7.10 Problems and Solutions (last update 29 December 2018)

1. Let X denote a discrete random variable with probability mass function (p.m.f.)

$$p(x) = \begin{cases} 2c & \text{for } x = 1 \\ 4c & \text{for } x = 2 \\ 6c & \text{for } x = 3 \\ 8c & \text{for } x = 4 \\ 7c & \text{for } x = 5 \\ 5c & \text{for } x = 6 \\ 3c & \text{for } x = 7 \\ 1c & \text{for } x = 8 \\ 0 & \text{otherwise.} \end{cases}$$

where c is a suitable constant. **Do not round off – express all values as rational fractions.**

a) Find the value of c for which this is a valid probability mass function.

b) Find the probability $\mathrm{P}(X \leq 5)$.

c) Find the probability $\mathrm{P}(2 < X \leq 5)$.

d) Find the expected value of X.

e) Find the expected value of X^2.

f) Find the variance of X.

Solution

a) We need to find the constant c such that each of these 8 non–zero values of $p_X(x)$ is positive and their sum $(2+4+6+8+7+5+3+1)c = 36c$ is one. The solution is $c = \frac{1}{36}$.

b)

$$\begin{aligned} P(X \leq 5) &= P(X = 1, 2, 3, 4, 5) = p_X(1) + p_X(2) + p_X(3) + p_X(4) + p_X(5) \\ &= \frac{2}{36} + \frac{4}{36} + \frac{6}{36} + \frac{8}{36} + \frac{7}{36} = \frac{27}{36} = \frac{3}{4} \end{aligned}$$

c)

$$P(2 < X \leq 5) = P(X = 3, 4, 5) = p_X(3) + p_X(4) + p_X(5) = \frac{6}{36} + \frac{8}{36} + \frac{7}{36} = \frac{21}{36} = \frac{7}{12}$$

d)

$$\begin{aligned} E(X) &= (1)\frac{2}{36} + (2)\frac{4}{36} + (3)\frac{6}{36} + (4)\frac{8}{36} + (5)\frac{7}{36} + (6)\frac{5}{36} + (7)\frac{3}{36} + (8)\frac{1}{36} \\ &= \frac{2+8+18+32+35+30+21+8}{36} = \frac{154}{36} = \frac{77}{18} \end{aligned}$$

e)

$$\begin{aligned} E(X^2) &= (1)^2\frac{2}{36} + (2)^2\frac{4}{36} + (3)^2\frac{6}{36} + (4)^2\frac{8}{36} + (5)^2\frac{7}{36} + (6)^2\frac{5}{36} + (7)^2\frac{3}{36} + (8)^2\frac{1}{36} \\ &= \frac{2+16+54+128+175+180+147+64}{36} = \frac{766}{36} = \frac{383}{18} \end{aligned}$$

f)

$$\text{var}(X) = E(X^2) - [E(X)]^2 = \frac{383}{18} - \left(\frac{77}{18}\right)^2 = \frac{965}{324}$$

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2. Consider a box containing 3 red balls and 7 white balls. Suppose that balls are drawn one at a time, at random, without replacement from this box until two red balls are obtained. Let X denote the number of the draw on which the second red ball is obtained.

a) Find the probability mass function (p.m.f.) of X.

b) Find the expected value of X.

c) Find the expected value of X^2.

d) Find the variance of X.

Solution

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a) The possible values of X are 2,3,4,5,6,7,8, and 9 ($\Omega_X = \{2,3,4,5,6,7,8,9\}$). For $x \in \Omega_X$, in order to have $X = x$ we must observe 1 red ball and $x-2$ white balls among the first $x-1$ draws and we must obtain a red ball on draw x. For example, letting R denote a red ball, W denote a white ball, $\{RR\}$ corresponds to $X = 2$ and $\{RWR, WRR\}$

