5: Uniform Series

The preceding chapter presents basic equivalence calculations that do not depend on patterns, such as uniform series or trends, that result in easier computations. This chapter shows procedures for uniform series, also referred to as *annuities,* and the next chapter presents trends. Topics of this chapter include compound and discounted amounts of series, multiple series, applications of series (bonds and loans), and multiple interest rates.

5.1 Compound Amounts

There are two situations involving compound amounts of series. In the first, the value of the cash flows in the series is known and the compound amount needs to be determined. In the second, the compound amount is known and the size of the series is unknown. This section determines formulas for these situations, presents the standard notation for the formulas, and illustrates the use of these new factors.

Series Compound Amount

Figure 5.1 shows the unknown compound amount E_s of a uniform series having s-r cash flows of known amount U. For example, if s were 6 and r were 4, then there would be 6-4 or 2 cash flows, at times 5 and 6. The compound amount formula indicates that the value of E_s is given by

$$E_s = U\gamma^{s-(r+1)} + U\gamma^{s-(r+2)} + \cdots + U\,, \qquad \text{(5-1)}$$

where γ equals $1+i$. Factoring U results in

$$E_s = U\,[\gamma^{s-(r+1)} + \gamma^{s-(r+2)} + \cdots + 1]\,, \qquad \text{(5-2)}$$

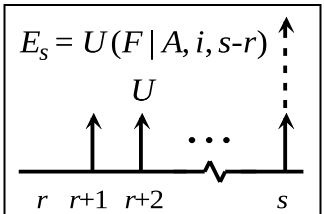

Figure 5.1 Series Compound Amount

and the term in brackets is a geometric series having the sum shown below:

$$E_s = U\,[\,(\gamma^{\,s-r} - 1)\,/\,i\,]\,. \qquad \text{(5-3)}$$

The standard notation for the sum in equation (5-3) factor is $(F|A, i, m)$:

$$\boxed{E_s = U\,(F|A, i, m)}\,, \qquad \text{(5-4)}$$

where m equals the number of cash flows in the series, s-r in this case. Its name is the *uniform series compound amount factor*, and it also is referred to as “F given A.” The notation indicates that an unknown *F*uture amount is sought, given an *A*nnuity, the interest rate, and the number of cash flows.

Examples of Series Compound Amounts

The following two examples illustrate the use of the series compound amount factor, first in a situation exactly resembling Figure 5.1 and then in a common multi-step problem.

Example 5.1 *Uniform Series Compound Amount*

Camille plans to purchase a car in three years. She intends to invest \$200 at the end of each month for the next 36 months. If the savings account pays ½% per month, then how much will she have immediately after the last deposit? Figure

5.2 shows the end-of-month deposits, with today labeled as time 0. The savings after the last deposit equal \$7,867.22:

$$\$7{,}867.22 = 200\,(F|A,\ ½\%,\ 36\text{-}0) \quad ♣ \qquad (5\text{-}5)$$

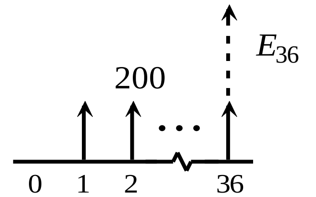

Figure 5.2 End-of-Month Deposits

Example 5.2 *Delayed Compound Amounts*

Camille wants to see how much will be in the account at time 36 if she makes her deposits at the beginning of each month. She also wishes to see how much will be in the account at time 48 if she waits an extra year but does not make deposits during that time.

Figure 5.3 shows the cash flows and the requested equivalents at times 36 and 48. The series compound amount factor is for the situations shown in Figures 5.1 and 5.2, where the compound amount is *at the time of the last cash flow*. Such series are referred to as *end-of-period* series. This is a *beginning-of-period* series relative to E_{36}, so the factor cannot be used. It is necessary to compute E_{35} first since it is at the time of the last cash flow. The number of payments for E_{35} equals 35 – (-1) or 36, the time of the last cash flow minus the time immediately *before* the first cash flow. Only *differences* in time are needed to count cash flows, even though a time such as -1 might seem odd at first. The amount in the account at time 35 is \$7,867.22:

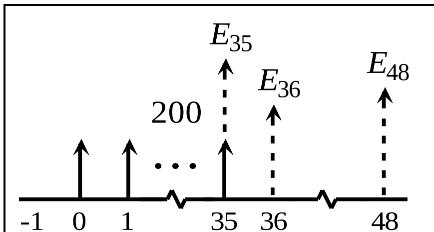

Figure 5.3 Beginning-of-Month Deposits

$$E_{35} = \$7{,}867.22 = 200\,(F|A,\ ½\%,\ 35-(-1)) \qquad (5\text{-}6)$$

This is the same as immediately after the last deposit in the preceding example, where everything happens one period later. Once E_{35} is known, then the remaining equivalents can be computed as single payment compound amounts:

$$E_{36} = \$7{,}906.56 = 7{,}867.22\,(F|P,\ ½\%,\ 36-35) \qquad (5\text{-}7)$$

$$E_{48} = \$8{,}394.32 = 7{,}867.22\,(F|P,\ ½\%,\ 48-35) \qquad (5\text{-}8)$$

There will be \$7,906.56 in the account at time 36 and \$8,394.32 at time 48. ♣

Sinking Funds

A *sinking fund* is an account into which deposits are made for the purpose of accumulating a given future sum. In Figure 5.4 the compound amount is c_s and the series is E_U. Equation (5-3) indicates that the compound amount equals the series multiplied by $[(\gamma^{s-r}-1)/i]$, so

