete abelian groups. Then $H^s_{\mathrm{cts}}(K; \mathrm{Map}_c(H, A)) = 0$, for $s > 0$.*

Proof. The space H/K is profinite and there is a K-equivariant homeomorphism $H \to K \times H/K$, where $K \times H/K$ has the diagonal action. Thus, there is an isomorphism $\mathrm{Map}_c(H, A) \cong \mathrm{Map}_c(K, \mathrm{Map}_c(H/K, A))$ of topological K-modules.

As described in [**37**, pg. 106], for any topological K-module M, $H^*_{\mathrm{cts}}(K; M)$ can be computed by taking the cohomology of the cochain complex obtained by taking the K-fixed points of the acyclic complex $X(K, M)$ given by

$$\mathrm{Map}_c(K, M) \xrightarrow{\partial^1} \mathrm{Map}_c(K^2, M) \xrightarrow{\partial^2} \mathrm{Map}_c(K^3, M) \xrightarrow{\partial^3} \cdots,$$

where $\mathrm{Map}_c(K^n, M)$ has a K-action given by $(k \cdot f)(k_1, ..., k_n) = k \cdot f(k^{-1}k_1, ..., k^{-1}k_n)$, and ∂^n is defined by $\partial^n(f)(k_1, ..., k_{n+1}) = \sum_{i=1}^{n+1} (-1)^{i+1} f(k_1, ..., \hat{k_i}, ..., k_{n+1})$, where $\hat{k_i}$ means that the ith entry is omitted. Note that this acyclic complex is defined when M is just a topological abelian group.

Let $M \cong \varprojlim_\beta M_\beta$ be a topological abelian group, where each M_β is a discrete abelian group. Consider the map $\delta \colon \mathrm{Map}_c(K^{n+1}, \mathrm{Map}_c(K, M))^K \to \mathrm{Map}_c(K^{n+1}, M)$, defined by $f \mapsto [(k_1, ..., k_{n+1}) \overset{\hat{f}}{\mapsto} f(k_1, ..., k_{n+1})(1)]$, where $\hat{f}$ is continuous, since δ is induced by the similarly-defined map $\mathrm{Map}_c(K^{n+1}, \mathrm{Map}_c(K, M_\beta))^K \to \mathrm{Map}_c(K^{n+1}, M_\beta)$. Because δ is an isomorphism that commutes with ∂^{n+1}, there is an isomorphism of cochain complexes

$X(K, \mathrm{Map}_c(K, M))^K \cong X(K, M)$. Thus,

$$H^s_{\mathrm{cts}}(K; \mathrm{Map}_c(H, A)) \cong H^s\big(X(K, \mathrm{Map}_c(K, \mathrm{Map}_c(H/K, A)))^K\big)$$
$$\cong H^s\big(X(K, \mathrm{Map}_c(H/K, A))\big) = 0,$$

for $s > 0$, since $X(K, \mathrm{Map}_c(H/K, A))$ is acyclic. □

The following useful result is Proposition 2.2 in [**23**].

Theorem 8.17. *Given a spectrum Z, define $\Lambda(Z)$ to be the filtered category of pairs (Y, u), where Y is a finite spectrum and $u \colon Y \to Z$. The morphisms are the obvious commutative triangles and all maps are in the stable category. Then, for any integer t, the canonical map $E_n^t(Z) \to \varprojlim_{(Y,u)\in\Lambda(Z)} E_n^t(Y)$ is an isomorphism of abelian groups and a homeomorphism of profinite spaces.*

In Lemma 7.12, we showed that when certain $\varprojlim^1$ terms vanish, smashing with a finite spectrum commutes with holim for a tower of spectra. As mentioned in Remark 7.13, we now verify that the appropriate $\varprojlim^1$ terms vanish for the cases where Lemma 7.12 is applied in Corollary 8.10 and Chapter 10. All of these cases follow from the lemma below, by: setting G and/or K equal to $\{e\}$, letting $X = S^0$, using the weak equivalence $E_n^{hG} \wedge M_I \simeq (\mathrm{colim}_i\, E_n^{hU_iG}) \wedge M_I$, and/or by applying the fact that the Spanier-Whitehead dual of a finite spectrum is again finite.

Lemma 8.18. *Let G and K be closed subgroups of G_n, and let X be a finite spectrum. When $j \geq 0$ and m is equal to 0 or 1, then the tower of abelian groups*

$$\{\pi_t((\operatorname{colim}_N \prod_{G^j/N} X) \wedge E_n^{hK} \wedge E_n^m \wedge M_I)\}_I$$

is Mittag-Leffler for every integer t, where we write $G^j \cong \varprojlim_{N \lhd_o G^j} G^j/N$. We let $G^0 = \{e\}$ and $E_n^0 = S^0$.

Proof. Observe that $\pi_t((\operatorname{colim}_N \prod_{G^j/N} X) \wedge E_n^{hK} \wedge E_n^m \wedge M_I)$ is isomorphic to the expression $\operatorname{colim}_N \prod_{G^j/N} \pi_t(E_n^{hK} \wedge M_I \wedge E_n^m \wedge X)$. Since filtered colimits and taking finite products are exact functors, it is enough to show that $\{\pi_t(E_n^{hK} \wedge M_I \wedge E_n^m \wedge X)\}$ is Mittag-Leffler. Since $\pi_t(E_n^{hK} \wedge M_I \wedge E_n^m \wedge X) \cong \operatorname{colim}_i \pi_t(E_n^{hU_iK} \wedge E_n^m \wedge M_I \wedge X)$, we only need to show that $\{\pi_t(E_n^{hU_iK} \wedge E_n^m \wedge M_I \wedge X)\}$ is Mittag-Leffler. If $m = 0$, then we are done, since $\pi_t(E_n^{hU_iK} \wedge M_I \wedge X)$ is a finite abelian group. When $m = 1$, $\pi_t(E_n^{hU_iK} \wedge E_n \wedge M_I \wedge X) \cong \pi_t(L_{K(n)}(E_n^{hU_iK} \wedge E_n) \wedge M_I \wedge X)$. By [**9**, Cor. 4.5], $L_{K(n)}(E_n^{hU_iK} \wedge E_n) \simeq \prod_{G_n/U_iK} E_n$. Thus, $\pi_t(L_{K(n)}(E_n^{hU_iK} \wedge E_n) \wedge M_I \wedge X) \cong \prod_{G_n/U_iK} \pi_t(E_n \wedge M_I \wedge X)$, which gives a Mittag-Leffler tower since $\pi_t(E_n \wedge M_I \wedge X)$ is a finite abelian group. $\square$

CHAPTER 9

Homotopy fixed points for $L_{K(n)}(E_n \wedge X)$ and its descent spectral sequence

Let $\{Z_i\}$ be a tower of discrete G-spectra, where G has finite virtual cohomological dimension. By Lemma 4.4.5, if each Z_i is fibrant in $\mathbf{Spt}_G$ or if $\{\pi_t(Z_i)\}$ is Mittag-Leffler for every t, then $Z = \operatorname{holim}_i Z_i$ is a continuous G-spectrum. In this chapter, we construct a descent spectral sequence for such continuous G-spectra (e.g. $\operatorname{holim}_I (F_n \wedge M_I \wedge X)_{f,G}$), and use it to obtain the descent spectral sequence for the homotopy fixed points of $L_{K(n)}(E_n \wedge X)$, when X satisfies a particular finiteness condition.

