4: Single Cash Flows and Equivalence

This chapter explains basic concepts of project economics by examining single cash flows. This means that each term in the balance equation is treated individually, and no assumptions are made that require special patterns of cash flows, such as the uniform series or trends shown in Figure 4.1. The role of single cash flows as equivalents is examined, where two sets of cash flows are equivalent if they result in the same final balance. Equivalence is a fundamental concept central to understanding material in the following chapters.

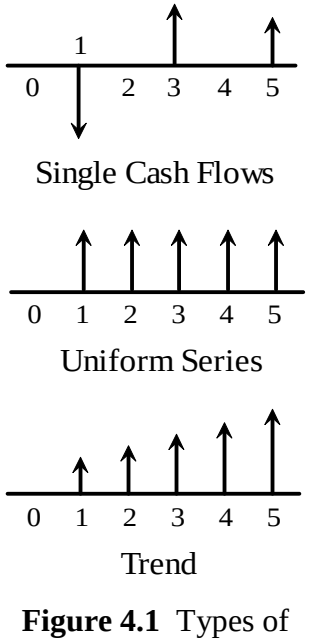

Figure 4.1 Types of Cash Flows

The next section provides basic concepts for two types of equivalents and explains how they are useful in solving practical problems. Then the following sections investigate these equivalents in more detail. All of this material is developed first within the context of an account paying a single rate of interest, where investment funds come from and return to this account. Then the results are provided for an account paying more than one rate of interest. Later chapters will extend all results to industrial and governmental environments.

4.1 Basic Concepts

This section presents basic concepts and applications of two types of equivalents, compound amounts and discounted amounts.

Compound Amounts

Compound amounts equal prior cash flows plus the interest they have earned. At 10% interest, the \$110 shown in Figure 4.2 equals \$100(1.1), the \$100 plus the interest it has earned from time 1 to time 2. Similarly, \$121 equals \$110(1.1) or \$100$(1.1)^2$, the \$100 plus the interest it has earned from time 1 to time 3. The \$110 and \$121 are compound amounts of the \$100, and the \$121 is a compound amount of the \$110.

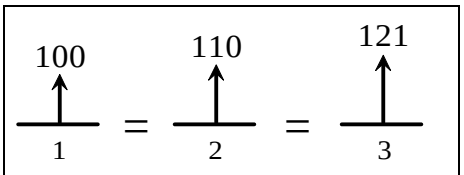

Figure 4.2 Equivalents at 10%

Compound amounts are equivalents, because any of these cash flows would contribute \$121 to the balance at time 3. Their contributions to any balance thereafter would also be equal, increasing by 10% per year. For example, any of these cash flows would contribute \$133.10 (121×1.1) to the balance at time 4.

Discounted Amounts

Discounting is the opposite of compounding because it removes interest instead of adding it. Compounding adds interest by multiplying by $(1+i)$ or γ the first year, γ^2 the second year, and so on. Discounting removes interest by dividing by these factors. For example,

$121 discounted for 1 period at 10% equals $121/1.1 or $110, so $110 is the discounted amount of $121 for a 1 year period. Usually the division is written using negative exponents, so $121(1.1)^{-2}$ or $100 is the discounted of amount $121 for a 2 year period.

A discounted amount grows into the original cash flow as it accrues interest, so it produces the same balance. For example, the $100 accrues interest to contribute $110 to an account after 1 period and $121 after 2 periods. Thus a discounted amount is equivalent to the original cash flow.

Applications

Figure 4.3 illustrates the type of problem in which equivalents are useful. A student realizes that depending on social security might mean working forever, so she plans to provide for herself. She estimates that she will need $50,000 per year beginning at age 66 and continuing through age 85. She needs to know how much to save at 8% each year from ages 22 through 65. This problem can be solved using the balance equation:

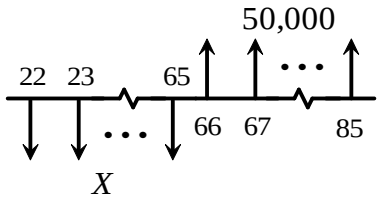

Figure 4.3 Retirement Plan

$$0 = 0 + X(1.08)^{85-22} + X(1.08)^{85-23} + \cdots + X(1.08)^{85-65} - 50{,}000(1.08)^{85-66} - \cdots - 50{,}000(1.08) - 50{,}000 \,. \qquad (4\text{-}1)$$

The final balance is $0, since there will be no funds left after the last withdrawal. The account starts with $0 in it, and then X is deposited yearly beginning at age 22 until the withdrawals start. The only unknown is X, so the equation is solvable, but the computations are formidable. Alternatively, such problems with patterns of cash flows are solved easily by breaking them into steps based on this chapter's coverage of equivalence and upcoming material on patterned flows. The unknown yearly deposit equals $1,375.28.

The following sections show how to compute compound amounts and discounted amounts that are equivalent to the original cash flows. This allows the original cash flows to be replaced by a single, equivalent cash flow that simplifies the problem.

4.2 Compound Amounts

This section shows how to compute equivalent compound amounts and use them in multi-step problems, and then it introduces standard notation.