$$c_s = E_U\,[(\gamma^{s-r}-1)/i]\,. \qquad (5\text{-}9)$$

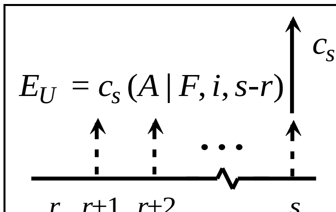

Figure 5.4 Sinking Fund Factor

Solving for E_U results in

$$E_U = c_s\,[\,i/(\gamma^{s-r}-1)]\,. \qquad (5\text{-}10)$$

This is the *sinking fund factor*, and its standard notation is

$$E_U = c_s\,(A \mid F, i, m)\ , \tag{5-11}$$

where an unknown *A*nnuity is sought, given a *F*uture amount, the interest rate, and the number of cash flows *m*. The factor also is known as "*A* given *F*."

Examples of Sinking Funds

The following two examples illustrate the use of the sinking fund factor, first in a situation exactly resembling Figure 5.4 and then in a common multi-step problem.

Example 5.3 *End-of-Month Sinking Fund*

Barrett is considering making deposits at the end of each month for three years into an account paying ½% per month to purchase a car for $10,000. How much must she invest each month? Figure 5.5 shows that the placement of the compound amount and the equivalent series corresponds to Figure 5.4, so the sinking fund factor can be used:

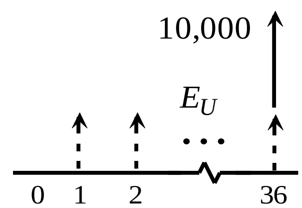

Figure 5.5 End-of-Month Sinking Fund

$$E_U = \$254.22 = 10{,}000(A|F, ½\%, 36\text{-}0) \tag{5-12}$$

If $254.22 is deposited at the end of each month, then it will accumulate to the desired compound amount of $10,000. ♣

Example 5.4 *Beginning-of-Month Sinking Fund*

Suppose Barrett still wants $10,000 at the end-of-month 36, but she plans to make beginning-of-month deposits at ½% per month. How much must the deposits be? The compound amount is not at the same time as the last cash flow of the series, so the sinking fund factor cannot be used in one step. First discount the $10,000 from time 36 to time 35 to determined how much is needed at time 35, and then use the sinking fund factor:

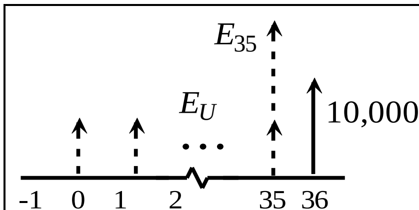

Figure 5.6 Beginning-of-Month Sinking Fund

$$E_{35} = \$9{,}950.25 = 10{,}000\,(P|F, ½\%, 36\text{-}35) \tag{5-13}$$

$$E_U = \$252.95 = 9{,}950.25\,(A|F, ½\%, 35 - (\text{-}1)\,) \tag{5-14}$$

As before, the number of cash flows in the series equals the time of the last one minus the time *before* the first one, -1 in this case. If $252.95 is deposited at times 0, 1, …, 35, then the compound amount of those deposits at time 35 will be $9,950.25. In turn, the $9,950.25 will grow into the desired $10,000 after one period. In general, if the compound amount is *after* the last cash flow of the series, then compute the discounted amount at the time of the last series flow before using the sinking fund factor. ♣

5.2 Discounted Amounts

There are two situations involving discounted amounts of series. In the first, the size of the series is known and the discounted amount must be computed. In the second, the discounted amount is known and the size of the series is unknown. This section determines formulas for these situations, presents the standard notation for the formulas, and illustrates the use of these new factors.

$E_r = U(P\,|\,A, i, s\text{-}r)$

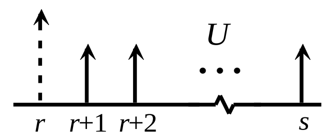

Figure 5.7 Series Discounted Amount

Series Discounted Amount

Figure 5.7 shows a series of future end-of-period cash flows having discounted amount E_r. An investor who pays E_r for the series recovers the original E_r plus interest at rate i. The investment equals the discounted amount of the series, so

$$E_r = U\gamma^{-(r+1-r)} + U\gamma^{-(r+2-r)} + \cdots + U\gamma^{-(s-r)} . \qquad (5\text{-}15)$$

Factor U to obtain

$$E_r = U\,[\,\gamma^{-1} + \gamma^{-2} + \cdots + \gamma^{-(s-r)}] , \qquad (5\text{-}16)$$

where the geometric series in the brackets has the sum shown below:

$$E_r = U\,[\,1 - (1+i)^{-(s-r)}] \,/\, i . \qquad (5\text{-}17)$$

This series sum is known as the uniform series present worth factor, and its standard notation is:

$$E_r = U\,(P|A, i, s\text{-}r) \ . \qquad (5\text{-}18)$$

A *P*rior discounted amount is sought, given a future series or *A*nnuity, interest rate i, and number of cash flows s-r.

Examples of Series Discounted Amounts

The following two examples illustrate the use of the series present worth factor, first in a situation exactly resembling Figure 5.7 and then in a common multi-step problem.