9.1. Towers of spectra and homotopy fixed points

Theorem 9.1.1. *Let G be a profinite group and let $\{Z_i\}$ be any tower in* $\mathbf{tow}(\mathbf{Spt}_G)$. *Then there is a conditionally convergent spectral sequence*

$$E_2^{s,t} \cong H^s_{\mathrm{cont}}(G; \{\pi_t(Z_i)\}) \Longrightarrow \pi_{t-s}\big(\operatorname{holim}_i \operatorname{holim}_\Delta (\Gamma^\bullet_G((Z_i)_{f,G}))^G\big). \tag{9.1.2}$$

Remark 9.1.3. This theorem is a special case of [**12**, Prop. 3.1.2] for $G-\mathbf{Sets}_{df}$. Also related to 9.1.2 is the ℓ-adic descent spectral sequence of algebraic K-theory ([**48**], [**34**, pg. 266]). For a "nice scheme" U, the K-theory presheaf K, and $\mathbb{H}^\bullet_{\text{ét}}(U;K)^\wedge_\ell =$

$\operatorname{holim}_n \mathbb{H}^{\bullet}_{\text{ét}}(U;K) \wedge M(\ell^n)$, this spectral sequence has the form

$$H^s_{\text{ét}}(U;\mathbb{Z}_\ell(t/2)) \Longrightarrow \pi_{t-s}\big(\mathbb{H}^{\bullet}_{\text{ét}}(U;K)^{\wedge}_{\ell}\big).$$

Proof. Note that $\operatorname{holim}_i \operatorname{holim}_\Delta (\Gamma^{\bullet}_G(Z_i)_f)^G \cong \operatorname{holim}_{\Delta\times\mathbb{N}} (\Gamma^{\bullet}_G(Z_i)_f)^G$, and $(\Gamma^{\bullet}_G(Z_i)_f)^G$ is a diagram of fibrant spectra. Then by Remark 6.9, there is a conditionally convergent spectral sequence

$$E_2^{s,t} = \lim{}^s{}_{\Delta\times\mathbb{N}}\, \pi_t\big((\Gamma^{\bullet}_G(Z_i)_f)^G\big) \Longrightarrow \pi_{t-s}(\operatorname{holim}_i \operatorname{holim}_\Delta (\Gamma^{\bullet}_G(Z_i)_f)^G).$$

We only need to identify the E_2-term:

$$E_2^{s,t} = \lim{}^s{}_{\Delta\times\mathbb{N}}\, \pi_t\big((\Gamma^{\bullet}_G(Z_i)_f)^G\big) \cong \lim{}^s{}_{\Delta\times\mathbb{N}}\, (\Gamma^{\bullet}_G\pi_t(Z_i))^G \cong H^s_{\text{cont}}(G;\{\pi_t(Z_i)\}),$$

by Lemma 8.3. □

Now we focus on two types of towers that give continuous G-spectra.

Definition 9.1.4. Let $\{Z_i\}$ be in $\mathbf{tow}(\mathbf{Spt}_G)$. Suppose that one of the following conditions is satisfied: (a) Z_i is fibrant in $\mathbf{Spt}_G$ for every i; or (b) $\{\pi_t(Z_i)\}$ is Mittag-Leffler for every integer t. Then we let $Z = \operatorname{holim}_i Z_i$ be the continuous G-spectrum associated to the tower. Also, we define the homotopy fixed point spectrum $Z^{hG} = \operatorname{holim}_i Z_i^{hG}$.

The following result shows that when G is a finite group and the tower $\{Z_i\}$ satisfies one of the conditions given in the definition, Z^{hG} agrees with the usual homotopy fixed points $Z^{h'G}$ (see Remark 6.3), so that Definition 9.1.4 generalizes the notion of homotopy fixed points for a particular kind of continuous G-spectrum.

Theorem 9.1.5. *Let $\{Z_i\}$ be as in Definition 9.1.4. Suppose that G is a finite group. Then the canonical map $Z^{hG} \to Z^{h'G}$ is a weak equivalence.*

Proof. Note that if each Z_i is fibrant in $\mathbf{Spt}_G$, we can let $(Z_i)_f = Z_i$. The map is defined in the following way:

$$Z^{hG} = \operatorname{holim}_i \lim_G (Z_i)_f \to \operatorname{holim}_i \operatorname{holim}_G (Z_i)_f \cong \operatorname{holim}_G (\operatorname{holim}_i (Z_i)_f) = Z^{h'G}.$$

By hypothesis, $Z = \operatorname{holim}_i Z_i \to \operatorname{holim}_i (Z_i)_f$ is a weak equivalence and a G-equivariant map, and $\operatorname{holim}_i (Z_i)_f$ is fibrant in $\mathbf{Spt}$. Thus, $Z^{h'G}$ is indeed the target of the above map. For a fixed i, the map $\operatorname{Hom}_G(*, (Z_i)_f) \cong \lim_G (Z_i)_f \to \operatorname{holim}_G (Z_i)_f$ is a weak equivalence between fibrant spectra, so that the above map is a weak equivalence. $\square$

By applying Remark 6.16 and Theorem 9.1.1, we obtain the following.

Corollary 9.1.6. *Let G be a profinite group with $\operatorname{vcd}(G) < \infty$, and let $\{Z_i\}$ be an object in $\mathbf{tow}(\mathbf{Spt}_G)$ that satisfies one of the conditions in Definition 9.1.4. Let $Z = \operatorname{holim}_i Z_i$ be a continuous G-spectrum and let $Z^{hG} = \operatorname{holim}_i Z_i^{hG}$ be its homotopy* *fixed point spectrum. Then there is a conditionally convergent descent spectral sequence*

(9.1.7) $$H^s_{\mathrm{cont}}(G; \{\pi_t(Z_i)\}) \Longrightarrow \pi_{t-s}(Z^{hG}).$$

Remark 9.1.8. As in Remark 7.2, there is a weak equivalence between the two possible interpretations of Z^{hG}: $\operatorname{holim}_i ((Z_i)_f)^G \xrightarrow{\simeq} \operatorname{holim}_i \operatorname{holim}_\Delta (\Gamma^{\cdot}_G (Z_i)_f)^G$.

In case (b) of Definition 9.1.4, we can simplify the E_2-term of spectral sequences 9.1.2 and 9.1.7.