Compound Amount Formula

Figure 4.4 shows the compound amount E_t of the cash flows c_r, …, c_s, where

$$0 \le r \le s \le t \le n \qquad (4\text{-}2)$$

and n is the length of the planning horizon. Developing a formula for E_t begins by writing one equation for the balance at time n for c_r, …, c_s and then another one for E_t. Then equate these balances since E_t is an equivalent, and solve for E_t. There can be cash flows other than c_r, …, c_s, so represent any flows that are not part of the equivalence calculation as x_j. The two balance equations are

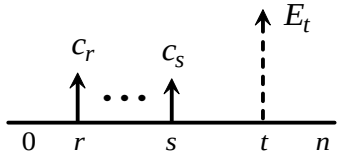

Figure 4.4 Compound Amount of Several Flows

$$B_n = A\gamma^n + x_0\,\gamma^{n\text{-}0} + \cdots + x_{n\text{-}1}\,\gamma^1 + x_n + c_r\,\gamma^{n\text{-}r} + \cdots + c_s\,\gamma^{n\text{-}s} \tag{4-3}$$

and

$$B_n = A\gamma^n + x_0\,\gamma^{n\text{-}0} + \cdots + x_{n\text{-}1}\,\gamma^1 + x_n + E_t\,\gamma^{n\text{-}t}\,. \tag{4-4}$$

Set the right-hand sides of these equations equal to each other and solve for E_t to obtain the *compound amount formula*:

$$\boxed{E_t \;=\; c_r\gamma^{\,t\text{-}r} + \cdots + c_s\,\gamma^{\,t\text{-}s}} \tag{4-5}$$

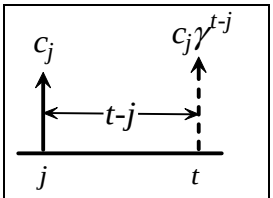

Figure 4.5 Compounding Exponent

Each exponent equals the number of times its cash flow has been compounded, as shown in Figure 4.5, and each term equals the amount to which its cash flow has grown by time *t*. A simple verbal expression of the compound amount formula is:

$$\textit{Compound Amount} = \Sigma\,\textit{Cash Flow}(1{+}i)^{\textit{Periods between cash flow and compound amount}} \tag{4-6}$$

The development of the compound amount formula insures that either E_t or the original cash flows will have the same effect on balances after time *t*, such as time *n*. This allows the equivalent to be substituted for the original cash flows in a multi-step problem. The following examples compute an initial equivalent compound amount, and then use it as part of a multi-step problem. In the next chapter, this same logic will be combined with special formulas for uniform series to solve easily problems such as the one in Figure 4.3.

Example 4.1 *One Step Compound Amount*

Figure 4.6 shows an equivalent compound amount at 10% interest. The compound amount formula indicates that the value is

$$E_7 = 4.95 = -100(1.1)^{7\text{-}4} + 55(1.1)^{7\text{-}5} + 65(1.1)^{7\text{-}6} \tag{4-7}$$

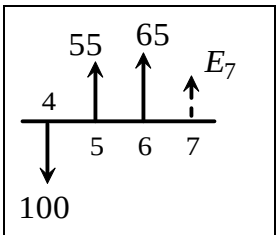

Figure 4.6 Initial Compound Amount

Either the original cash flows or the equivalent will contribute the same amount to the balance at time 7, so they are equivalent for any initial deposit *A*:

$$B_7 = A(1.1)^7 - 100(1.1)^{7\text{-}4} + 55(1.1)^{7\text{-}5} + 65(1.1)^{7\text{-}6} \tag{4-8}$$

or

$$B_7 = A(1.1)^7 + 4.95 \qquad \clubsuit \tag{4-9}$$

Example 4.2 *Multiple Step Compound Amount*

Compute the compound amount at time 10 of the cash flows shown in Figure 4.6 by using the equivalent at time 7. Replace the cash flows in Figure 4.6 with their equivalent, as shown in Figure 4.7. It is immaterial whether the $4.95 is in the account as a consequence of the original cash flows plus their interest or as a new cash flow at time 7. In either case, use the compound amount formula to determine the contribution of this equivalent to the balance at time 10:

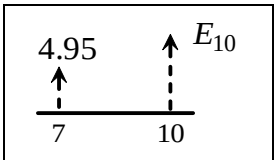

Figure 4.7 Final Compound Amount

$$E_{10} = 6.59 = 4.95(1.1)^{10\text{-}7} \tag{4-10}$$

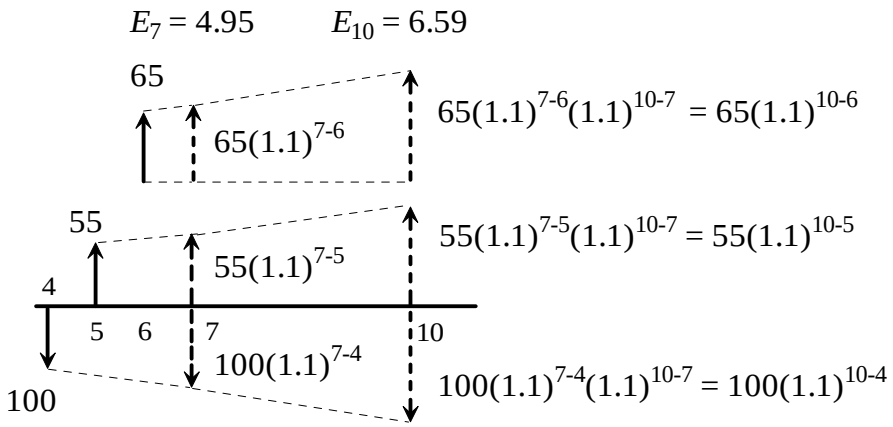

Figure 4.8 Compound Amount as an Intermediate Solution

The balance equation verifies the contribution of 6.59 by the original cash flows to the balance at time 10 since

$$B_{10} = A(1.1)^{10} - 100(1.1)^{10-4} + 55(1.1)^{10-5} + 65(1.1)^{10-6} \qquad (4\text{-}11)$$

or $A(1.1)^{10} + 6.59$. Figure 4.8 shows how E_7 grows into E_{10}. Each component of E_{10} ultimately has an exponent equal to 10 minus the time of its cash flow. ♣