Example 5.5 *Uniform Series Discounted Amount*

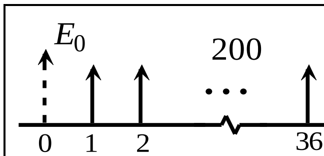

Figure 5.8 End-of-Month Payments

Camille computed that she must make 36 end-of-month deposits of $200 at ½% per month to accumulate $7,867.22 for a car. If she should borrow the money at ¾% with 36 end-of-month payments of $200, then how much could she borrow for the car? This situation exactly matches the one depicted in Figure 5.7, so the series present worth factor can be used:

$$E_0 = \$6{,}289.36 = 200(P|A, \tfrac{3}{4}\%, 36\text{-}0) \ . \qquad (5\text{-}19)$$

If Camille chooses saving $200 instead of repaying a loan at a rate of $200 per month, then she will have $1,577.86 (7,867.22 - 6,289.36) more for a car. ♣

Example 5.6 *Discounted Amount with Delayed Series*

Consider the situation shown in Figure 5.9, where the loan payments of \$200 have been delayed for 12 months. If interest remains at ¾% per month, then how much can be borrowed at time 0? The series present worth factor is for the situation in which the discounted amount occurs one period before the start of the payments, at time 12 in this case. So first discount the series to time 12, and then discount the resulting single equivalent to time 0:

$$E_{12} = \$6{,}289.36 = 200(P|A, ¾\%, 48\text{-}12) \qquad (5\text{-}20)$$

$$E_0 = \$5{,}749.97 = 6{,}289.36(P|F, ¾\%, 12\text{-}0) \qquad (5\text{-}21)$$

Notice that the number of payments is 48-12, not 48-13. As before, \$6,289.36 can be borrowed in exchange for 36 immediately following end-of-month payments. In turn, a debt of \$6,289.36 at time 12 is the result of an original debt of \$5,749.97 at time 0. Thus \$5,749.97 can be borrowed if this delayed payment plan is used. ♣

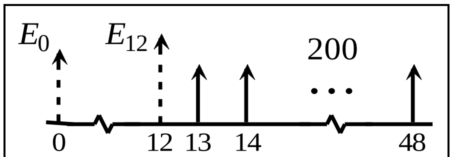

Figure 5.9 Delayed End-of-Month Payments

In general, both borrowing instead of saving and delaying payments decrease the amount available for a purchaser. Interest works for a saver and against a borrower. Moreover, rates for borrowed money typically are higher than savings rates, since many lenders, such as finance companies, actually borrow money to have the capital to lend.

Capital Recovery

Investors give up capital in hopes of recovering it with interest, as illustrated in Figure 5.10. Given an investment (or a loan) of amount c_r , what size series E_U would recover the capital with interest at rate *i*? Equation (5-17) indicates that the relationship between a discounted amount (*DA*) and an equivalent uniform series (*US*) is

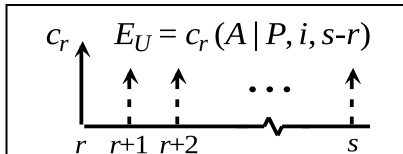

Figure 5.10 Capital Recovery

$$DA = US\,[\,1 - (1{+}i)^{-(s-r)}]\,/\,i\,. \qquad (5\text{-}22)$$

This can be solved for an unknown uniform series,

$$US = DA \times\ i\,/\,[\,1 - (1{+}i)^{-(s-r)}]\,, \qquad (5\text{-}23)$$

and expressing it in terms of Figure 5.10 results in

$$E_U = c_r\,\{\,i\,/\,[\,1 - (1{+}i)^{-(s-r)}]\,\}\,. \qquad (5\text{-}24)$$

The standard notation for this is

$$E_U = c_r(A|P, i, s\text{-}r)\,, \qquad (5\text{-}25)$$

and the factor is known as the capital recovery factor. It is used to compute an unknown series or *A*nnuity, given a *P*rior amount, the interest rate *i*, and the number of series flows *s-r*. Notice that the prior amount occurs one period *before* the first series flow, and the number of series flows equal the time of the last flow minus the time *before* the first flow.

Examples of Capital Recovery

The following two examples illustrate the use of the capital recovery factor, first in a situation exactly resembling Figure 5.10 and then in a common multi-step problem.

Example 5.7 *Capital Recovery*

Barrett determined that if she deposited \$254.22 at the end of each month in an account paying ½% per month for 3 years, then she would have \$10,000 for a car. If instead of saving, she borrows the \$10,000 at ¾% per month for 3 years, then what would her payment be at the end of each month? Figure 5.11 illustrates the loan, and the monthly payments are

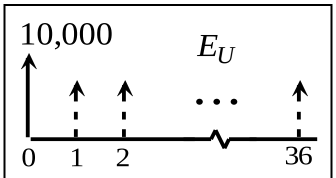

Figure 5.11 Loan

$$E_U = \$318.00 = 10{,}000(A|P, 3/4\%, 36\text{-}0)\ . \tag{5-26}$$

Barrett's notes will cost \$63.78 (\$318.00 - \$254.22) more than her saving deposits. ♣

Example 5.8 *Capital Recovery with Delayed Payments*

If Barrett should be able to delay her monthly payments in the preceding example by a year, then how much would she have to pay per month? The capital recovery factor assumes that the initial transaction occurs one period before the start of the series, so the first step is to compute the debt as of time 12 using the single payment compound amount factor:

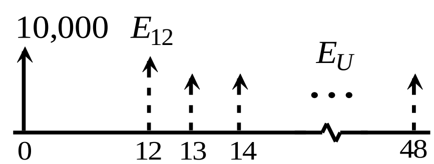

Figure 5.12 Loan with Delayed Payments

$$E_{12} = \$10{,}938.07 = 10{,}000(F|P, 3/4\%, 12\text{-}0)\ . \tag{5-27}$$

Now the capital recovery factor can compute the amount of the payments:

$$E_U = \$347.83 = 10{,}938.07\ (A|P, 3/4\%, 48\text{-}12)\ . \tag{5-28}$$

In this case, delaying payments will cost Barrett \$94.88 (347.83 – 252.95) more per month than saving for the car. ♣

5.3 Multiple Series

The preceding sections show how to convert a series into a single cash flow, a compound or discounted amount, and vice-versa. Another common form of problem involves multiple series. For example, a person or a company planning for a series of future expenditures might accumulate the funds with a series of deposits. This section provides two examples that illustrate the use of equivalents as intermediate steps in solving such problems.

Example 5.9 *Known Deposits, Unknown Withdrawals*

Mac realizes that early planning is necessary for his eventual retirement. Figure 5.13 shows that he wants to determine his yearly withdrawals *X* from 39 through

63 years in the future if he makes yearly deposits of $3,000 beginning in 2 years and ending in 30 years. The gap between the deposits and withdrawals is in anticipation of helping his children through college and having no extra funds for savings. He thinks that he can earn 8% per year on his investments.

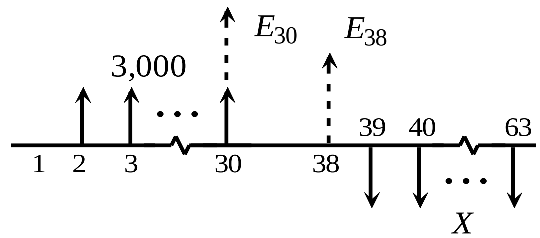

Figure 5.13 Known Deposits, Unknown Withdrawals

The best way to solve multi-step problems like this is to plan each step in terms of the factors. Careful attention must be paid to the positioning of each equivalent.

- Use $(F\,|\,A,\ i,\ m)$ to compute the compound amount E_{30}.
- Use $(F\,|\,P,\ i,\ m)$ to calculate the compound amount E_{38}, one period before the first withdrawal.
- E_{38} is positioned so $(A|P,\ i,\ m)$ can compute X.

Now implement this logic, being particularly careful with the *s-r* terms:

$$E_{30} = \$311{,}897.81 = 3{,}000\,(F|A,\ 8\%,\ 30\text{-}1) \tag{5-29}$$

$$E_{38} = \$577,\ 301.08 = E_{30}\,(F|P,\ 8\%,\ 38\text{-}30) \tag{5-30}$$

$$X = \$54{,}080.86 = E_{38}\,(A|P,\ 8\%,\ 63\text{-}38) \tag{5-31}$$

An early program of regular, modest savings of $3,000 per year can produce a significant annuity of $54,080.86 over time. ♣

Example 5.10 *Known Withdrawals, Unknown Deposits*

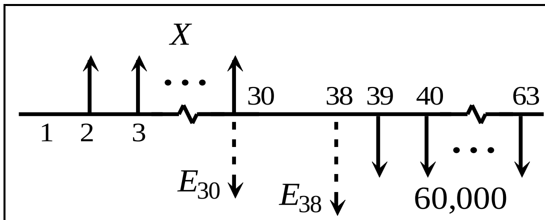

Figure 5.14 Known Withdrawals, Unknown Deposits

Suppose that Mac decides that he would like to have $60,000 per year in his retirement, as shown in Figure 5.14. How much would he have to deposit each year? The solution strategy for this problem is given below. Notice the positioning of the equivalents.

- Use $(P\,|\,A,\ i,\ m)$ to compute the discounted amount E_{38}. This amount must be in the account at time 38 to make the withdrawals.
- Use $(P|F,\ i,\ m)$ to calculate the discounted amount E_{30}, placed at the time of the last deposit. This amount will grow to E_{38}.
- E_{30} is positioned so $(A|F,\ i,\ m)$ can compute X.

Now write and solve the equations, paying attention to the *s-r* terms:

$$E_{38} = \$640{,}486.57 = 60{,}000\,(P|A,\ 8\%,\ 63\text{-}38) \tag{5-32}$$

$$E_{30} = \$346{,}034.97 = E_{38}\,(P|F,\ 8\%,\ 38\text{-}30) \tag{5-33}$$

$$X = \$3{,}328.35 = E_{30}(A|F, 8\%, 30\text{-}1) \tag{5-34}$$

A yearly savings of \$3,328.35 will produce the annuity of \$60,000 per year. ♣

5.4 Bonds

This section applies series factors to solve commonly occurring problems dealing with bonds, and the next section uses them to solve loan problems.

Both government and private industry use bonds to borrow directly from the public. Banks also borrow from the public in the form of savings accounts, so the interest rate on their loans must be higher than the rate paid on savings accounts to make a profit. Institutions that sell bonds bypass the profit margin of banks, but they have administrative costs associated with selling the bond issue and making payments on the bonds.

A bondholder typically receives an income stream of periodic payments plus a final payment, as illustrated in Figure 5.15. This bond has just been issued at time 0, and it has a *maturity date* of 10 years when its \$1,000 *redemption value* must be paid to the bondholder. The redemption value is also known as the *face or par value*. This value usually is printed on the face (front) of the bond, and it is the bond's selling price under par conditions when the interest rate paid by the bond equals the rate of return generally available in the bond market.