Theorem 9.1.9. *Let $\{Z_i\}$ be a tower in* $\mathbf{tow}(\mathbf{Spt}_G)$ *such that $\{\pi_t(Z_i)\}$ is Mittag-Leffler for every integer t, so that $Z = \operatorname{holim}_i Z_i$ is a continuous G-spectrum. Then in spectral sequences 9.1.2 and 9.1.7, $E_2^{s,t} \cong H^s_{\mathrm{cts}}(G; \pi_t(Z))$, the cohomology of continuous cochains.*

Proof. By [**24**, Thm. 2.2], $H^s_{\mathrm{cont}}(G; \{\pi_t(Z_i)\}) \cong H^s_{\mathrm{cts}}(G; \varprojlim_i \pi_t(Z_i)) \cong H^s_{\mathrm{cts}}(G; \pi_t(Z))$. □

Because of the finite vcd hypothesis in Corollary 9.1.6, spectral sequence 9.1.7 has abutment $\pi_*(\operatorname{holim}_i Z_i^{hG})$. We point out that the functor

$$\operatorname{holim}_i (-)^{hG} \colon \mathrm{Ho}(\mathbf{tow}(\mathbf{Spt}_G)) \to \mathrm{Ho}(\mathbf{Spt}), \quad \{Z_i\} \mapsto \operatorname{holim}_i Z_i^{hG}$$

is the total right derived functor $\mathbf{R}\big(\varprojlim_i (-)^G\big)$, in the following sense, generalizing the homotopy fixed points $X^{hG} = (\mathbf{R}(-)^G)(X)$, for a discrete G-spectrum X.

Lemma 9.1.10. *Let* $\{Z_i\}$ *be a tower of discrete* G*-spectra. Let* $\{(Z_i)_f\} \to \{(Z_i)'_f\}$ *be a trivial cofibration with* $\{(Z_i)'_f\}$ *fibrant in* $\mathbf{tow}(\mathbf{Spt}_G)$*. Then*

$$\operatorname{holim}_i ((Z_i)_f)^G \xrightarrow{\simeq} \operatorname{holim}_i ((Z_i)'_f)^G \xleftarrow{\simeq} \varprojlim_i ((Z_i)'_f)^G = \mathbf{R}\big(\varprojlim_i (-)^G\big)(\{Z_i\}),$$

where $\operatorname{holim}_i Z_i^{hG} = \operatorname{holim}_i ((Z_i)_f)^G$.

Proof. The first weak equivalence follows from the fact that for each i, $((Z_i)_f)^G \to ((Z_i)'_f)^G$ is a weak equivalence between fibrant spectra. The second weak equivalence is verified as in the proof of 4.4.5, since $\{((Z_i)'_f)^G\}$ is a tower in $\mathbf{Spt}$ of fibrations between fibrant spectra. Finally, the equality comes from the fact that the composition $\{Z_i\} \to \{(Z_i)'_f\}$ is a trivial cofibration in $\mathbf{tow}(\mathbf{Spt}_G)$. $\square$

Remark 9.1.11. By the above observation, we can rewrite spectral sequence 9.1.7 in a more conceptual way:

$$R^s\big(\varprojlim_i (-)^G\big)\{\pi_t(Z_i)\} \Longrightarrow \pi_{t-s}\big(\mathbf{R}\big(\varprojlim_i (-)^G\big)(\{Z_i\})\big).$$

9.2. The continuous G_n-spectrum $L_{K(n)}(E_n \wedge X)$

We apply the results of the previous section to continuous G-spectra associated with E_n, where G is a closed subgroup of G_n. Corollary 9.1.6 immediately gives

Theorem 9.2.1. *Let* G *be a closed subgroup of* G_n *and let* X *be an arbitrary spectrum with trivial* G*-action. Then* $\operatorname{holim}_I (F_n \wedge M_I \wedge X)_f$ *is a continuous* G*-spectrum and there* *is a conditionally convergent descent spectral sequence*

$$H^s_{\mathrm{cont}}(G; \{\pi_t(E_n \wedge M_I \wedge X)\}) \Longrightarrow \pi_{t-s}(\operatorname{holim}_I (F_n \wedge M_I \wedge X)^{hG}).$$

Remark 9.2.2. For arbitrary X, we do not know that $\operatorname{holim}_I (F_n \wedge M_I \wedge X)$ is a continuous G_n-spectrum, so that, in general, $\operatorname{holim}_I (F_n \wedge M_I \wedge X)^{hG}$ is the homotopy fixed point spectrum $\big(\operatorname{holim}_I (F_n \wedge M_I \wedge X)_f\big)^{hG}$ and *not* $\big(\operatorname{holim}_I (F_n \wedge M_I \wedge X)\big)^{hG}$.

Let X be a spectrum with trivial G-action. When X satisfies a certain condition, we define the homotopy fixed point spectrum $L_{K(n)}(E_n \wedge X)^{hG}$ and construct its descent spectral sequence.

Definition 9.2.3. If the tower $\{\pi_t(E_n \wedge M_I \wedge X)\}_I$ is Mittag-Leffler for all $t \in \mathbb{Z}$, then we say that the spectrum X is E_n-Mittag-Leffler.

Remark 9.2.4. An arbitrary finite spectrum X is E_n-Mittag-Leffler, since the tower $\{\pi_t(E_n \wedge M_I \wedge X)\}_I$ consists of finite abelian groups, by Lemma 8.9. However, an E_n-Mittag-Leffler spectrum need not be finite. For example, for $j \geq 1$, let $X = E_n^j$. Then

$$\pi_t(E_n \wedge M_I \wedge X) \cong \pi_t(E_n^{j+1} \wedge M_I) \cong \pi_t(L_{K(n)}(E_n^{j+1}) \wedge M_I) \cong \mathrm{Map}^{\ell}_c(G_n^j, \pi_t(E_n)/I),$$

by [**9**, pg. 10]. Since $\{\pi_t(E_n)/I\}_I$ is Mittag-Leffler, 8.7 implies that E_n^j is E_n-Mittag-Leffler.

Let X be E_n-Mittag-Leffler. Then there is an isomorphism $L_{K(n)}(E_n \wedge X) \cong \operatorname{holim}_I (E_n \wedge M_I \wedge X)$ in Ho(**Spt**), and a weak equivalence $\operatorname{holim}_I (F_n \wedge M_I \wedge X) \to \operatorname{holim}_I (E_n \wedge M_I \wedge X)$, since $\pi_t(\operatorname{holim}_I (E_n \wedge M_I \wedge X)) \cong \varprojlim_I \pi_t(E_n \wedge M_I \wedge X) \cong \pi_t(\operatorname{holim}_I (F_n \wedge M_I \wedge X))$. Similarly, if X is a finite spectrum, $E_n \wedge X \simeq L_{K(n)}(E_n \wedge X)$, by 8.10. Since $F_n \wedge M_I \wedge X$ is a discrete G_n-spectrum, we have

Theorem 9.2.5. *If X is an E_n-Mittag-Leffler spectrum with trivial G-action, then $L_{K(n)}(E_n \wedge X) \cong \operatorname{holim}_I (F_n \wedge M_I \wedge X)$ is a continuous G-spectrum. If X is a finite spectrum, then $E_n \wedge X$ is a continuous G-spectrum.*

Definition 9.2.6. If X is an E_n-Mittag-Leffler spectrum, then

$$\big(L_{K(n)}(E_n \wedge X)\big)^{hG} = \operatorname{holim}_I ((F_n \wedge M_I)_f \wedge X)^{hG}.$$

We write $L_{K(n)}(E_n \wedge X)^{hG}$, in place of $\big(L_{K(n)}(E_n \wedge X)\big)^{hG}$, to avoid excessive parentheses. If X is a finite spectrum, then we can write

$$(E_n \wedge X)^{hG} = \operatorname{holim}_I ((F_n \wedge M_I)_f \wedge X)^{hG},$$

in place of $L_{K(n)}(E_n \wedge X)^{hG}$. As usual, $F_n \wedge M_I \to (F_n \wedge M_I)_f$ is a trivial cofibration with $(F_n \wedge M_I)_f$ fibrant in $\mathbf{Spt}_G$.