corresponds to $X = 3$. Thus

$$p_X(x) = \begin{cases} \frac{\binom{3}{1}\binom{7}{x-2}}{\binom{10}{x-1}} \cdot \frac{2}{10-x+1} & \text{for } x = 2, 3, \ldots, 9 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 4/60 & \text{if } x = 2 \\ 7/60 & \text{if } x = 3 \\ 9/60 & \text{if } x = 4 \\ 10/60 & \text{if } x = 5 \\ 10/60 & \text{if } x = 6 \\ 9/60 & \text{if } x = 7 \\ 7/60 & \text{if } x = 8 \\ 4/60 & \text{if } x = 9 \\ 0 & \text{otherwise} \end{cases}$$

b) Exploiting the symmetry in this p.m.f., we have

$$E(X) = \frac{[2+9] \cdot 4 + [3+8] \cdot 7 + [4+7] \cdot 9 + [5+6] \cdot 10}{60} = \frac{11}{2} = 5.5$$

That was the hard way to compute this! This distribution is symmetric on $\Omega_X = \{2, 3, 4, 5, 6, 7, 8, 9\}$ so $E(X) = \frac{2+9}{2} = 5.5$.

c) Similarly

$$E(X^2) = \frac{[4+81] \cdot 4 + [9+64] \cdot 7 + [16+49] \cdot 9 + [25+36] \cdot 10}{60} = \frac{341}{10} = 34.1$$

d) Thus

$$\text{var}(X) = E(X^2) - [E(X)]^2 = 34.1 - (5.5)^2 = 3.85$$

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3. Assume that the probability that a letter will be delivered within three working days is .9. Suppose that 10 letters inviting friends for a dinner party are sent out on Tuesday. Everyone who receives the invitation by Friday (*i.e.*, within 3 working days) will come. Those who do not receive the invitation by Friday will not come. Let X denote the number of friends who come to dinner.

(a) What is the name of the distribution of X?
(b) Find the probability that at least 7 friends will come.
(c) What are the expected value and variance of X.
(d) If the caterer charges a base fee of \$100 plus \$10 for each guest who comes to the dinner party. What is the expected value and variance of the catering cost?

Solution

a) We will assume that the events "letter 1 is delivered within 3 days", . . ., "letter 10 is delivered within 3 days" are independent so that we can model these events as Bernoulli trials with success probability $p = .9$. Under this assumption, the distribution of X is binomial with $n = 10$ and $p = .9$.

b) First we provide a direct calculation using the p.m.f.

$$\begin{aligned} P(X \geq 7) &= P(X = 7) + P(X = 8) + P(X = 9) + P(X = 10) \\ &= \binom{10}{7}(.9)^7(.1)^3 + \binom{10}{8}(.9)^8(.1)^2 + \binom{10}{9}(.9)^9(.1)^1 + \binom{10}{10}(.9)^{10}(.1)^0 \\ &\approx .0574 + .1937 + .3874 + .3487 = .9872 \end{aligned}$$

Using the c.d.f and a calculator, we have

$$P(X \geq 7) = 1 - P(X \leq 6) \approx 1 - .0128 = .9872$$

c) $E(X) = np = 10(.9) = 9$ and $\text{var}(X) = np(1-p) = 10(.9)(.1) = .9$.

d) Let Y denote the catering cost (in dollars), then
$Y = 100 + 10X$,
$E(Y) = 100 + 10E(X) = 190$, and
$\text{var}(Y) = (10)^2\text{var}(X) = 90$.
Thus the expected value of the catering cost is \$190 and the variance of the catering cost is \$90.

4. A jewelry dealer is considering the sale of an antique necklace. She estimates that there is a 20% chance that she will make a profit of \$250 on the sale, a 40% chance that she will make a profit of \$150 on the sale, a 30% chance that she will break even on the sale, and a 10% chance that she will lose \$50 on the sale. What is her expected profit?

Solution

Let X denote her profit and let $p(x)$ denote the probability that $X = x$. The values of x, $p(x)$, $xp(x)$ and the desired expected profit are given in the following table.

x	$p(x)$	$xp(x)$
250	.20	50
150	.40	60
0	.30	0
-50	.10	-5
		$E(X) = 50 + 60 + 0 - 5 = 105$

Thus, her expected profit is \$105.

