5.8 Problems and Solutions (last update 30 May 2018)

1.The lineup or batting order for a baseball team is a list of the nine players on the team indicating the order in which they will bat during the game.

a) How many lineups are possible?

b) If lineup is restricted so that the catcher is listed first and the pitcher is listed last, how many lineups are possible?

Solution

We can use an ordered 9–tuple of player names to represent a lineup. For example, if we use the letters a,b,c,d,e,f,g,h,i to represent the player names, then two possible lineups are $(a, b, c, d, e, f, g, h, i)$ and $(a, c, e, g, i, b, d, f, h)$.

a) With no restrictions there are 9 choices for the first player in the lineup, 8 choices for the second, and so on. Thus there are $9! = 9 \cdot 8 \cdot 7 \cdots 1 = 362880$ possible lineups.

b) With the restriction that the catcher is listed first and the pitcher is listed last, there is one choice for the first player in the lineup and there is one choice for the last player in the lineup. Thus, after accounting for these two players in the lineup, the problem reduces to determining how many lineups of the 7 remaining players are possible. The answer is clearly $7! = 7 \cdot 6 \cdot 5 \cdots 1 = 5040$. In terms of the 9 positions in the lineup we have $1 \cdot 7! \cdot 1 = 5040$.

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2. In how many ways can one plumber, one electrician, and one carpenter be selected when there are 5 choices of plumber, 3 choices of electrician, and 7 choices of carpenter?

Solution

We can use an ordered triple of names to represent a selection with the first name indicating the choice of the plumber, the second the electrician, and the third the carpenter. For example, if we use the numbers to represent the different tradesmen, then $(1, 3, 5)$ indicates the selection of plumber number 1, electrician number 3, and carpenter number 6. Since there are 5, 3, and 7 choices for the respective tradesmen, there are $5 \cdot 3 \cdot 7 = 105$ ways in which the three tradesmen can be selected.

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3. (Akritas) The clock rate of a CPU (central processing unit) chip refers to the frequency, measured in megahertz (MHz), at which it functions reliably. CPU manufacturers typically categorize (bin) CPUs according to their clock rate and charge more for CPUs that operate at higher clock rates. A chip manufacturing facility will test and bin each of the next 10 CPUs in four clock rate categories denoted by G1, G2, G3, and G4.

a) How many possible outcomes of this binning process are there?

b) How many of the outcomes have three CPUs classified as G1, two classified as G2, two classified as G3, and three classified as G4?

c) If the outcomes of the binning process are equally likely, what is the probability of the event described in part (b)?

Solution

We can use an ordered ten–tuple to represent the outcome of the binning process. The elements of the ten–tuple are chosen from the four clock rates G1, G2, G3, and G4 and the position in the ten–tuple indicates the CPU. For example, (G1,G1,G3,G3,G3,G3,G4,G4,G2,G2) indicates the the first two CPU's are classified as G1, the next four are classified as G3, the next two are classified as G4, and the last two are classified as G2.

a) Since there are four ways to classify each CPU and there are 10 CPU's, there are $4^{10} = 1048576$ possible outcomes of this binning process.

b) To form an outcome with three CPUs classified as G1, two classified as G2, two classified as G3, and three classified as G4 we need to first select 3 of the 10 CPU's to be classified as G1, then select 2 of the remaining 7 CPU's to be classified as G2, then select 2 of the remaining 5 CPU's to be classified as G3, and finally select 3 of the remaining 3 CPU's to be classified as G4. This can be done in $\binom{10}{3}\binom{7}{2}\binom{5}{2}\binom{3}{3} = 120 \cdot 21 \cdot 10 \cdot 1 = 25200$ ways.

c) Under the assumption of equally likely outcomes the probability that three CPUs are classified as G1, two are classified as G2, two are classified as G3, and three are classified as G4 is equal to the number of outcomes favorable to this event divided by the number of possible outcomes. That is, this probability is

$$\frac{\binom{10}{3}\binom{7}{2}\binom{5}{2}\binom{3}{3}}{4^{10}} = \frac{25200}{1048576} \approx .0240$$

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4. A president, treasurer, and secretary, all different, are to be chosen from a club consisting of 10 people. How many different choices of officers are possible if

a) There are no restrictions?

b) A and B will not serve together?

c) C and D will serve together or not at all?

d) E must be an officer?

e) F will serve only if he is president?