Remark 9.2.7. We explain why we use $((F_n \wedge M_I)_f \wedge X)^{hG}$, in the above definition, instead of $(F_n \wedge M_I \wedge X)^{hG}$. We want to be able to define the map $E_n^{h'G} \wedge X \to (E_n \wedge X)^{h'G}$.

If we use $(F_n \wedge M_I \wedge X)^{hG}$, then this map requires a map of diagrams

$$\{(F_n \wedge M_I)_f \wedge X\}_I \longrightarrow \{(F_n \wedge M_I \wedge X)_f\}_I.$$

This requires, for example, a commutative diagram

$$\begin{array}{ccc} (F_n \wedge M_I)_f \wedge X & \longrightarrow & (F_n \wedge M_I \wedge X)_f \\ \downarrow & & \downarrow \\ (F_n \wedge M_J)_f \wedge X & \longrightarrow & (F_n \wedge M_J \wedge X)_f. \end{array}$$

However, there is no natural map $(F_n \wedge M_I)_f \wedge X \to (F_n \wedge M_I \wedge X)_f$, and thus, no such commutative diagram. Thus, we cannot use $(F_n \wedge M_I \wedge X)^{hG}$ above *and* have the desired map. But since $(F_n \wedge M_I \wedge X)^{hG} \to ((F_n \wedge M_I)_f \wedge X)^{hG}$ is a weak equivalence between fibrant spectra, the two definitions are equivalent.

Note that by definition, $L_{K(n)}(E_n \wedge X)^{hG} \cong \operatorname{holim}_\Delta \operatorname{holim}_I (\Gamma^\bullet_G((F_n \wedge M_I)_f \wedge X)_f)^G$.

Theorem 9.2.8. *Let G be a closed subgroup of G_n. If X is an E_n-Mittag-Leffler spectrum with trivial G-action, then there is a conditionally convergent descent spectral sequence*

$$H^s_{\text{cts}}(G; \pi_t(L_{K(n)}(E_n \wedge X))) \Longrightarrow \pi_{t-s}\big(L_{K(n)}(E_n \wedge X)^{hG}\big). \tag{9.2.9}$$

In particular, if X is a finite spectrum, then there is a descent spectral sequence

$$H^s_c(G; \pi_t(E_n \wedge X)) \Longrightarrow \pi_{t-s}((E_n \wedge X)^{hG}). \tag{9.2.10}$$

Proof. This follows from Corollary 9.1.6 and Theorem 9.1.9 by considering the tower $\{(F_n \wedge M_I)_f \wedge X\}_I$. The G-equivariant isomorphism $\pi_t((F_n \wedge M_I)_f \wedge X) \cong \pi_t(F_n \wedge M_I \wedge X)$ gives the desired E_2-term. When X is finite, 8.13 showed that $H^s_{\text{cts}}(G; \pi_t(E_n \wedge X)) \cong H^s_c(G; \pi_t(E_n \wedge X))$. $\square$

The above descent spectral sequences for $L_{K(n)}(E_n \wedge X)^{hG}$ and $(E_n \wedge X)^{hG}$ can be obtained in a way that is slightly different from the above. This second method of derivation of the spectral sequence is required for Theorem 10.6, because of the spectral sequence comparison argument used in its proof. Thus, we give the needed alternative proof.

Proof of Theorem 9.2.8. Consider the cosimplicial fibrant spectrum

$$C^\bullet_X = \text{holim}_I \, (\Gamma^\bullet_G((F_n \wedge M_I)_f \wedge X)_f)^G.$$

Let $Z_I = (F_n \wedge M_I)_f \wedge X$ and note that $L_{K(n)}(E_n \wedge X)^{hG} \cong \text{holim}_\Delta \, \text{holim}_I \, (\Gamma^\bullet_G(Z_I)_f)^G$. Then spectral sequence 6.10 takes the form

$$E_2^{s,t} = \pi^s \pi_t\big(\text{holim}_I \, (\Gamma^\bullet_G(Z_I)_f)^G\big) \Longrightarrow \pi_{t-s}\big(L_{K(n)}(E_n \wedge X)^{hG}\big).$$

Note that, for $k \geq 0$, $\varprojlim^1{}_I \operatorname{Map}_c(G^k, \pi_{t+1}(Z_I)) = 0$, since $\{\pi_{t+1}(Z_I)\}_I$ is Mittag-Leffler and $\operatorname{Map}_c(G^k, -)$ is an exact functor. Therefore,

$$\pi_t(\operatorname{holim}_I \operatorname{Map}_c(G^k, (Z_I)_f)) \cong \varprojlim{}_I \operatorname{Map}_c(G^k, \pi_t(Z_I)).$$

Thus, $\pi_t\big(\operatorname{holim}_I (\operatorname{Map}_c(G^{k+1}, (Z_I)_f))^G\big) \cong \varprojlim{}_I \operatorname{Map}_c(G^{k+1}, \pi_t(Z_I))^G$, and

$$\pi_t\big(\operatorname{holim}_I (\Gamma^\bullet (Z_I)_f)^G\big) \cong \varprojlim{}_I (\Gamma^\bullet \pi_t(Z_I))^G.$$

Now we identify the E_2 term:

$$\begin{aligned} E_2^{s,t} &\cong \pi^s\big(\varprojlim{}_I (\Gamma^\bullet \pi_t(Z_I))^G\big) \cong \pi^s\big(\lim_G \varprojlim{}_I \Gamma^\bullet \pi_t(Z_I)\big) \cong \pi^s\big((\Gamma^\bullet (\varprojlim{}_I \pi_t(Z_I)))^G\big) \\ &\cong \pi^s((\Gamma^\bullet \pi_t(L_{K(n)}(E_n \wedge X)))^G) \cong H^s_{\mathrm{cts}}(G; \pi_t(L_{K(n)}(E_n \wedge X))), \end{aligned}$$

by applying 8.15 and the fact that, since $\varprojlim^1{}_I \pi_{t+1}(Z_I) = 0$, $\pi_t(L_{K(n)}(E_n \wedge X)) \cong \varprojlim{}_I \pi_t(Z_I)$ is an inverse limit of discrete G_n-modules. $\square$

Remark 9.2.11. In the above proof, note that $E_2^{s,t} \cong H^s\big(\varprojlim{}_I (-)^G \{\Gamma^* \pi_t(Z_I)\}_I\big)$. One can also show that $E_2^{s,t} \cong H^s_{\mathrm{cts}}(G; \pi_t(L_{K(n)}(E_n \wedge X)))$ by showing that the exact sequence

$$0 \to \{\pi_t(Z_I)\} \to \{\Gamma^* \pi_t(Z_I)\} \to \cdots$$

in $\operatorname{DMod}(G)^{\mathbb{N}}$ is a $(\varprojlim{}_I (-)^G)$-acyclic resolution of $\{\pi_t(Z_I)\}$, by applying [**24**, Thm. 2.2].