Notation for Compound Amounts of Single Cash Flows

It is fairly easy to use algebraic notation for the compound amounts of single cash flows, such as $c_j\gamma^{n-j}$, but that notation is awkward for more complex cash flows, such as uniform series or trends. The American Society for Engineering Education uses the following notation for *single payment compound amount factor:*

$$\boxed{(F|P,i,m) = (1+i)^m} \qquad (4\text{-}12)$$

It computes the compound amount after m periods of a single cash flow. It frequently is referred to as "*F given P*," and its letters indicate the operation being performed: computing a *F*uture compound amount, *given* (designated by the vertical bar) a *P*rior cash flow, an interest rate i, and a number of compounding periods m. Figure 4.9 illustrates its use. Notice that the last parameter is written as the difference of the future and prior positions, clearly indicating the logic. This is a good practice since it makes it easier to check for errors. Appendix A contains the values of this factor and other ones in several tables. Each table is for a different interest rate. For example, there is one table for 5% and another one for 10%. The tables contain columns for each factor and rows for commonly used compounding periods.

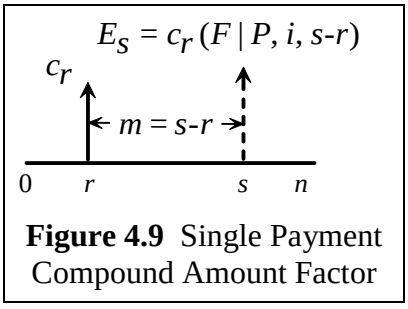

Figure 4.9 Single Payment Compound Amount Factor

The tables contain fewer significant digits than calculators or computers, so answers can differ slightly depending on whether the formulas or the tabular values are used. As a rough rule of thumb, answers should be within roughly 1% of each other, regardless of the computational procedure. This is noted in the following example.

Example 4.3 *Single Payment Compound Amount Factor Notation*

With interest at 10%, use the standard notation to compute E_{10}, the equivalent at time 10 of the cash flows at times 4, 5, and 6 shown in Figure 4.6. In each case, an amount further in the *F*uture is being computed given *P*rior cash flows, so "*F* given *P*" is the appropriate factor. The number of periods for each factor equals the number of compounding periods between each cash flow and the compound amount, so the compound amount equation becomes:

$$E_{10} = -100(F|P, 10\%, 10-4) + 55(F|P, 10\%, 10-5) + 65(F|P, 10\%, 10-6). \quad (4\text{-}13)$$

Using the values of the factors Appendix A results in

$$E_{10} = 6.58 = -100(1.7716) + 55(1.6105) + 65(1.4641)\,. \quad (4\text{-}14)$$

This is 0.15% smaller than the prior result of 6.59 computed with a calculator. ♣

4.3 Discounted Amounts

This section first develops a general formula for discounted amounts, and then it shows how discounted amounts are used to compute the amount that can be borrowed for a specified repayment. The last subsection introduces standard notation.

Discounted Amount Formula

The development of a general discounted amount formula is very similar to that of the compound amount formula. Figure 4.10 shows the placement of the equivalent and the actual cash flows, where

$$0 \le r \le s \le t \le n \quad (4\text{-}15)$$

and n is the length of the planning horizon. Represent any cash flows other than $c_s, \ldots, c_t$ as x_j, so the balance equation for the original cash flows is

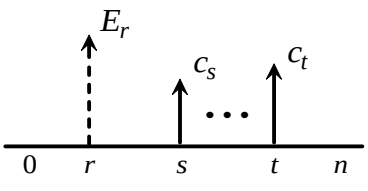

Figure 4.10 Discounted Amount of Several Flows

$$B_n = A\gamma^n + x_0\,\gamma^{n-0} + \cdots + x_{n-1}\,\gamma^1 + x_n + c_s\,\gamma^{n-s} + \cdots + c_t\,\gamma^{n-t}\,, \quad (4\text{-}16)$$

and the balance equation for the equivalent discounted amount is

$$B_n = A\gamma^n + x_0\,\gamma^{n-0} + \cdots + x_{n-1}\,\gamma^1 + x_n + E_r\,\gamma^{n-r}\,. \quad (4\text{-}17)$$

Equate the balances and solve for E_r to obtain the *discounted amount formula*:

$$\boxed{E_r = c_s\,\gamma^{-(s-r)} + \cdots + c_t\,\gamma^{-(t-r)}} \quad (4\text{-}18)$$

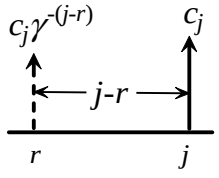

Figure 4.11 Discounting Exponent

Each exponent equals the negative of the number of times its cash flow has been discounted, as shown in Figure 4.11, and each term shows the amount to which its cash flow is reduced by time r. A simple verbal expression of the discounted amount formula is:

$$\textit{Discounted Amount} = \Sigma\,\textit{Cash Flow}(1+i)^{-\textit{Periods between cash flow and discounted amount}} \quad (4\text{-}19)$$

The development of the discounted amount formula insures that either E_r or the original cash flows will have the same effect on later balances, such as at time n. This

allows the equivalent to be substituted for the original cash flows in a multi-step problem, as illustrated by the following two examples.