Solution

We can use an ordered triple of names to represent a selection with the first name indicating the choice for president, the second the choice for treasurer, and the third the choice for secretary. For example, using the letters A,B,C,D,E,F,G,H,I,J to represent the club members, (D, A, J) indicates the selection of D as president, A as treasurer, and J as secretary.

a) With no restrictions there are 10 choices for president, 9 for treasurer, and 8 for secretary. Thus there are $10 \cdot 9 \cdot 8 = 720$ different choices of officers.

b) If A and B will not serve together, then we need to consider two cases. First, if neither A nor B is an officer, then there are $8 \cdot 7 \cdot 6 = 336$ choices. Second, if one of A or B is an officer, then there are $\binom{3}{1} \cdot 2 \cdot 8 \cdot 7 = 336$ choices. In this second case, the $\binom{3}{1}$ corresponds to the selection of the office to be occupied by A or B, the 2 corresponds to the choice of A or B for the selected office, and the $8 \cdot 7$ corresponds to the choice of the other two officers. Thus there are $336 + 336 = 672$ choices of officers when A and B will not serve together.

c) If C and D will serve together or not at all, then we need to consider two cases. First, if neither C nor D is an officer, then there are $8 \cdot 7 \cdot 6 = 336$ choices. Second, if both C and D are officers, then there are $\binom{3}{2} \cdot 2 \cdot 1 \cdot 8 = 48$ choices. In this second case, the $\binom{3}{2}$ corresponds to the selection of the two offices to be occupied by C and D, the $2 \cdot 1$ corresponds to the choices of C and D for the the 2 selected offices, and the 8 corresponds to the choice of the final officer. Thus there are $336 + 48 = 384$ choices of officers when C and D will serve together or not at all.

d) If E must be an officer, then there are $\binom{3}{1} \cdot 1 \cdot 9 \cdot 8 = 216$ choices. In this case, the $\binom{3}{1}$ corresponds to the selection of the office to be occupied by E and the $9 \cdot 8$ corresponds to the choices of the other two officers.

e) If F will serve only if he is president, then we need to consider two cases. First, if F is not an officer, then there are $9 \cdot 8 \cdot 7 = 504$ choices. Second, if F is president, then there are $1 \cdot 9 \cdot 8 = 72$ choices. Thus there are $504 + 72 = 576$ choices of officers when F serves as President or not at all.

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5. A child has 12 blocks, of which 6 are red, 3 are green, 2 are blue, and 1 is black. If the child puts the blocks in a line, how many arrangements are possible?

Solution

We will assume that the blocks of the same color are indistinguishable (all look the same). That is, the elementary outcomes will be envisioned as ordered sequences of the form $(RRRRRRGGGBBK)$, where R, G, B, and K denote the respective colors red, green, blue, and black. We will count by selecting positions in the sequence (line) sequentially for the four colors. Initially there are 12 positions to choose from. Thus, there are $\binom{12}{6}$ choices for the 6 red balls. Once this is done, there are 6 positions to choose from, so we have $\binom{6}{3}$ choices for the green balls. Continuing in this way, there are $\binom{3}{2}$ position choices for the blue balls and there is $\binom{1}{1} = 1$ choice for the black ball. Hence, there are

$$\binom{12}{6}\binom{6}{3}\binom{3}{2}\binom{1}{1} = 924 \cdot 20 \cdot 3 \cdot 1 = 55440$$

possible arrangements of the 12 blocks.

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6. If 4 Americans, 3 French people, and 3 British people are to be seated in a row.

a) How many seating arrangements are possible for these 10 individuals when there is no restriction on where each person may be seated?

b) How many seating arrangements are possible when people of the same nationality must be seated next to each other?

c) How many seating arrangements are possible when the 4 Americans must sit next to each other but there is no restriction on the others?