Remark 9.2.12. For $j \geq 1$, consider $L_{K(n)}(E_n \wedge E_n^j)^{hG}$, where G acts nontrivially only on the leftmost factor. By [**9**, 3.19, 3.24], $\pi_t(L_{K(n)}(E_n^{j+1})) \cong \mathrm{Map}_c^{\ell}(G_n^j, \pi_t(E_n)) \cong \mathrm{Map}_c^{\ell}(G_n, \mathrm{Map}_c(G_n^{j-1}, \pi_t(E_n))) \cong \mathrm{Map}_c(G_n, \mathrm{Map}_c(G_n^{j-1}, \pi_t(E_n)))$. Thus, in 9.2.9, $E_2^{s,t} \cong H^s_{\mathrm{cts}}(G; \pi_t(L_{K(n)}(E_n^{j+1})))$ vanishes for $s > 0$ by 8.16, and equals $\mathrm{Map}_c(G_n^j, \pi_t(E_n))^G$ when $s = 0$. Therefore, when $G = G_n$, $\pi_*\big(L_{K(n)}(E_n \wedge E_n^j)^{hG_n}\big) \cong \mathrm{Map}_c(G_n^{j-1}, \pi_*(E_n))$, and hence, $L_{K(n)}(E_n \wedge E_n^j)^{hG_n} \cong L_{K(n)}(E_n^j)$, in the stable category, although we do not concern ourselves with defining the map that induces the given isomorphism. This generalizes the formula $E_n^{h'G_n} \cong L_{K(n)}(S^0)$.

CHAPTER 10

The descent spectral sequence is an Adams spectral sequence

Let X be a finite spectrum. We compare $(E_n \wedge X)^{hG}$ with the spectrum $E_n^{hG} \wedge X$ of Devinatz and Hopkins, and show that, in this case, the descent spectral sequence is strongly convergent. First of all, we make explicit the relationship between these two homotopy fixed point spectra.

We start by noting that there is an isomorphism

$$E_n^{hG} \wedge X \cong (\operatorname{holim}_I ((\operatorname{colim}_i E_n^{hU_iG}) \wedge M_I)) \wedge X$$

in Ho(**Spt**). Consider the canonical map $E_n^{h'G} \wedge X \to (E_n \wedge X)^{hG}$ defined by

$$\big(\operatorname{holim}_I (F_n \wedge M_I)^{hG}\big) \wedge X \to \operatorname{holim}_I \big((F_n \wedge M_I)^{hG} \wedge X\big) \to \operatorname{holim}_I ((F_n \wedge M_I)_f \wedge X)^{hG}.$$

Definition 10.1. The map $\widehat{\psi}\colon (\operatorname{holim}_I ((\operatorname{colim}_i E_n^{hU_iG}) \wedge M_I)) \wedge X \to (E_n \wedge X)^{hG}$ is defined to be the composition $(\operatorname{holim}_I ((\operatorname{colim}_i E_n^{hU_iG}) \wedge M_I)) \wedge X \to E_n^{h'G} \wedge X \to (E_n \wedge X)^{hG}$.

To go further, we need some results from [**9**, AI]. We begin by briefly discussing the $K(n)_*$-local E_n-Adams spectral sequence. For more details, the interested reader is referred to [**9**, AI] and [**32**].

Definition 10.2. Let X be a $K(n)_*$-local spectrum and suppose that the sequence $* \to X \to I^0 \to I^1 \to \cdots$, in Ho(**Spt**), is E_n-exact with each I^s a $K(n)_*$-local E_n-injective spectrum. Then this sequence is said to be a $K(n)_*$-local E_n-resolution of X; such a resolution always exists. Then the $K(n)_*$-local E_n-Adams spectral sequence is the conditionally and strongly convergent spectral sequence $E_1^{s,t} = [Z, I^s]^t \Rightarrow [Z, X]^{t+s}$, where Z in **Spt** is equivalent to a CW-spectrum [**9**, Prop. AI.3].

Theorem 10.3 ([**9**, Prop. AI.5])**.** *Let X be a $K(n)_*$-local spectrum and let $C^\bullet$ be a cosimplicial fibrant spectrum in* **Spt** *with an augmentation $X \to C^\bullet$ such that*

$$* \to X \to C^0 \to C^1 \to C^2 \to \cdots \tag{10.4}$$

is a $K(n)_*$-local E_n-resolution of X. Then for any spectrum Z (equivalent to a CW-spectrum), the $K(n)_*$-local E_n-Adams spectral sequence

$$E_2^{s,t} \cong H^s([Z, C^*]^t) \Longrightarrow [Z, X]^{t+s}$$

is isomorphic to the Bousfield-Kan spectral sequence $E_r^{*,*}(Z, C^\bullet)$ given by

$$E_2^{s,t}(Z, C^\bullet) \cong H^s([Z, C^*]^t) \Longrightarrow [Z, \operatorname{holim}_\Delta C^\bullet]^{t+s}.$$

Corollary 10.5 ([**9**, Cor. AI.8]). *Let X and $C^\bullet$ be as in Theorem 10.3. Then the spectral sequence $E_r^{*,-*}(S^0, C^\bullet)$ of the form*

$$E_2^{s,-t} = \pi^s \pi_t(C^\bullet) \Longrightarrow \pi_{t-s}(\operatorname{holim}_\Delta C^\bullet)$$

is strongly convergent and is isomorphic to the $K(n)_*$-local E_n-Adams spectral sequence with abutment $\pi_*(X)$. Thus, $X \to \operatorname{holim}_\Delta C^\bullet$ is a weak equivalence.

We use these two results to obtain

Theorem 10.6. *Let X be a finite spectrum. Then descent spectral sequence 9.2.10 for $(E_n \wedge X)^{hG}$ is strongly convergent and is isomorphic to the $K(n)_*$-local E_n-Adams spectral sequence with abutment $\pi_*(E_n^{hG} \wedge X)$. Furthermore, the map*

$$\widehat{\psi}\colon (\operatorname{holim}_I((\operatorname{colim}_i E_n^{hU_iG}) \wedge M_I)) \wedge X \longrightarrow (E_n \wedge X)^{hG}$$

is a weak equivalence and $E_n^{hG} \wedge X \cong (E_n \wedge X)^{hG}$ in $\mathrm{Ho}(\mathbf{Spt})$.

Proof. Let $W = (\operatorname{holim}_I((\operatorname{colim}_i E_n^{hU_iG}) \wedge M_I)) \wedge X$. Since

$$W \simeq \operatorname{holim}_I(((\operatorname{colim}_i E_n^{hU_iG}) \wedge M_I) \wedge X) \simeq L_{K(n)}((\operatorname{colim}_i E_n^{hU_iG}) \wedge X),$$

W is $K(n)_*$-local. We write $C^\bullet$ for the cosimplicial spectrum

$$C_X^\bullet = \operatorname{holim}_I(\Gamma_G^\bullet((F_n \wedge M_I)_f \wedge X)_f)^G,$$

when doing so causes no confusion. The canonical map $\mathrm{Tot}(\prod^* C^\bullet) \to \mathrm{Tot}_0(\prod^* C^\bullet) \cong \prod_j C^j \to C^0$ gives an augmentation $\alpha\colon W \to C^\bullet$ defined by the map $W \to (E_n \wedge X)^{hG} \cong \mathrm{holim}_\Delta C^\bullet_X = \mathrm{Tot}(\prod^* C^\bullet) \to C^0$.