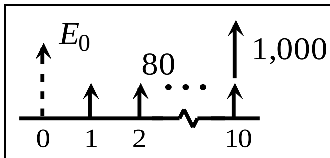

Figure 5.15 New Bond

The bond's interest rate is called its *coupon rate*, and it is 8% in this case. The bond's *coupon* is the \$80 payable yearly, where the coupon equals the face value multiplied by the coupon rate:

$$\textit{Coupon} = \textit{Face Value} \times \textit{Coupon Rate} \tag{5-35}$$

Bondholders used to clip off parts of the bond known as coupons and send them to the issuer to receive the periodic interest payments.

Prudent buyers determine a bond's cash flows and then discount them at the interest rate that the buyer wants to earn. This is the most that the buyer should offer for the bond, as illustrated by the following two examples.

Example 5.11 *Purchase of a New Bond*

How much would an investor wanting to earn 9% per year pay for the bond shown in Figure 5.15? The discounted amount at 9% of the income stream is:

$$E_0 = \$935.82 = 80(P|A, 9\%, 10\text{-}0) + 1{,}000(P|F, 9\%, 10\text{-}0) \tag{5-36}$$

If the bond should be purchased for \$935.82 and held to maturity, then the investor would earn exactly 9%, so this is the most the investor would pay for the bond. This would be a *below par* or face value offer. When the rate of return desired by investors equals the bond's coupon rate, then the bond sells *at par* or face value:

$$E_0 = \$1{,}000.00 = 80(P|A, 8\%, 10\text{-}0) + 1{,}000(P|F, 8\%, 10\text{-}0) \tag{5-37}$$

If the rate of return desired by bond investors is lower than the bond's coupon rate, such as at 7%, then the bond sells *above par* or face value:

$$E_0 = \$1{,}070.24 = 80\,(P|A,\ 7\%,\ 10\text{-}0) + 1{,}000\,(P|F,\ 7\%,\ 10\text{-}0) \quad ♣ \qquad (5\text{-}38)$$

Example 5.12 *Purchase of an Existing Bond*

Bonds are bought and sold just like other financial instruments, such as stocks. If the bond shown in Figure 5.15 has just paid its coupon at time 3, then what is the most that an investor wishing to earn 9% would be willing to offer? The remaining payments are shown in Figure 5.16, and their discounted value at time 3 equals the most that an investor wanting to earn 9% would be willing to offer:

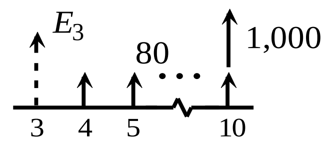

Figure 5.16 Existing Bond

$$E_3 = \$949.67 = 80\,(P|A,\ 9\%,\ 10\text{-}3) + 1{,}000\,(P|F,\ 9\%,\ 10\text{-}3) \qquad (5\text{-}39)$$

The investor's desired rate of return is greater than the coupon rate, so the offer is below par. ♣

5.5 Loans

Series factors are quite useful for several types of problems involving loans other than bonds. Example 5.5 calculated the amount that can be borrowed in exchange for a payment of a given size, and Example 5.7 determined the payment necessary for a loan of a given amount. Three other common computations for loans involve balloon notes, determining how much interest is paid each year, and early repayment.

Balloon Loan

A balloon loan occurs when the last payment is much larger than the preceding ones, expanding like a balloon. This happens when the borrower wants to keep the notes or periodic payments as low as possible, but anticipates being able to pay the larger balloon payment later. Thus a borrower might pay $100 per month for 3 years, and then an extra $5,000 with the last payment of $100. The extra payment equals the balance due at the end of the loan. This is illustrated by the following example.

Example 5.13 *Balloon Note*

Kate's company has just transferred her to another city where she is planning on buying a house. She does not think that she will be in the city for over 5 years, and she has discovered that she can get favorable terms on the loan shown in Figure 5.17. It has monthly payments calculated using a 30 year period, but paid for only 5 years, plus a balloon note at the end of 5 years. What is her month note and balloon payment for a loan of $80,000 at ¾% per month?

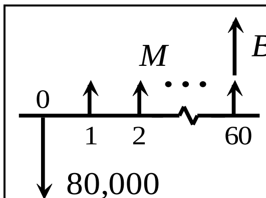

Figure 5.17 Balloon Loan

The monthly note M is based on a 30 year or 360 month period, so its value is $643.70:

$$M = \$643.70 = 80{,}000\,(A|P,\ ¾\%,\ 360\text{-}0) \qquad (5\text{-}40)$$

The amount of the balloon note B is the balance due on the loan immediately after the $643.70 payment at time 60. The balance equation indicates that

$$B = \$76{,}704.11 = 80{,}000\,(F|P,\ ¾\%,\ 60\text{-}0) - 643.70\,(F|A,\ ¾\%,\ 60\text{-}0)\ . \quad (5\text{-}41)$$

In this case, it is known that the $643.70 would pay off the loan if the payments continued through time 360, so discounting $643.70 at times 61, 62, …, 360 also results in

$$B = \$76{,}704.11 = 643.70\,(P|A,\ ¾\%,\ 360\text{-}60)\ . \quad (5\text{-}42)$$

The total payment at time 60 is $M + B$ or $77,347.81. ♣

Principal and Interest

Computing the amount of interest paid each year is important because interest for business purposes and for home ownership is a tax deduction. For example, paying $10,000 in interest on a home loan reduces income taxes by $2,800 for someone in the 28% tax bracket. Each loan payment consists of two components, interest and principal:

$$\textit{Loan Payment} = \textit{Interest} + \textit{Principal} \quad (5\text{-}43)$$

The interest component equals preceding balance multiplied by the interest rate, and the rest of the payment reduces the debt or principal. The sum of the principal components of several payments is the amount by which they reduce the debt. This debt reduction also equals the difference in balances before and after the payments. Everything else is interest, so the sum of the interest payments is given by:

$$\Sigma \textit{Interest} = \Sigma \textit{Loan Payments} - \textit{Debt Reduction}\ . \quad (5\text{-}44)$$

This is illustrated in the following example.