Example 4.4 *One Step Discounted Amount*

Figure 4.12 shows that E_2 is the discounted amount at time 2 of the cash flows at times 4, 5, and 6. The discounted amount formula with interest at 10% provides the value of E_2,

$$E_2 = \$3.07 = -100(1.1)^{-(4-2)} + 55(1.1)^{-(5-2)} + 65(1.1)^{-(6-2)} . \quad (4\text{-}20)$$

A single cash flow of \$3.07 at time 2 produces the same balance as the original cash flows. For example, equation (4-9) shows that the balance of the original cash flows at time 7 is $A(1.1)^7 + 4.95$. Applying the balance equation to the discounted amount produces the same result:

$$B_7 = A(1.1)^7 + 3.07(1.1)^{7-2} , \quad (4\text{-}21)$$

$$B_7 = A(1.1)^7 + 4.95. \quad (4\text{-}22)$$

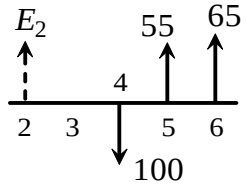

Figure 4.12 Initial Discounted Amount

Thus E_2 is equivalent to the original cash flows, as well as the compound amounts computed earlier. They all result in the same balance. ♣

Example 4.5 *Multiple Step Discounted Amount*

Compute the discounted amount at time 0 of the cash flows shown in Figure 4.12 by using the equivalent at time 2. As before, replace the cash flows in Figure 4.12 with their equivalent, as shown in Figure 4.13. Now use the discounted amount formula to determine the equivalent at time 0 of \$3.07 at time 2:

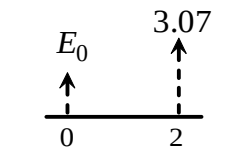

Figure 4.13 Final Discounted Amount

$$E_0 = 2.54 = 3.07(1.1)^{-(2-0)} \quad (4\text{-}23)$$

This new equivalent produces the same balance at time 7 as the earlier ones:

$$B_7 = A(1.1)^7 + 2.54(1.1)^{7-0} , \quad (4\text{-}24)$$

$$B_7 = A(1.1)^7 + 4.95 . \quad (4\text{-}25)$$

Figure 4.14 shows how E_2 discounts into E_0. Each component of E_0 ultimately has an exponent equal to the negative of 7 minus the time of its cash flow. ♣

Discounted Amounts and Future Payments

A common application of discounted amounts is to determine how much should be paid for a future income stream. For example, a bank lending money for a car or a home is buying a contract for future payments. Bonds and annuities are also routine contracts in which future payments are purchased. Similarly, someone purchasing a business hopes to buy a stream of future income. This section shows why purchasing future payments for their discounted amount allows the buyer to earn interest at the rate used for discounting, known as the *discount rate*.

It is obvious that someone wanting to earn a rate of return of 10% would buy a payment of \$110 after 1 period for its discounted amount, \$100. In general, consider pur-

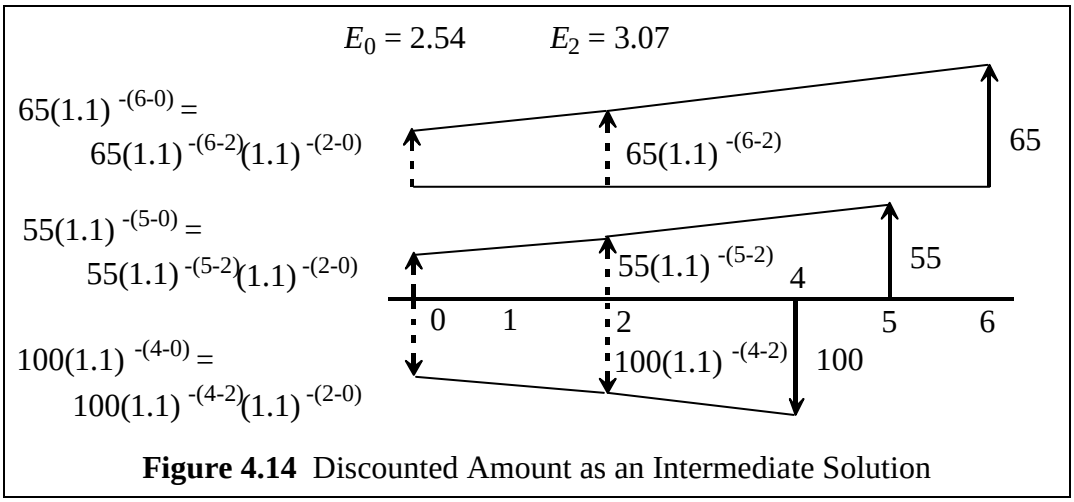

Figure 4.14 Discounted Amount as an Intermediate Solution

chasing the future stream of payments $c_s, \ldots, c_t$ shown in Figure 4.10 for their discounted amount E_r. The equation for the discounted amount can be written as

$$E_r = c_s \gamma^{-(s-r)} + c_{s+1} \gamma^{-(s+1-r)} + \cdots + c_{t-1} \gamma^{-(t-1-r)} + c_t \gamma^{-(t-r)} \ . \qquad (4\text{-}26)$$