Consider the sequence

$$(10.7) \qquad * \to W \xrightarrow{\alpha} C^0 \to C^1 \to C^2 \to \cdots .$$

We claim that this sequence is a $K(n)_*$-local E_n-resolution of W.

Suppose the claim is true. Then Theorems 10.3 and 10.5 give the conclusions of the theorem, and we only need to verify the claim.

First of all, we show that each C^i is E_n-injective. Note that by 8.7, the tower $\{\pi_t(\mathrm{Map}_c(G^i, F_n \wedge M_I \wedge X))\}_I$ is Mittag-Leffler. Then

$$C^i \cong \mathrm{holim}_I \, \mathrm{Map}_c(G^i, ((F_n \wedge M_I)_f \wedge X)_f) \simeq \mathrm{holim}_I \, \mathrm{colim}_N \prod_{G^i/N} (F_n \wedge M_I \wedge X)$$
$$\simeq \mathrm{holim}_I \left(\left(\mathrm{colim}_N \prod_{G^i/N} X \right) \wedge E_n \wedge M_I \right) \simeq L_{K(n)}\left((\mathrm{colim}_N \prod_{G^i/N} X) \wedge E_n \right),$$

so that C^i is $K(n)_*$-local and a retract of $L_{K(n)}\left((\mathrm{colim}_N \prod_{G^i/N} X) \wedge E_n \right)$. Thus, C^i is $K(n)_*$-local E_n-injective.

It only remains to show that 10.7 is E_n-exact. As in [**9**, pp. 37-38], it suffices to show that

$$(10.8) \quad 0 \to [Z, L_{K(n)}(W \wedge E_n)]^t \to [Z, L_{K(n)}(C^0 \wedge E_n)]^t \to [Z, L_{K(n)}(C^1 \wedge E_n)]^t \to \cdots$$

is exact for any Z equivalent to a CW-spectrum.

Note that $L_{K(n)}(C^i \wedge E_n) \simeq \operatorname{holim}_I \big(\big(\operatorname{colim}_N \prod_{G^i/N} X\big) \wedge E_n^2 \wedge M_I\big)$, and since X is a finite spectrum, this is weakly equivalent to $\big(\operatorname{holim}_I (\operatorname{colim}_N \prod_{G^i/N} S^0) \wedge E_n^2 \wedge M_I\big) \wedge X \simeq L_{K(n)}(C^i_{S^0} \wedge E_n) \wedge X$. Thus, $[Z, L_{K(n)}(C^i \wedge E_n)]^t \cong [Z \wedge DX, L_{K(n)}(C^i_{S^0} \wedge E_n)]^t$.

Recall the category $\Lambda = \Lambda(Z \wedge DX)$ described in Theorem 8.17. Let

$$\tau_i \colon [Z \wedge DX, L_{K(n)}(C^i_{S^0} \wedge E_n)]^t \to \varprojlim_{(Y,u)\in\Lambda} [Y, L_{K(n)}(C^i_{S^0} \wedge E_n)]^t$$

be the canonical map, where each Y is a finite spectrum. Since τ_i is a natural map and cosimplicial abelian groups give associated cochain complexes,

$$\tau \colon [Z \wedge DX, L_{K(n)}(C^*_{S^0} \wedge E_n)]^t \to \varprojlim_{(Y,u)\in\Lambda} [Y, L_{K(n)}(C^*_{S^0} \wedge E_n)]^t$$

is a map of cochain complexes.

Note that

$$[Y, L_{K(n)}(C^i_{S^0} \wedge E_n)]^t \cong \pi_{-t}(\operatorname{holim}_I ((\operatorname{colim}_N \textstyle\prod_{G^i/N} S^0) \wedge E_n^2 \wedge M_I \wedge DY)).$$

Also, since $\pi_{-t}(E_n^2 \wedge M_I \wedge DY) \cong \pi_{-t}(F_n^2 \wedge M_I \wedge DY)$ is a discrete G_n-module,

$$\begin{aligned}\pi_{-t}((\operatorname{colim}_N \textstyle\prod_{G^i/N} S^0) \wedge E_n^2 \wedge M_I \wedge DY) &\cong \Gamma^i_G(\operatorname{Map}^\ell_c(G_n, \pi_{-t}(E_n \wedge M_I \wedge DY))) \\ &\cong \Gamma^i_G(\operatorname{Map}_c(G_n, \pi_{-t}(E_n \wedge M_I \wedge DY))).\end{aligned}$$

By 8.9 and 8.7, $\varprojlim^1{}_I \Gamma^i_G(\mathrm{Map}_c(G_n, \pi_{-t+1}(E_n \wedge M_I \wedge DY))) = 0$. This implies that $\pi_{-t}(\mathrm{holim}_I\, ((\mathrm{colim}_N \prod_{G^i/N} S^0) \wedge E_n^2 \wedge M_I \wedge DY)) \cong \Gamma^i_G \mathrm{Map}_c(G_n, \pi_{-t}(E_n \wedge DY))$. Thus, there is a natural isomorphism

$$[Y, L_{K(n)}(C^i_{S^0} \wedge E_n)]^t \cong \mathrm{Map}_c(G^i, \mathrm{Map}_c(G_n, \pi_{-t}(E_n \wedge DY))).$$

Therefore, τ is the map of cochain complexes

$$[Z \wedge DX, L_{K(n)}(C^*_{S^0} \wedge E_n)]^t \to \mathrm{Map}_c(G^*, \mathrm{Map}_c(G_n, E_n^t(Z \wedge DX))),$$

and $\mathrm{Map}_c(G^i, \mathrm{Map}_c(G_n, E_n^t(Z \wedge DX))) \cong \mathrm{Map}_c(G^i \times G_n, E_n^t(Z \wedge DX))$. Now consider the canonical map (obtained as above)

$$\hat{\tau} \colon [V, L_{K(n)}(C^*_{S^0} \wedge E_n)]^t \to \mathrm{Map}_c(G^* \times G_n, E_n^t(V))$$

of cochain complexes, for an arbitrary spectrum V (equivalent to a CW-spectrum). Since $L_{K(n)}(C^i_{S^0} \wedge E_n)^t(-)$ and $\mathrm{Map}_c(G^i \times G_n, E_n^t(-))$ are cohomology theories satisfying the product axiom, to show that $\hat{\tau}$ is an isomorphism for all V, it suffices to check this for $V = S^0$ [**9**, pg. 25]. Setting $V = S^0$ yields

$$[S^0, L_{K(n)}(C^i_{S^0} \wedge E_n)]^t \cong \mathrm{Map}_c(G^i, \mathrm{Map}_c(G_n, \pi_{-t}(E_n))),$$

showing that $\hat{\tau}$ is an isomorphism for any V. In particular, $V = Z \wedge DX$ implies that τ is an isomorphism of cochain complexes for any Z, and

$$[Z, L_{K(n)}(C^i \wedge E_n)]^t \cong [Z \wedge DX, L_{K(n)}(C^i_{S^0} \wedge E_n)]^t \cong \mathrm{Map}_c(G^i, \mathrm{Map}_c(G_n, E_n^t(Z \wedge DX))).$$