Solution

For this problem we will assume that the people are distinguishable with A_1, A_2, A_3, A_4 denoting the Americans, F_1, F_2, F_3 denoting the French people, and B_1, B_2, B_3 denoting the British people. A seating assignment will be represented by an ordered ten–tuple of these ten letters such as $(A_1, A_2, A_3, A_4, B_1, B_2, B_3, F_1, F_2, F_3)$.

a) With no restriction on how the people are seated, there are $10! = 3628800$ arrangements, since there are 10 choices for the first person, 9 for the next, and so on.

b) If people of the same nationality must be seated next to each other, then we can start by ordering 3 objects (the group of Americans, the group of French people, and the group of British people). There are $3!$ ways to do this. Next we need to order the people of each nationality within their group. Since there are 4 Americans, 3 French people, and 3 British people, there are $4!3!3!$ ways to do this. Hence there are $3!4!3!3! = 5184$ ways to seat the ten people when people of the same nationality must be seated next to each other.

c) Here the 4 Americans must sit next to each other but there is no restriction on the others. Thus we begin by ordering 7 objects, the group of Americans, the 3 French people, and the 3 British people. There are $7!$ ways to do this. Next we order the Americans among themselves. There are $4!$ ways to do this. Thus, there are $7!4! = 5040 \cdot 24 = 120960$ ways to seat the ten people when the 4 Americans must sit next to each other but there is no restriction on the others.

7. For years, telephone area codes in the United States and Canada consisted of a sequence of three digits. The first digit was an integer from 2 through 9; the second digit was either 0 or 1; and the third digit was any integer from 1 through 9.
a) How many area codes were possible?
b) How many area codes ending in 6 were possible?

Solution

a) With these restrictions there are 8 choices (2,3,4,5,6,7,8, or 9) for the first digit of the area code, 2 choices (0 or 1) for the second digit, and 9 choices (1,2,3,4,5,6,7,8, or 9) for the third digit. Thus there are $8 \cdot 2 \cdot 9 = 144$ possible area codes.

b) If we require the last digit to be a six, then there are still 8 choices (2,3,4,5,6,7,8, or 9) for the first digit of the area code and 2 choices (0 or 1) for the second digit, but, there is only one choice (6) for the third digit. Thus, $8 \cdot 2 \cdot 1 = 16$ of the 144 possible area codes end in 6.

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8. In how many ways can 4 novels, 3 biology books, and 2 mathematics books be arranged on a bookshelf if

a) the books can be arranged in any order;

b) the biology books must be together and the novels must be together;

c) the novels must be together but the other books can be arranged in any order?

Solution

For this problem we will assume that the books of the same type are distinguishable. That is, the elementary outcomes will be envisioned as ordered sequences of the form $(N_1N_2N_3N_4B_1B_2B_3M_1M_2)$, where $N_1 \ldots N_5$ represent the 5 different novels, B_1, B_2, B_3 represent the 3 different biology books, M_1, M_2 represent the 2 different mathematics books, and these 9 symbols can appear in every possible order.

a) With no restriction on how the books are arranged, there are $9! = 362880$ arrangements, since there are 9 choices for the first book, 8 for the next, and so on.

b) Since the 4 novels must be positioned together and the 3 biology books must be positioned together, we can start by arranging 4 objects (the group of novels, the group of biology books, and the 2 mathematics books) on the bookshelf. There are $4!$ ways to do this. Next we need to order the 4 novels and order the 3 biology books. There are $4!3!$ ways to do this. Hence there are $4!4!3! = 3456$ ways to arrange the books when the biology books must be together and the novels must be together.

c) Here the 4 novels must be positioned together. Thus we can start by arranging 6 objects (the group of novels and the 5 other books) on the bookshelf. There are $6!$ ways to do this. Next we need to order the 4 novels. There are $4!$ ways to do this. Hence there are $6!4! = 17280$ ways to arrange the books when the only restriction is that the novels must be together.

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9. Two pollsters will canvas a neighborhood with 20 houses. Each pollster will visit 10 houses. No house will be visited more than once. How many different assignments of pollsters to houses are possible?