Multiply both sides by γ^{t-r} and move the term with E_r to the right side to obtain:

$$0 = -E_r \gamma^{t-r} + c_s \gamma^{t-s} + c_{s+1} \gamma^{t-(s+1)} + \cdots + c_{t-1} \gamma + c_t \ . \qquad (4\text{-}27)$$

This is the same as a balance equation at time t for \$0 in initial assets, a loan of amount E_r at rate i, and payments of $c_s, \ldots, c_t$ that repay the loan since the balance is \$0. A purchaser or the lender earns interest at a rate of i per period, and a loan of a discounted amount will be repaid by the original cash flows.

Another useful interpretation of discounted amounts results from looking at equation (4-27) from the viewpoint of a person making a deposit E_r in a savings account in exchange for future withdrawals or payments of $c_s, \ldots, c_t$ that empty the account. A deposit equal to the discounted amount allows withdrawals of the original cash flows.

Example 4.6 *Initial Loan Amount*

How much would a bank wanting to earn 1% per month on its loans be willing to lend today in exchange for payments of \$50 at year 1, \$70 at year 2, \$80 at year 3, and \$90 at year 4? This is shown in Figure 4.15, where the time axis is expressed in terms of the compounding period, months. The loan equals the discounted amount:

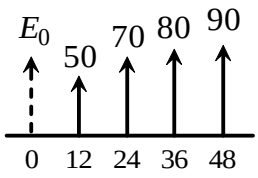

Figure 4.15 Loan

$$E_0 = \$211.24 = 50(1.01)^{-12} + 70(1.01)^{-24} + 80(1.01)^{-36} + 90(1.01)^{-48} \quad \clubsuit \qquad (4\text{-}28)$$

Example 4.7 *Loan Payoff*

Suppose that the borrower decides to pay off the loan immediately after paying \$70 at year 2. How much is owed at that time? The payments of \$80 at year 3 and \$90 at year 4 are worth their discounted amount at year 2 to the bank, as shown in Figure 4.16:

$$E_{24} = \$141.88 = 80(1.01)^{-(36-24)} + 90(1.01)^{-(48-24)} . \quad (4\text{-}29)$$

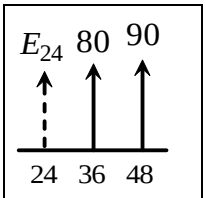

Figure 4.16 Payoff

If the bank loans the \$141.88 to someone else, then it still receives payments of \$80 at year 3 and \$90 at year 4. Issuing another loan involves administrative costs that banks prefer to avoid, so they sometimes charge a penalty for early repayments.

Another method to compute the payoff amount is to determine the loan's balance after the payment at year 2:

$$B_{24} = \$141.88 = 211.24(1.01)^{24-0} - 50(1.01)^{24-12} - 70 \quad (4\text{-}30)$$

Either procedure yields the same result, but discounting frequently involves fewer computations. ♣

Example 4.8 *Stock Purchase*

A student who wishes to earn 10% per year is considering purchasing a stock. The best estimates of the future cash flows are that the dividends paid yearly to stockholders will be \$1.50 per share in 1 year, \$1.75 in 2 years, and \$2.50 in 3 years. He plans on selling the stock immediately after receiving the dividend at year 3 for \$28.00 per share. What is the most that he should pay for each share of the stock? The estimated cash flows are shown in Figure 4.17. If today is time 0, then the most that should be paid for each share of the stock is:

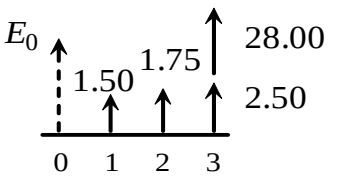

Figure 4.17 Stock

$$E_0 = \$25.73 = 1.50(1.1)^{-1} + 1.75(1.1)^{-2} + (2.50 + 28.00)(1.1)^{-3} \quad ♣ \quad (4\text{-}31)$$

Notation for Discounted Amounts of Single Cash Flows

Discounting provides equivalents closer to the present, so the discount factor known as the *single payment present worth factor*. The American Society for Engineering Education uses the notation:

$$\boxed{(P|F,i,m) = (1+i)^{-m}} \quad (4\text{-}32)$$

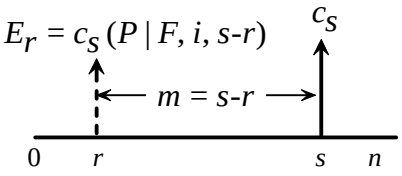

Figure 4.18 Single Payment Present Worth Factor

The factor frequently is referred to as "*P given F*." These letters indicate the quantity that is being computed: a *P*rior discounted amount, given a *F*uture cash flow, an interest rate *i*, and a number of compounding periods *m*. Figure 4.18 illustrates its use. Writing *m* as *s*-*r* clearly shows the discounting logic and is good practice.