Therefore, since $\pi_{-t}(C^*)$ gives continuous cochains, the sequence $[Z, L_{K(n)}(C^* \wedge E_n)]^t$ from 10.8 is the cochain complex of continuous cochains

$$\Gamma_{G_n}(E_n^t(Z \wedge DX)) \to \mathrm{Map}_c(G, \Gamma_{G_n} E_n^t(Z \wedge DX)) \to \mathrm{Map}_c(G^2, \Gamma_{G_n} E_n^t(Z \wedge DX)) \to \cdots .$$

Thus, since $\mathrm{Map}_c(G_n, E_n^t(Z \wedge DX)) \cong \varprojlim_{(Y,u),I} \mathrm{Map}_c(G_n, \pi_{-t}(E_n \wedge M_I \wedge DY))$ is an inverse limit of discrete G_n-modules, 8.15 implies that

$$H^s\big([Z, L_{K(n)}(C^*_X \wedge E_n)]^t\big) \cong H^s_{\mathrm{cts}}(G; \mathrm{Map}_c(G_n, E_n^t(Z \wedge DX))),$$

which vanishes for $s > 0$ by 8.16, and equals $\mathrm{Map}_c(G_n, E_n^t(Z \wedge DX))^G$, when $s = 0$.

Now consider the portion of 10.8

$$0 \to [Z, L_{K(n)}(W \wedge E_n)]^t \to [Z, L_{K(n)}(C^0 \wedge E_n)]^t \tag{10.9}$$

that is induced by α. This can be rewritten as

$$0 \to [Z \wedge DX, L_{K(n)}(E_n^{hG} \wedge E_n)]^t \to [Z \wedge DX, L_{K(n)}(C^0_{S^0} \wedge E_n)]^t.$$

As in [**9**, proof of Prop. 5.4], the map

$$\pi_* L_{K(n)}(E_n^{hG} \wedge E_n) \to \pi_* L_{K(n)}(C^0_{S^0} \wedge E_n) \cong \pi_* L_{K(n)}(E_n \wedge E_n)$$

is an injection with image $\mathrm{Map}_c(G_n, E_{n*})^G \subset \mathrm{Map}_c(G_n, E_{n*})$. Therefore, as in [**9**, pp. 37-39], 10.9 is an injection with image $\mathrm{Map}_c(G_n, E_n^t(Z \wedge DX))^G \subset \mathrm{Map}_c(G_n, E_n^t(Z \wedge DX))$. This completes the proof that 10.8 is an exact sequence for any Z. □

The following corollary shows that the $K(n)_*$-localization of any finite spectrum is a homotopy fixed point spectrum associated to E_n.

Corollary 10.10. *If X is a finite spectrum, then $L_{K(n)}(X) \cong (E_n \wedge X)^{hG_n}$, in the stable category.*

Proof. We have: $(E_n \wedge X)^{hG_n} \cong E_n^{hG_n} \wedge X \cong L_{K(n)}(S^0) \wedge X \cong L_{K(n)}(X)$. □

Putting these results together gives

Corollary 10.11. *Let X be a finite spectrum. Using the continuous G_n-action on $E_n \wedge X$, there is a strongly convergent descent spectral sequence*

$$H^s_c(G_n; \pi_t(E_n \wedge X)) \Longrightarrow \pi_{t-s}(L_{K(n)}X)$$

that is isomorphic to the $K(n)_*$-local E_n-Adams spectral sequence with $\pi_*(E_n^{hG_n} \wedge X)$ as abutment.

This Corollary shows that spectral sequence 1.1 can indeed be realized as a descent spectral sequence for $E_n \wedge X$ with actual homotopy fixed point spectrum $L_{K(n)}(X)$.

One can also view the $E(n)_*$-localization of any finite type n complex as a homotopy fixed point spectrum:

Theorem 10.12. *If X is a finite type n spectrum, then $L_n X \cong (F_n \wedge X)^{hG_n}$.*

Proof. We have: $L_n X \cong L_n S^0 \wedge X \cong L_{K(n)}(X) \cong (E_n \wedge X)^{hG_n} \cong E_n^{hG_n} \wedge X \cong (F_n \wedge X)^{hG_n}$. □

Remark 10.13. Recall that by [**9**, Prop. 5.7], if X is a CW-spectrum such that, for each $E(n)$-module spectrum M, there exists a k with $I_n^k M_*(X) = 0$, then there is a $K(n)_*$-local E_n-Adams spectral sequence of the form

$$H_c^*(G; \pi_*(E_n \wedge X)) \Longrightarrow \pi_*(L_{K(n)}(E_n^{hG} \wedge X)).$$

Also, as discussed in Chapter 3, if $\varprojlim_I \pi_*(E_n \wedge M_I \wedge X)$ is finitely generated over E_{n*}, then there is a spectral sequence

$$H_c^*(S_n; \varprojlim_I \pi_*(E_n \wedge M_I \wedge X))^{\mathrm{Gal}} \Longrightarrow \pi_*(L_{K(n)} X).$$

Our descent spectral sequence was motivated by these earlier spectral sequences. However, we do not know precisely the relationships among the X's that appear in the three spectral sequences.

References

[1] J. Michael Boardman. Conditionally convergent spectral sequences. In *Homotopy invariant algebraic structures (Baltimore, MD, 1998)*, pages 49–84. Amer. Math. Soc., Providence, RI, 1999.

[2] A. K. Bousfield. The localization of spectra with respect to homology. *Topology*, 18(4):257–281, 1979.

[3] A. K. Bousfield and D. M. Kan. *Homotopy limits, completions and localizations.* Springer-Verlag, Berlin, 1972. Lecture Notes in Mathematics, Vol. 304.

[4] Ethan S. Devinatz. Morava's change of rings theorem. In *The Čech centennial (Boston, MA, 1993)*, pages 83–118. Amer. Math. Soc., Providence, RI, 1995.

[5] Ethan S. Devinatz. Morava modules and Brown-Comenetz duality. *Amer. J. Math.*, 119(4):741–770, 1997.

[6] Ethan S. Devinatz. The generating hypothesis revisited. In *Stable and unstable homotopy (Toronto, ON, 1996)*, pages 73–92. Amer. Math. Soc., Providence, RI, 1998.

[7] Ethan S. Devinatz. A Lyndon-Hochschild-Serre spectral sequence for certain homotopy fixed point spectra. *Available at the Hopf topology archive at http://hopf.math.purdue.edu*, 2002.

[8] Ethan S. Devinatz and Michael J. Hopkins. Homotopy fixed point spectra for closed subgroups of the Morava stabilizer groups. *Available at the Hopf topology archive at http://hopf.math.purdue.edu*, 1999.

[9] Ethan S. Devinatz and Michael J. Hopkins. Homotopy fixed point spectra for closed subgroups of the Morava stabilizer groups. *Available at the Hopf topology archive at http://hopf.math.purdue.edu*, 2002.

[10] J. D. Dixon, M. P. F. du Sautoy, A. Mann, and D. Segal. *Analytic pro-p groups.* Cambridge University Press, Cambridge, second edition, 1999.

[11] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May. *Rings, modules, and algebras in stable homotopy theory.* American Mathematical Society, Providence, RI, 1997. With an appendix by M. Cole.