5. A manufacturer ships parts in lots of 1000 and makes a profit of \$50 per lot sold. The purchaser, however, subjects the product to a sampling plan as follows: 10 parts are selected at random with replacement. If none of these parts is defective, the lot is purchased; if one part is defective, the lot is purchased but the manufacturer returns \$10 to the buyer; if two or more parts are found to be defective, the entire lot is returned at a net loss of \$25 to the manufacturer. What is the manufacturers expected profit if 10% of the parts are defective?

Solution

As noted, we will assume that 10% of the parts in the lot of 1000 are defective. Let X denote the number of defective parts in the random sample of size $n = 10$ and let Y denote the corresponding profit. Since the sample is selected with replacement, X follows the binomial distribution with parameters $n = 10$ and $p = .10$.

$$P(X = 0) = \binom{10}{0} \cdot .1^0 \cdot .9^{10} = 1 \cdot 1 \cdot .9^{10} = .9^{10}$$

$$P(X = 1) = \binom{10}{1} \cdot .1^1 \cdot .9^9 = 10 \cdot .1 \cdot .9^9 = .9^9$$

$$P(X \geq 2) = 1 - P(X = 0) - P(X = 1) = 1 - .9^{10} - .9^9$$

Note that: if $X = 0$, then $Y = 50$; if $X = 1$, then $Y = 40$; and, if $X \geq 2$, then $Y = -25$. Thus, the manufacturer's expected profit, $E(Y)$, is

$$E(Y) = 50 \cdot .9^{10} + 40 \cdot .9^9 - 25\left[1 - .9^{10} - .9^9\right] \approx 26.33$$

That is, the manufacturer's expected profit is \$26.33.

6. A jury trial ended in a hung jury with 8 jurors voting guilty and 4 jurors voting for acquittal. Suppose that the 12 jurors exit the courtroom in random order. How many of the first four jurors to leave the courtroom would be expected to have voted for acquittal? In other words, what is the expected value of X, when X denotes the number of jurors out of the first four jurors to leave the courtroom who voted for acquittal?

Solution

Since X, the number of jurors out of the first four jurors to leave the courtroom who voted for acquittal, is determined by randomly selecting four different jurors from a collection of $A + B = 12$ jurors of whom $B = 8$ voted guilty and $A = 4$ voted for acquittal, X follows the hypergeometric distribution with $A = 4$, $B = 8$, and $n = 4$. We know that the expected value of such a hypergeometric random variable is

$$E(X) = n \cdot \left(\frac{A}{A + B} \right) = 4 \cdot \left(\frac{4}{4 + 8} \right) = \frac{4}{3}$$

That is, in the on average in the long run sense of expected values, out of the first four jurors who exit we would expect 4/3 (approximately one) to have voted for acquittal. In other words, if we imagine repeating this process of asking the first four jurors how they voted in such a way that every possible order in which the twelve could exit occurs, then the average number who voted for acquittal would be 4/3.

7. A salesman has scheduled two sales appointments. The probability that he will be able to close the deal at his first appointment is .75. The probability that he will be able to close the deal at his second appointment is .45. If he closes the deal at the first appointment he will earn a commission of $1,000 and if he closes the deal at the second appointment he will earn a commission of $1,500. Assuming that the outcomes of the deals at the two appointments are independent, what is the salesman's expected profit?

Solution

The details of this problem are outlined in the following table, where X is his profit. You may find a tree diagram helpful here.

outcome	probability	x	$xp(x)$
close first deal only	(.75)(1-.45) = .4125	1000+0=1000	412.5
close first and second deals	(.75)(.45) = .3375	1000+1500=2500	843.75
close second deal only	(1-.75)(.45) = .1125	0+1500=1500	168.75
close neither deal	(1-.75)(1-.45) = .1375	0+0=0	0

$$E(X) = 412.5 + 843.75 + 168.75 + 0 = 1425$$

As shown above his expected profit is $1425.