Example 4.9 *Single Payment Present Worth Factor Notation*

With interest at 10%, use the standard notation to compute E_0, the equivalent at time 0 of the cash flows at times 4, 5, and 6 shown in Figure 4.12. In each case, a *P*rior amount is being computed given *F*uture cash flows, so "*P* given *F*" is the appropriate factor. The number of periods for each factor equals the number of compounding periods between each cash flow and the discounted amount, so the discounted amount equation becomes:

$$E_0 = -100(P|F, 10\%, 4-0) + 55(P|F, 10\%, 5-0) + 65(P|F, 10\%, 6-0). \quad (4\text{-}33)$$

Using the values of the factors Appendix A results in

$$E_0 = 2.54 = -100(0.6830) + 55(0.6209) + 65(0.5645), \quad (4\text{-}34)$$

which agrees with the previous calculator-based solution, in this case. ♣

4.4 Equivalents for Multiple Interest Rates

Sometimes interest rates change over time, so this section shows how to calculate compound and discounted amounts in that situation. The concepts for the constant rate and multiple rate cases are the same:

- Equivalents produce the same balance.
- Compound and discounted amounts are equivalents.
- Compound amounts equal one or more prior cash flows plus interest.
- Discounted amounts equal one or more future cash flows with interest removed.

The logic for developing the formulas is exactly the same and always starts with the balance equation. For the multiple rate case the balance at time n is

$$B_n = A\gamma_1\gamma_2\cdots\gamma_n + c_0\gamma_1\gamma_2\cdots\gamma_n + c_1\gamma_2\gamma_3\cdots\gamma_n + c_2\gamma_3\gamma_4\cdots\gamma_n + \cdots + c_{n-1}\gamma_n + c_n, \quad (4\text{-}35)$$

where γ_j equals $1+i_j$. Figure 4.19 shows that i_j is the rate during period j, from time $j-1$ to time j. The following sections summarize the development of the compound and discounted amount formulas for multiple interest rates.

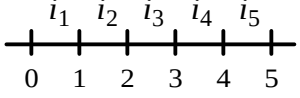

Figure 4.19 Multiple Rates

Compound Amounts

Figure 4.4 shows the compound amount E_t of several prior cash flow, $c_r,\ldots, c_s$; any other cash flows that might be present are denoted as x_j. As before, write the balance equations for the cash flows,

$$B_n = A\gamma_1\gamma_2\cdots\gamma_n + x_0\gamma_1\gamma_2\cdots\gamma_n + x_1\gamma_2\gamma_3\cdots\gamma_n + x_2\gamma_3\gamma_4\cdots\gamma_n + \cdots + x_{n-1}\gamma_n + x_n + c_r\gamma_{r+1}\gamma_{r+2}\cdots\gamma_n + \cdots + c_s\gamma_{s+1}\gamma_{s+2}\cdots\gamma_n \quad (4\text{-}36)$$

and the equivalent,

$$B_n = A\gamma_1\gamma_2\cdots\gamma_n + x_0\gamma_1\gamma_2\cdots\gamma_n + x_1\gamma_2\gamma_3\cdots\gamma_n + x_2\gamma_3\gamma_4\cdots\gamma_n + \cdots + x_{n-1}\gamma_n + x_n + E_t\gamma_{t+1}\gamma_{t+2}\cdots\gamma_n, \quad (4\text{-}37)$$

then set them equal to each other and solve for E_t to obtain the *multiple interest rate compound amount formula*:

$$\boxed{E_t = c_r\gamma_{r+1}\gamma_{r+2}\cdots\gamma_t + \cdots + c_s\gamma_{s+1}\gamma_{s+2}\cdots\gamma_t} \quad (4\text{-}38)$$

Example 4.10 *Compound Amount for Multiple Rates*

The compound amount at time 7 for the situation shown in Figure 4.20 is

$$E_7 = -100(1.09)(1.10)(1.11) + 55(1.10)(1.11) + 65(1.11) \quad (4\text{-}39)$$

or \$6.22. Each factor γ_j equals $1+i_j$, and it can be written as $(F|P, i_j, 1)$ using the standard notation, but the notation is not helpful in this instance. ♣

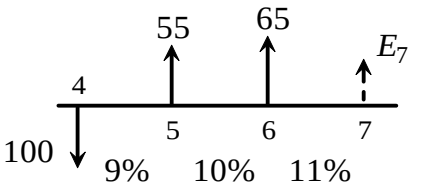

Figure 4.20 Multiple Rate Compound Amount

Example 4.11 *Multiple Step Compound Amount for Multiple Rates*

Use the results of the previous example to compute the compound amount at time 15 when the interest rate is 11% from time 7 through 10 and 12% from time 10 through 15. The original cash flows can be replaced by their equivalent, since either one contributes \$6.22 to the balance at time 7. This results in the problem shown in Figure 4.21. Apply the multiple rate compound amount formula with one factor for each period to obtain

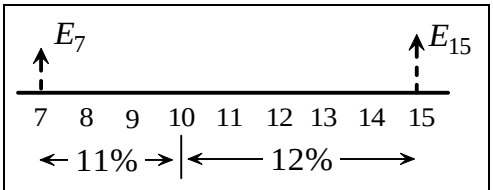

Figure 4.21 Multi-Step, Multiple Rate Compound Amount

$$E_{15} = 6.22(1.11)(1.11)(1.11)(1.12)(1.12)(1.12)(1.12)(1.12)\,, \tag{4-40}$$

so

$$E_{15} = 6.22(1.11)^{10-7}\,(1.12)^{15-10}\,. \tag{4-41}$$

In this situation, it is convenient to use standard notation, so the solution becomes

$$E_{15} = 6.22(F|P, 11\%, 10\text{-}7)(F|P, 12\%, 15\text{-}10) \tag{4-42}$$

or \$14.99. When compounding a cash flow or equivalent over regions in which the interest rate is constant, use one factor to compound it up to the end of the first region, another factor to compound it in the next region, and so forth. ♣