Solution

There are $\binom{20}{10}$ ways to choose 10 of the 20 houses for the first pollster to visit. The second pollster will visit the other 10 houses. Thus, there are $\binom{20}{10} = 184756$ possible assignments of pollsters to houses.

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10. A box contains 24 light bulbs, of which two are defective. If a person selects 10 light bulbs at random, without replacement, what is the probability that both defective light bulbs will be selected?

Solution

There are $\binom{24}{10}$ ways to choose a group of 10 light bulbs without replacement from a group of 24 light bulbs (a combination of 24 bulbs taken 10 at a time). The group of 24 light bulbs consists of 2 defective bulbs and 22 non–defective bulbs. The event of interest is the selection of both of the defective bulbs and 8 non–defective bulbs. This can be done in $\binom{2}{2}\binom{22}{8}$ ways (choose 2 defective bulbs from the 2 defective bulbs and then choose 8 non–defective bulbs from the 22 non–defective bulbs. Since the bulbs were selected at random the $\binom{24}{10}$ possible outcomes are equally likely and the desired probability is

$$P(\text{both defective bulbs are selected}) = \frac{\binom{2}{2}\binom{22}{8}}{\binom{24}{10}} = \frac{1 \cdot 319770}{1961256} \approx .1630$$

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11. Suppose that two defective refrigerators have been included in a shipment of six refrigerators. Now suppose that the buyer begins to test the refrigerators one at a time.
a) What is the probability that the last defective refrigerator is found on the fourth test?
b) What is the probability that no more than four refrigerators need to be tested before both defective refrigerators are found?

Solution

For this problem we can think of each possible outcome as an ordered arrangement (6–tuple) of 2 $D's$ and 4 $N's$, where D indicates a defective refrigerator and N indicates a non–defective refrigerator and the location of the letter indicates when the refrigerator was tested. For example, $NDNDNN$ indicates that the second and fourth refrigerators tested were found to be defective while the others were found to not be defective. There are $\binom{6}{2} = 15$ such outcomes and we will assume that they are equally likely. Note that we can think of $\binom{6}{2} = 15$ as the number of ways to select 2 of the 6 positions for the two $D's$.

a) In order for the last defective refrigerator to be found on the fourth test the first 3 refrigerators tested must consist of two non–defective refrigerators and one defective refrigerator and the fourth refrigerator tested must be the second defective refrigerator. That is, the first three elements of the 6-tuple of the outcome must contain two $N's$ and one D; the fourth element of the 6-tuple must be a D; and, the last three elements must be $N's$. There are $\binom{3}{1} = 3$ ways to select one of the first 3 positions for a D and there is one way to place the remaining D in the fourth position. Hence, the probability that the last defective refrigerator is found on the fourth test is

$$P(\text{last defective on fourth test}) = \frac{\binom{3}{1}}{\binom{6}{2}} = \frac{3}{15} = \frac{1}{5}.$$

b) The outcomes (6–tuples) that are favorable for the event that no more than four refrigerators need to be tested before both defective refrigerators are found are those for which the two $D's$ are located in the first 4 positions. There are $\binom{4}{2} = 6$ ways to choose 2 of the first 4 positions for the two $D's$. Once this (placing 2 $D's$ into two of the first 4 positions) is done, the 4 $N's$ are placed in the remaining 4 positions. Thus, the probability that no more than four refrigerators need to be tested before both defective refrigerators are found is

$$P(\text{both defectives found by the fourth test}) = \frac{\binom{4}{2}}{\binom{6}{2}} = \frac{6}{15} = \frac{2}{5}.$$

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12. Consider two collections of 4 cards, one that contains 4 red cards numbered from 1 to 4 and one that contains 4 blue cards numbered from 1 to 4. Suppose that one card is selected at random from the 4 red cards and one card is selected at random from the 4 blue cards.

Let A denote the event that the number on the red card is larger than the number on the blue card.

Let B denote the event that the number on the red card is greater than two.

Let C denote the event that the number on the blue card is odd.

a) List the 16 possible outcomes of this experiment as ordered pairs with the first element representing the number on the red card and the second element representing the number on the blue card, *e.g.* the ordered pair $(1, 4)$ indicates that the red card numbered 1 was selected and the blue card numbered 4 was selected.