Example 5.14 *Principal and Interest Payments*

Thibodeaux borrowed $100,000 at ¾% per month over 30 years for a house, so his monthly payment is $804.62,

$$\$804.62 = 100{,}000\,(A|P,\ ¾\%,\ 360\text{-}0)\ . \quad (5\text{-}45)$$

Slightly over a year has passed since the purchase, and he is preparing his income tax return. During the current tax year he made payments 3, 4, …, 14 on the house. How much interest did he pay? The principal components of payments 3, 4, …, 14 reduce the debt from the balance immediately after payment 2 to the balance immediately after payment 14. Compute the balances by discounting the payments remaining at times 2 and 14 to obtain E_2 and E_{14}, shown in Figure 5.18, or use the balance equation. The debt reduction is $E_2 - E_{14}$, so

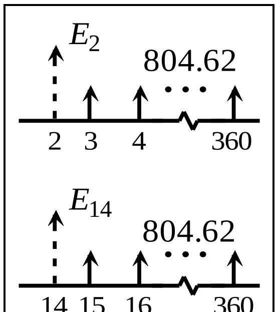

Figure 5.18 Remaining Payments

$$\textit{Debt Reduction} = 804.62\,(P|A,\ ¾\%,\ 360\text{-}2) - 804.62\,(P|A,\ ¾\%,\ 360\text{-}14) \quad (5\text{-}46)$$

or $693.48. The sum of the loan payments is $804.62×12 or $9,655.44, so the interest payment is

$$\textit{Interest} = \$8{,}961.96 = \$9{,}655.44 - 693.48 \quad (5\text{-}47)$$

Of the $9,655.44 paid, $8,961.96 is for interest and $693.48 is for principal. ♣

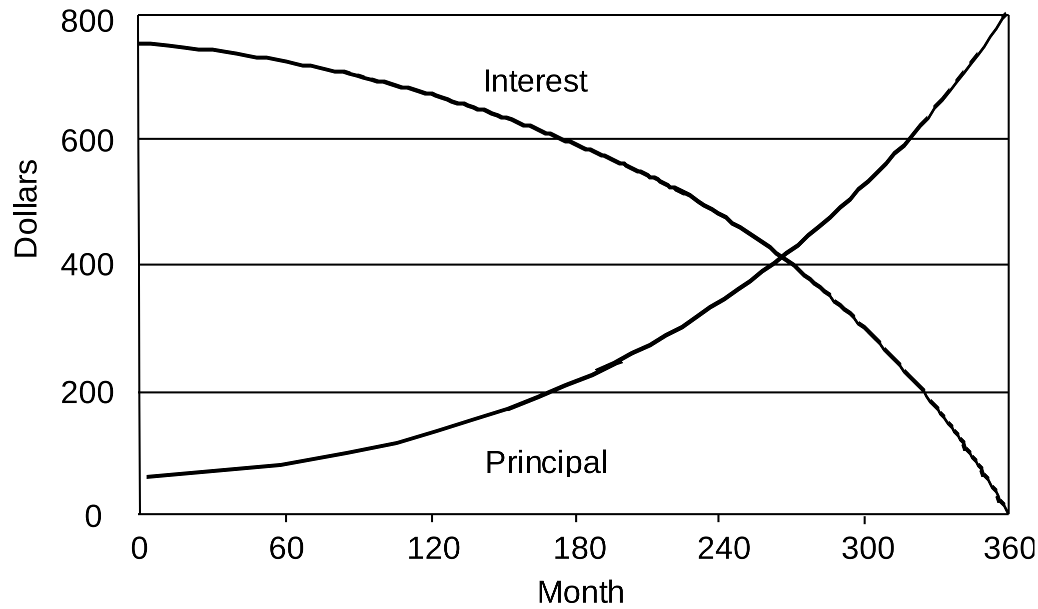

Figure 5.19 Principal and Interest Components

Early Repayment

Computing monthly balances on a spreadsheet for Example 5.14 results in Figure 5.19. The interest payments on the loan are larger than the principal payments for over 22 years, and the total of the interest payments is $189,664.14. Making extra, early principal payments can reduce interest charges substantially. Some lenders allow borrowers to make extra principal payments, whereas others do not or charge a penalty. The following example illustrates the impact of an early principal payment.

Example 5.15 *Early Loan Repayment*

Suppose that Thibodeaux's loan allows him to make early principal payments without penalty. He must continue paying at least $804.62 per month, but the loan will be paid off earlier. What will be the effect of his paying an extra $10,000 at time 12 and continuing to pay $804.62 at all other times?