Discounted Amounts

Figure 4.10 shows the discounted amount E_r of several future cash flow, $c_s, \ldots, c_t$; any other cash flows that might be present are denoted as x_j. Write the balance equations for the cash flows,

$$B_n = A\gamma_1\gamma_2\cdots\gamma_n + x_0\gamma_1\gamma_2\cdots\gamma_n + x_1\gamma_2\gamma_3\cdots\gamma_n + x_2\gamma_3\gamma_4\cdots\gamma_n + \cdots + x_{n-1}\gamma_n + x_n$$
$$+ c_r\gamma_{r+1}\gamma_{r+2}\cdots\gamma_n + \cdots + c_s\gamma_{s+1}\gamma_{s+2}\cdots\gamma_n \tag{4-43}$$

and the equivalent,

$$B_n = A\gamma_1\gamma_2\cdots\gamma_n + x_0\gamma_1\gamma_2\cdots\gamma_n + x_1\gamma_2\gamma_3\cdots\gamma_n + x_2\gamma_3\gamma_4\cdots\gamma_n + \cdots + x_{n-1}\gamma_n + x_n$$
$$+ E_r\gamma_{r+1}\gamma_{r+2}\cdots\gamma_n\,, \tag{4-44}$$

then set them equal to each other and solve for E_r to obtain the *multiple interest rate discounted amount formula*:

$$\boxed{E_r = c_s(\gamma_s\gamma_{s-1}\cdots\gamma_{r+1})^{-1} + \cdots + c_t(\gamma_t\gamma_{t-1}\cdots\gamma_{r+1})^{-1}} \tag{4-45}$$

Previously developed results for loans and deposits involving discounted amounts in the single rate case are still applicable. A deposit equal to the discounted amount al-

lows withdrawals of the original cash flows, and a loan of a discounted amount will be repaid by the original cash flows.

Example 4.12 *Discounted Amount for Multiple Rates*

The discounted amount at time 12 for the situation shown in Figure 4.22 is

$$E_{12} = -100[(1.08)(1.07)]^{-1} + 55[(1.09)(1.08)(1.07)]^{-1} + 65[(1.10)(1.09)(1.08)(1.07)]^{-1} \quad (4\text{-}46)$$

or \$4.04. Writing each factor as $(P|F, i_j, 1)$ is not helpful in this instance. ♣

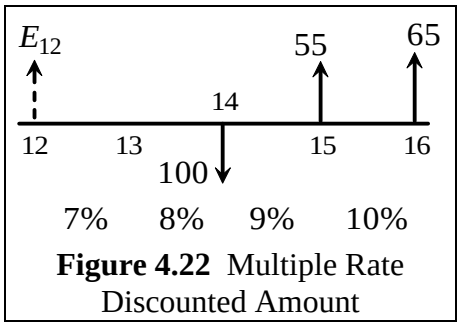

Figure 4.22 Multiple Rate Discounted Amount

Example 4.13 *Multi-Step Discounted Amount for Multiple Rates*

Use the results of the previous example to compute the discounted amount at time 4 when the interest rate is 5% from time 4 through 7 and 6% from time 7 through 12. The original cash flows can be replaced by their equivalent, since either \$4.04 at time 12 or the original cash flows contribute the same amount to the balance at time 16. This results in the problem shown in Figure 4.23. Apply the multiple rate discounted amount formula with one factor for each period to obtain

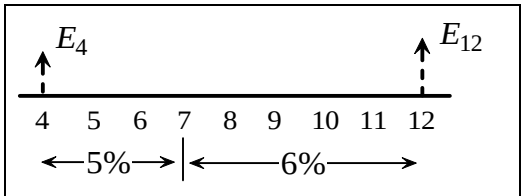

Figure 4.23 Multi-Step, Multiple Rate Discounted Amount

$$E_4 = 4.04[(1.06)(1.06)(1.06)(1.06)(1.06)(1.05)(1.05)(1.05)]^{-1}, \quad (4\text{-}47)$$

so

$$E_4 = 4.04(1.06)^{-(12-7)}(1.05)^{-(7-4)}. \quad (4\text{-}48)$$

In this situation, it is convenient to use standard notation, so the solution becomes

$$E_4 = 4.04(P|F, 6\%, 12\text{-}7)(P|F, 5\%, 7\text{-}4) \quad (4\text{-}49)$$

or \$2.61. When discounting a cash flow or equivalent over regions in which the interest rate is constant, use one factor to discount it back to the beginning of the first region, another factor to discount it in the next region, and so forth. ♣

4.5 Summary

This chapter uses single cash flows to explain fundamental concepts of equivalence. Two sets of cash flows are equivalent if they result in the same balance, and compound and discounted amounts are the two basic types of equivalents.

Compounding produces a future equivalent by adding interest to a cash flow. A compound amount is computed using the compound amount formula that sums of each cash flow's contribution to a future balance. A common application of compounding is determining how much will be in a savings account.