Using this collection of 16 ordered pairs as a representation of the sample space Ω, describe each of the following events both in words and as subsets of Ω:

b) $A \cap B$

c) $A \cup C$

d) $B \cap (A \cup C)$

e) $B^c \cap C^c$

f) $A \cap B \cap C$.

Solution

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a) The 16 possible outcomes of this experiment expressed as ordered pairs with the first element representing the number on the red card and the second element representing the number on the blue card are:
$(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2), (4, 3), (4, 4)$
The event A "the number on the red card is larger than the number on the blue card" is

$$A = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$$

The event B "the number on the red card is greater than two" is

$$B = \{(3, 1), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2), (4, 3), (4, 4)\}$$

The event C "the number on the blue card is odd" is

$$C = \{(1, 1), (1, 3), (2, 1), (2, 3), (3, 1), (3, 3), (4, 1), (4, 3)\}$$

b) $A \cap B$ is the event "red larger than blue and red greater than two"

$$A \cap B = \{(3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$$

c) $A \cup C$ is the event "red larger than blue or blue odd"

$$A \cup C = \{(1, 1), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2), (3, 3), (4, 1), (4, 2), (4, 3)\}$$

d) $B \cap (A \cup C)$ is the event "in addition to red greater than two we also have red larger than blue or blue odd (or both)", *i.e.*, "red greater than two and red larger than blue" or "red greater than two and blue odd"

$$B \cap (A \cup C) = \{(3,1),(3,2),(3,3),(4,1),(4,2),(4,3)\}$$

e) $B^c \cap C^c$ is the event "red not greater than two and blue not odd", *i.e.*, "red is two or less and blue even"

$$B^c \cap C^c = \{(1,2),(1,4),(2,2),(2,4)\}$$

f) $A \cap B \cap C$ is the event "red larger than blue and red greater than two and blue odd"

$$A \cap B \cap C = \{(3,1),(4,1),(4,3))\}$$

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13. This problem is concerned with a simple visual messaging system based on an ordered arrangement of ten colored flags. A message is formed by placing ten colored flags on a single flagpole. The message is read by noting the colors of the flags starting at the top of the pole and moving down. Suppose that the ten flags consist of two red flags, three blue flags, and five green flags.

a) How many different messages can be formed from these ten flags?
b) How many different messages can be formed from these ten flags if the first and last flags must be red?
c) How many different messages can be formed from these ten flags if the fifth and sixth flags must be red?
d) How many different messages can be formed from these ten flags if the every other flag, starting with the first must be green?

Solution

We will assume that flags of the same color are indistinguishable (all look the same). That is, the elementary outcomes will be envisioned as ordered sequences of the form $(RRBBBGGGGG)$, where R, B, and G denote the respective colors red, blue, and green. We will count by selecting positions in the sequence (line) sequentially for the three colors.

a) How many different messages can be formed from these ten flags?
Initially there are 10 positions to choose from. Thus, there are $\binom{10}{2}$ choices for the 2 red

flags. Once this is done, there are 8 positions to choose from, so we have $\binom{8}{3}$ choices for the blue flags. At this point, there are 5 positions and 5 green flags, thus, there are $\binom{5}{5} = 1$ choices for the green flags.

$$\binom{10}{2}\binom{8}{3}\binom{5}{5} = 45 \cdot 56 \cdot 1 = 2520$$

b) How many different messages can be formed from these ten flags if the first and last flags must be red?
If the first and last flags are red, then we only need a sequence of length 8 representing the 8 open positions. Arguing as before, in this case, the numbers of messages is

$$\binom{8}{3}\binom{5}{5} = 56 \cdot 1 = 56$$

c) How many different messages can be formed from these ten flags if the fifth and sixth flags must be red?
As far as counting is concerned, knowing that the fifth and sixth flags are red is equivalent to knowing that the first and last flags are red. Thus we need a sequence of length 8 representing the 8 open positions and, as before, the numbers of messages is

$$\binom{8}{3}\binom{5}{5} = 56 \cdot 1 = 56$$

d) How many different messages can be formed from these ten flags if the every other flag, starting with the first must be green? If every other flag is green, then there are 5 open positions and, arguing as before, the number of messages is

$$\binom{5}{2}\binom{3}{3} = 10 \cdot 1 = 10$$

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14. Consider three groups, group A, group B, and group C, each consisting of ten people. A subgroup of three people is to be selected from the combined group of thirty people.

a) How many choices of three people are possible?

b) How many choices of three people are there in which all three people come from group A?

c) How many choices of three people are there in which all three people come from the same group of ten?

d) How many choices of three people are there in which one of the three people comes from group A, one comes from group B, and one comes from group C?