After the payment of $804.62 at time 12, Thibodeaux owes the discounted amount of the remaining payments,

$$\$99{,}316.48 = 804.62\,(P|A,\ ¾\%,\ 360\text{-}12), \qquad (5\text{-}48)$$

and the extra payment of $10,000 reduces this to $89,316.48. The payments remain at $804.62 per month until an unknown month *s*. Use trial-and-error to determine the last value of *s* for which the balance is greater than or equal to zero,

$$B_S = 89{,}316.48\,(F|P,\ ¾\%,\ s\text{ -}12) - 804.62\,(F|A,\ ¾\%,\ s\text{-}12) \geq 0, \qquad (5\text{-}49)$$

or compute the balance each month on a spreadsheet using the recursive balance formula. The balance is $124.81 after the payment when *s* equals 251, so a final payment at time 252 of $124.81($F|P$, ¾%, 252-251) or $125.75 sets the balance to $0.00. Thibodeaux saves $678.87 (804.62 - 125.75) at time 252 and eliminates the remaining 108 (360 - 252) payments of \$804.62.

If he invests the eliminated payments at ½% per month, then he will have

$$E_{360} = 678.87(F|P, ½\%, 360\text{-}252) + 804.62(F|A, ½\%, 360\text{-}252) \qquad (5\text{-}50)$$

or \$116,014.76 in savings at time 360. If he had invested the \$10,000 at ½% per month instead of making an extra principal payment, then he would have had \$10,000($F|P$, ½%, 360-12) or \$56,726.96. He will have \$59,287.89 (116,014.76 - 56,726.96) more at time 360 by paying early. This is on a before tax basis. If Thibodeaux is in the 28% tax bracket, then it can be shown that he still will have \$33,007.97 more at time 360 on an after tax basis. ♣

5.6 Multiple Interest Rates

All of the preceding series formulas depend on factoring the cash flow term and recognizing the remaining terms as a geometric series. If the interest rate fluctuates, then the remaining terms do not form a geometric series, and there are no convenient formulas. This section provides relationships between series and compound or discounted amounts when interest rates change, as well as showing how series factors can still be used over regions where the interest rate is constant.

Compound Amounts

Figure 5.20 shows a uniform series, its compound amount, and the interest rates for each period. The multiple interest rate compound amount formula indicates that the relationship between the compound amount CA and the series U is

$$CA = U\gamma_{r+2}\gamma_{r+3}\cdots\gamma_s + U\gamma_{r+3}\cdots\gamma_t + \cdots + U\,. \qquad (5\text{-}51)$$

No simplifications are possible after factoring U, so

$$\boxed{CA = U(\gamma_{r+2}\gamma_{r+3}\cdots\gamma_s + \gamma_{r+3}\cdots\gamma_t + \cdots + 1)}\,. \qquad (5\text{-}52)$$

and

$$\boxed{U = CA\,/\,(\gamma_{r+2}\gamma_{r+3}\cdots\gamma_s + \gamma_{r+3}\cdots\gamma_t + \cdots + 1)}\,. \qquad (5\text{-}53)$$

The interest rate i_{r+1} is not needed for these formulas.

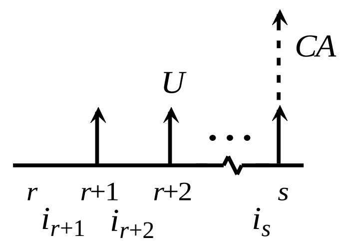

Figure 5.20 Series and Compound Amount with Multiple Rates

Example 5.16 *Multiple Rate Series and Compound Amount*

Consider the uniform series and compound amount shown in Figure 5.21. If the size of the series is known to be \$1,000, then the unknown compound amount is

$$CA = 1{,}000\,[\,(1.06)(1.065)(1.07) + (1.065)(1.07) + 1.07 + 1] \qquad (5\text{-}54)$$

or \$4,417.47. A known compound amount of \$2,000 implies that the unknown series is

$$U = 2{,}000\,/\,[\,(1.06)(1.065)(1.07) + (1.065)(1.07) + 1.07 + 1] \qquad (5\text{-}55)$$

or \$452.74. ♣

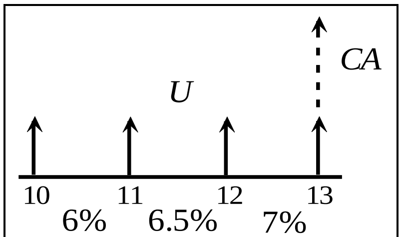

Figure 5.21 Example of Series and Compound Amount with Multiple Rates

Discounted Amounts

Figure 5.22 shows a uniform series, its discounted amount, and the interest rates each period. The multiple interest rate discounted amount formula indicates that the relationship between the discounted amount DA and the series U is

$$DA = U(\gamma_{r+1})^{-1} + \cdots + U(\gamma_{r+1}\gamma_{r+2}\cdots\gamma_s)^{-1} . \quad (5\text{-}56)$$

No simplifications are possible after factoring U, so

$$\boxed{DA = U[(\gamma_{r+1})^{-1} + \cdots + (\gamma_{r+1}\gamma_{r+2}\cdots\gamma_s)^{-1}]} \quad (5\text{-}57)$$

and

$$\boxed{U = DA / [(\gamma_{r+1})^{-1} + \cdots + (\gamma_{r+1}\gamma_{r+2}\cdots\gamma_s)^{-1}]} . \quad (5\text{-}58)$$