Discounting yields a prior equivalent by removing interest from future cash flows. In general, discounted amounts equal the amount that an investor will pay for future payments. They are computed using the discounted amount formula, and they are used to determine loan amounts, the remaining debt on a loan, or how much to deposit in an account.

Equivalents play an important role in multi-step problems involving patterns of cash flows that will be studied in the next chapter. Formulas are developed that easily compute equivalents of various patterns. Then replacing the original cash flows with their equivalent breaks a potentially complex problem into a sequence of simple steps.

Standard notation for the single payment compound amount and present worth factors allows tables in the appendix to be used. The notation indicates what is being computed, a *F*uture or a *P*rior amount, followed by what is given: $(F|P, i, m)$ and $(P|F, i, m)$. This notation is moderately useful for single payments and extremely useful for patterns of cash flows.

All of the foregoing results are extended to the multiple rate case at the end of the chapter. The concepts are the same for the constant and multiple rate cases, but the multiple rate formulas are more awkward. However, if the interest rates are constant over different regions, then the appropriate single payment factor can be used for each region.

Questions

Section 1: Basic Concepts

1.1 Define equivalence, and give examples of two basic types of equivalents.

1.2 Suppose that interest is 7% per period.

a) If there is a cash flow of \$100 at time 1, as shown in Figure 4.2, then what are the compound amounts at times 2 and 3? (\$107, \$114.49)

b) If there is a cash flow of \$114.49 at time 3, then what are the discounted amounts at times 2 and 1? (\$107, \$100)

1.3 Consider the situation shown in Figure 4.3. Write (but do not solve) the balance equation if the interest rate is 10% per year, withdrawals of \$60,000 are to be made at years 62 through 85, and unknown deposits are to be made at years 25 through 61.

Section 2: Compound Amounts

2.1 An investor deposits \$1,000 at time 6 in an initially empty account paying 7%, and then makes withdrawals of \$400 and \$500 at times 7 and 8. Draw the cash flow diagram showing any equivalents for the following parts of this problem as dotted arrows.

a) Compute the compound amount of these cash flows at time 10 using the compound amount formula and a calculator. (\$248.33)

b) Use the preceding result to calculate the compound amount at time 15. (\$348.29)

c) Evaluate the balance equation for time 15 to confirm that the equivalent for time 15 is correct.

2.2 Rework the entire preceding problem by writing the equations using standard notation and using the tabular values. What are the percent errors caused by using the tables? (a: 0.008%, b: 0.014%, c: 0.009%)

Section 3: Discounted Amounts

3.1 Consider the same cash flows used for the preceding two problems. Draw cash flow diagrams and show equivalents as dotted arrows for the following parts of this problem. The interest rate remains 7%.

a) Compute the discounted amount at time 3 using the discounted amount formula and a calculator. ($154.65)

b) Use the preceding result to compute the discounted amount at time 0. ($126.24)

c) Compute the balance at time 15 corresponding to a single cash flow E_3 and also to a single flow E_0 to verify that they are the same as the original cash flows. ($348.29)

3.2 A person wishes to make a loan at 1% per month. The payments that can be afforded are \$90 at year 1, \$80 at year 2, \$70 at year 3, and \$50 at year 4. Draw the cash flow diagram, showing any equivalents as dotted arrows. How much can the person afford to borrow? ($222.81)

3.3 Suppose that the loan in the preceding problem is paid off immediately after the second payment is made.

a) Determine the amount paid using discounted amounts. ($101.50)

b) Determine the amount paid using the balance equation. ($101.50)

c) Consider an investment that returns \$90 at year 1, \$80 at year 2, \$70 at year 3, and \$50 at year 4. If the person wants to earn 1% per month, then what is the most that should be paid for the investment? ($222.81)

3.4 Rework problem 3.2 by writing the equation using standard notation and using the tabular values. What are the percent errors caused by using the tables? (0%, which is unusual)

3.5 An investor who wishes to earn 8% per year is considering purchasing a stock. The best estimates of the future cash flows are that the dividends paid yearly to stockholders will be \$2.50 per share in 1 year, \$2.75 in 2 years, and \$3.50 in 3 years. She plans on selling the stock after receiving the dividend at year 3 for \$48.00 per share. What is the most that she should pay for each share of the stock? ($45.55)

Section 4: Equivalents for Multiple Interest Rates

4.1 An investor deposits \$1,000 at time 6 in an initially empty account, and then makes withdrawals of \$400 and \$500 at times 7 and 8. The account pays 6% from time 6 to time 7, 6.5% from 7 to 8, 7.0% from 8 to 9, 8% from 9 to 15, 9% from 15 to 20. Draw the cash flow diagram and show any equivalents as dotted arrows for the following parts of this problem.

a) Use the γ_j notation to calculate the compound amount at time 9. ($217.10)

b) Use the standard notation to compute the compound amount at time 20 by compounding the equivalent at time 9. ($530.08)

4.2 A person wants to repay a loan with payments of \$700 at time 11 and \$800 at time 12. Interest is anticipated to be 7% from time 0 to time 4, 8% from 4 to 9, then 8.5% from 9 to 10, 9.5% from 10 to 11, and 10% from 11 to 12.

a) Use the γ_j notation to calculate the amount that can be borrowed at time 9. ($1,201.33)

b) Use the standard notation to compute the discounted amount at time 0 by discounting the equivalent at time 9. ($623.75)