Solution

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a) How many choices of three people are possible?

Since there are 30 people, there are $\binom{30}{3} = 4060$ ways to choose a group of 3 people.

b) How many choices of three people are there in which all three people come from group A?

Since there are 10 people in each of the groups and we want 3 from group A, we need to choose 3 from group A, 0 from group B, and 0 from group C. Thus, there are $\binom{10}{3}\binom{10}{0}\binom{10}{0} = 120 \cdot 1 \cdot 1 = 120$ ways to choose a 3 people from group A.

c) How many choices of three people are there in which all three people come from the same group of ten?

The number of ways we could choose 3 people from the same group is obtained by adding the number of ways to choose 3 people from group A, the number of ways to choose 3 people from group B, and the number of ways to choose 3 people from group C. Since all three group contain 10 people, the answer is $3 \cdot \binom{10}{3}\binom{10}{0}\binom{10}{0} = 360$.

d) How many choices of three people are there in which one of the three people comes from group A, one comes from group B, and one comes from group C?

Since there are 10 people in each of the groups and we want 1 from each group, we need to choose 1 from group A, 1 from group B, and 1 from group C. Thus, there are $\binom{10}{1}\binom{10}{1}\binom{10}{1} = 10 \cdot 10 \cdot 10 = 1000$ ways to choose a group of 3 people containing 1 from each group.

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15. Suppose that five cards are selected at random without replacement from a standard deck of 52 playing cards.

a) Find the probability that the five cards selected include exactly three face cards (Jack, Queen, King).
b) Find the probability that the five cards selected include exactly three hearts and exactly two spades.
c) Find the probability that the five cards selected include exactly two clubs and exactly two diamonds.
d) Find the probability that the five cards selected consist of three sevens and two non–sevens which are not of the same kind.

Solution

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a) Find the probability that the five cards selected include exactly three face cards (Jack, Queen, King). We need to choose 3 of the 12 face cards and 2 of the 40 other cards.

$$\frac{\binom{12}{3}\binom{40}{2}}{\binom{52}{5}} = \frac{220 \cdot 780}{2598960}$$

b) Find the probability that the five cards selected include exactly three hearts and exactly two spades. We need to choose 3 of the 13 hearts, 2 of the 13 spades, and 0 of the 26 other cards.

$$\frac{\binom{13}{3}\binom{13}{2}\binom{26}{0}}{\binom{52}{5}} = \frac{286 \cdot 78 \cdot 1}{2598960}$$

c) Find the probability that the five cards selected include exactly two clubs and exactly two diamonds. We need to choose 2 of the 13 clubs, 2 of the 13 diamonds, and 1 of the 26 other cards.

$$\frac{\binom{13}{2}\binom{13}{2}\binom{26}{1}}{\binom{52}{5}} = \frac{78 \cdot 78 \cdot 26}{2598960}$$

d) Find the probability that the five cards selected consist of three sevens and two non–sevens which are not of the same kind. First we need to choose 2 of the 12 face–values (kinds) other than seven. There are $\binom{12}{2} = 66$ ways to do this. Once these two face–values are selected we need to choose 3 of the 4 sevens, 1 of the 4 cards for each of the two non–seven face–values selected earlier, and 0 of the other 40 cards.

$$\binom{12}{2}\frac{\binom{4}{3}\binom{4}{1}\binom{4}{1}\binom{40}{0}}{\binom{52}{5}} = \frac{66 \cdot 4 \cdot 4 \cdot 4 \cdot 1}{2598960}$$

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16. A box contains 100 balls of which 20 are red, 30 are white, and 50 are blue. If 6 balls are selected at random with replacement from this box, find the following.

a) The probability that exactly 3 of the 6 balls are red.

b) The probability that 2 of the 6 balls are red and 4 are white.

c) The probability that 1 of the 6 balls is red, 2 are white, and 3 are blue.