[12] Thomas Geisser and Lars Hesselholt. Topological cyclic homology of schemes. In *Algebraic K-theory (Seattle, WA, 1997)*, volume 67 of *Proc. Sympos. Pure Math.*, pages 41–87. Amer. Math. Soc., Providence, RI, 1999.

[13] P. G. Goerss and J. F. Jardine. Localization theories for simplicial presheaves. *Canad. J. Math.*, 50(5):1048–1089, 1998.

[14] Paul G. Goerss. Homotopy fixed points for Galois groups. In *The Čech centennial (Boston, MA, 1993)*, pages 187–224. Amer. Math. Soc., Providence, RI, 1995.

[15] Paul G. Goerss and Michael J. Hopkins. André-Quillen (co)-homology for simplicial algebras over simplicial operads. In *Une dégustation topologique [Topological morsels]: homotopy theory in the Swiss Alps (Arolla, 1999)*, pages 41–85. Amer. Math. Soc., Providence, RI, 2000.

[16] Paul G. Goerss and Michael J. Hopkins. Moduli spaces of commutative ring spectra. *manuscript, available at the website http://www.math.nwu.edu/~pgoerss*, 2003.

[17] Paul G. Goerss and John F. Jardine. *Simplicial homotopy theory.* Birkhäuser Verlag, Basel, 1999.

[18] M. J. Hopkins and B. H. Gross. The rigid analytic period mapping, Lubin-Tate space, and stable homotopy theory. *Bull. Amer. Math. Soc. (N.S.)*, 30(1):76–86, 1994.

[19] Michael J. Hopkins, Mark Mahowald, and Hal Sadofsky. Constructions of elements in Picard groups. In *Topology and representation theory (Evanston, IL, 1992)*, pages 89–126. Amer. Math. Soc., Providence, RI, 1994.

[20] Mark Hovey. Bousfield localization functors and Hopkins' chromatic splitting conjecture. In *The Čech centennial (Boston, MA, 1993)*, pages 225–250. Amer. Math. Soc., Providence, RI, 1995.

[21] Mark Hovey. *Model categories.* American Mathematical Society, Providence, RI, 1999.

[22] Mark Hovey, Brooke Shipley, and Jeff Smith. Symmetric spectra. *J. Amer. Math. Soc.*, 13(1):149–208, 2000.

[23] Mark Hovey and Neil P. Strickland. Morava K-theories and localisation. *Mem. Amer. Math. Soc.*, 139(666):viii+100, 1999.

[24] Uwe Jannsen. Continuous étale cohomology. *Math. Ann.*, 280(2):207–245, 1988.

[25] J. F. Jardine. Simplicial presheaves. *J. Pure Appl. Algebra*, 47(1):35–87, 1987.

[26] J. F. Jardine. Stable homotopy theory of simplicial presheaves. *Canad. J. Math.*, 39(3):733–747, 1987.

[27] J. F. Jardine. *Generalized étale cohomology theories*. Birkhäuser Verlag, Basel, 1997.

[28] Michel Lazard. Groupes analytiques p-adiques. *Inst. Hautes Études Sci. Publ. Math.*, (26):389–603, 1965.

[29] Saunders Mac Lane and Ieke Moerdijk. *Sheaves in geometry and logic*. Springer-Verlag, New York, 1994. A first introduction to topos theory, Corrected reprint of the 1992 edition.

[30] Mark Mahowald and Hal Sadofsky. v_n telescopes and the Adams spectral sequence. *Duke Math. J.*, 78(1):101–129, 1995.

[31] M. A. Mandell, J. P. May, S. Schwede, and B. Shipley. Model categories of diagram spectra. *Proc. London Math. Soc. (3)*, 82(2):441–512, 2001.

[32] Haynes R. Miller. On relations between Adams spectral sequences, with an application to the stable homotopy of a Moore space. *J. Pure Appl. Algebra*, 20(3):287–312, 1981.

[33] Haynes R. Miller, Douglas C. Ravenel, and W. Stephen Wilson. Periodic phenomena in the Adams-Novikov spectral sequence. *Ann. Math. (2)*, 106(3):469–516, 1977.

[34] Stephen A. Mitchell. Hypercohomology spectra and Thomason's descent theorem. In *Algebraic K-theory (Toronto, ON, 1996)*, pages 221–277. Amer. Math. Soc., Providence, RI, 1997.

[35] Stephen A. Mitchell. K-theory hypercohomology spectra of number rings at the prime 2. In *Une dégustation topologique [Topological morsels]: homotopy theory in the Swiss Alps (Arolla, 1999)*, volume 265 of *Contemp. Math.*, pages 129–157. Amer. Math. Soc., Providence, RI, 2000.

[36] Jack Morava. Noetherian localisations of categories of cobordism comodules. *Ann. of Math. (2)*, 121(1):1–39, 1985.

[37] Jürgen Neukirch, Alexander Schmidt, and Kay Wingberg. *Cohomology of number fields*, volume 323 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2000.

[38] Douglas C. Ravenel. *Nilpotence and periodicity in stable homotopy theory.* Princeton University Press, Princeton, NJ, 1992. Appendix C by Jeff Smith.

[39] Charles Rezk. Notes on the Hopkins-Miller theorem. In *Homotopy theory via algebraic geometry and group representations (Evanston, IL, 1997)*, pages 313–366. Amer. Math. Soc., Providence, RI, 1998.

[40] Luis Ribes and Pavel Zalesskii. *Profinite groups.* Springer-Verlag, Berlin, 2000.

[41] Stefan Schwede. S-modules and symmetric spectra. *Math. Ann.*, 319(3):517–532, 2001.

[42] Jean-Pierre Serre. Groupes analytiques p-adiques. In *Séminaire Bourbaki, 1963/64, Fasc. 2, Exposé 270*, page 10. Secrétariat mathématique, Paris, 1964.

[43] Jean-Pierre Serre. *Cohomologie Galoisienne.* Springer-Verlag, Berlin, 1973. Cours au Collège de France, Paris, 1962–1963, Avec des textes inédits de J. Tate et de Jean-Louis Verdier, Quatrième édition, Lecture Notes in Mathematics, Vol. 5.

[44] N. P. Strickland. On the p-adic interpolation of stable homotopy groups. In *Adams Memorial Symposium on Algebraic Topology, 2 (Manchester, 1990)*, pages 45–54. Cambridge Univ. Press, Cambridge, 1992.

[45] N. P. Strickland. Gross-Hopkins duality. *Topology*, 39(5):1021–1033, 2000.

[46] Peter Symonds and Thomas Weigel. Cohomology of p-adic analytic groups. In *New horizons in pro-p groups*, pages 349–410. Birkhäuser Boston, Boston, MA, 2000.

[47] John Tate. Relations between K_2 and Galois cohomology. *Invent. Math.*, 36:257–274, 1976.

[48] R. W. Thomason. Algebraic K-theory and étale cohomology. *Ann. Sci. École Norm. Sup. (4)*, 18(3):437–552, 1985.

[49] R. W. Thomason. The finite stable homotopy type of some topoi. *J. Pure Appl. Algebra*, 47(1):89–104, 1987.

[50] John S. Wilson. *Profinite groups.* The Clarendon Press Oxford University Press, New York, 1998.