Solution

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a) The probability that exactly 3 of the 6 balls are red. This is a binomial probability with 6 trials/draws and success (red) probability $p = 20/100$.

$$\binom{6}{3}\left(\frac{20}{100}\right)^3\left(\frac{80}{100}\right)^3$$

b) The probability that 2 of the 6 balls are red and 4 are white. This is a multinomial probability with the 6 trials/draws split into 2, 4, and 0 and with probabilities 20/100, 30/100, and 50/100. We need 2 red balls, 4 white balls, and 0 blue balls.

$$\binom{6}{2}\binom{4}{4}\left(\frac{20}{100}\right)^2\left(\frac{30}{100}\right)^4\left(\frac{50}{100}\right)^0$$

c) The probability that 1 of the 6 balls is red, 2 are white, and 3 are blue. This is multinomial as in part b. We need 1 red ball, 2 white balls, and 3 blue balls.

$$\binom{6}{1}\binom{5}{2}\binom{3}{3}\left(\frac{20}{100}\right)^1\left(\frac{30}{100}\right)^2\left(\frac{50}{100}\right)^3$$

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17. **This is the without replacement version of problem 16.** A box contains 100 balls of which 20 are red, 30 are white, and 50 are blue. If 6 balls are selected at random without replacement from this box, find the following.

a) The probability that exactly 3 of the 6 balls are red.

b) The probability that 2 of the 6 balls are red and 4 are white.

c) The probability that 1 of the 6 balls is red, 2 are white, and 3 are blue.

Solution

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a) The probability that exactly 3 of the 6 balls are red. We need to choose 3 of the 20 red balls and 3 of the 80 non–red balls.

$$\frac{\binom{20}{3}\binom{80}{3}}{\binom{100}{6}} = \frac{1140 \cdot 82160}{1192052400}$$

b) The probability that 2 of the 6 balls are red and 4 are white. We need to choose 2 of the 20 red balls, 4 of the 30 white balls, and 0 of the 50 blue balls.

$$\frac{\binom{20}{2}\binom{30}{4}\binom{50}{0}}{\binom{100}{6}} = \frac{190 \cdot 27405 \cdot 1}{1192052400}$$

c) The probability that 1 of the 6 balls is red, 2 are white, and 3 are blue. We need to choose 1 of the 20 red balls, 2 of the 30 white balls, and 3 of the 50 blue balls.

$$\frac{\binom{20}{1}\binom{30}{2}\binom{50}{3}}{\binom{100}{6}} = \frac{20 \cdot 435 \cdot 19600}{1192052400}$$

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18. Consider an elevator with five passengers. Suppose that this elevator stops at ten floors. Also assume that each passenger chooses the floor at which he or she gets off at random and independently of the other passengers. Find the probability that no two passengers get off at the same floor. **Hint:** You can think of the assignment of floors to passengers as the selection, at random and with replacement, of 5 numbers from the set $\{1, 2, \ldots, 10\}$.

Solution

This problem can be viewed as a variation of the Birthday Problem with 10 replacing 365 and 5 serving as the number of people in the group. We will use an ordered 5–tuple to represent the floor choices of the 5 passengers, *e.g.*, $(1, 3, 7, 2, 8)$ indicates that passenger 1 got off at floor 1, passenger 2 got off at floor 3, passenger 3 got off at floor 7, passenger 4 got off at floor 2, and passenger 5 got off at floor 8.

There are $10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$ ways for the 5 passengers to get off the elevator.
There are $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6$ ways for the 5 passengers to get off at 5 different floors.
Thus, if each passenger chooses a floor at random and independently of the other passengers, then the probability that no two passengers get off at the same floor is

$$\frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{10 \cdot 10 \cdot 10 \cdot 10 \cdot 10} = \frac{30240}{100000} = .3024$$