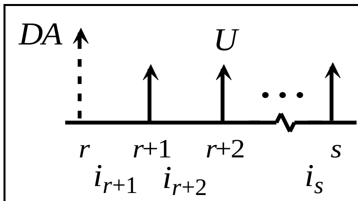

Figure 5.22 Series and Discounted Amount with Multiple Rates

Example 5.17 *Multiple Rate Series and Discounted Amount*

Consider the uniform series and discounted amount shown in Figure 5.23. If the size of the series is known to be $1,000, then the unknown discounted amount is

$$DA = 1{,}000[(1.06)^{-1} + (1.06\times1.065)^{-1} + (1.06\times1.065\times1.07)^{-1} + (1.06\times1.065\times1.07\times1.075)^{-1}] \quad (5\text{-}59)$$

or $3,427.19. A known discounted amount of $2,000 implies that the unknown series is

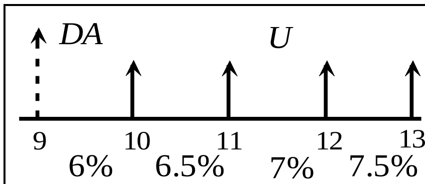

Figure 5.23 Discounted Series Example with Multiple Rates

$$U = 2{,}000 / [(1.06)^{-1} + (1.06\times1.065)^{-1} + (1.06\times1.065\times1.07)^{-1} + (1.06\times1.065\times1.07\times1.075)^{-1}] \quad (5\text{-}60)$$

or $583.56. ♣

Regions with Constant Rates

The development of the series factors assumes constant rates over the region in which they are used, so factors can be used within regions of constant rates. Equivalents must be placed at the boundaries of regions where necessary, just as with single cash flows. This is illustrated by the following examples.

Example 5.18 *Series Compound Amount with Regions*

Figure 5.24 shows the compound amount of a series that spans two interest regions. The factors can be used up to region boundaries, so compute the equivalent of the cash flows in the first region,

$$E_7 = U(F|A, 5\%, 7\text{-}4) , \quad (5\text{-}61)$$

and then the compound amount CA:

$$CA = E_7(F|P, 6\%, 12\text{-}7) \quad (5\text{-}62)$$

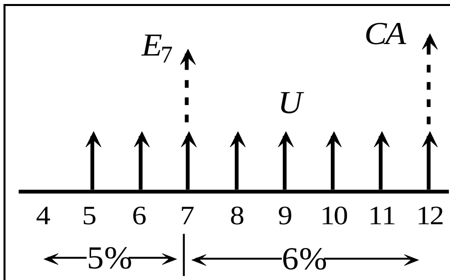

Figure 5.24 Series Compound Amount with Regions

$$+ U(F|A, 6\%, 12\text{-}7) .$$

Substituting for E_7 in equation (5-62) provides the relationship between the series and the compound amount,

$$CA = U(F|A, 5\%, 7\text{-}4)(F|P, 6\%, 12\text{-}7) + U(F|A, 6\%, 12\text{-}7) \tag{5-63}$$

or

$$CA = 9.8558U \tag{5-64}$$

If the series amount should be $1,000, then substitute its value to obtain the compound amount of $9,855.80 (9.8558×1,000). If the compound amount should be $20,000, then solve for the series value of $2,029.26 (20,000 / 9.8558). ♣

Example 5.19 *Series Discounted Amount with Regions*

Figure 5.25 shows the discounted amount of a series that spans two interest regions. Compute the equivalent of the cash flows in the second region at its boundary,

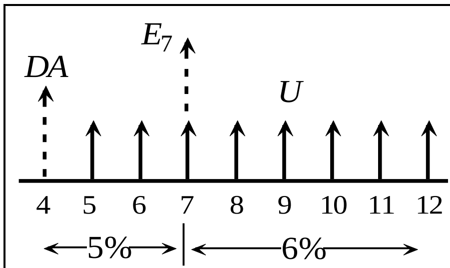

Figure 5.25 Series Discounted Amount with Regions

$$E_7 = U(P|A, 6\%, 12\text{-}7) , \tag{5-65}$$

and then the discounted amount DA:

$$DA = E_7(P|F, 5\%, 7\text{-}4) + U(P|A, 5\%, 7\text{-}4) . \tag{5-66}$$

Substitute for E_7 in equation (5-66) to obtain

$$DA = U(P|A, 6\%, 12\text{-}7)(P|F, 5\%, 7\text{-}4) + U(P|A, 5\%, 7\text{-}4) \tag{5-67}$$

or

$$DA = 6.3620U \tag{5-68}$$

If the series amount should be $1,000, then substitute its value to obtain the discounted amount of $6,362.00 (6.3620×1,000). If the discounted amount should be $20,000, then solve for the series value of $3,143.67 (20,000 / 6.3620). ♣

5.7 Summary

This chapter introduces factors for uniform series. A common aspect of these factors is the argument for the number of cash flows in the series. It always equals the time of the last cash flow minus the time *before* the first cash flow.

The uniform series compound amount factor computes the compound amount, and the sinking funds factor calculates the size of the series. Both of these factors require the compound amount to be positioned at the time of the last cash flow, and any other positioning results in a multi-step problem.

The uniform series present worth factor discounts a series, and the capital recovery factor computes the periodic flow required to recover an investment with interest. The discounted amount must be placed one period before the first cash flow, or a multi-step problem occurs.

Problems involving multiple series use single equivalents as intermediate steps. Compound amounts must be positioned at the time of the last cash flow of a series, and discounted amounts have to be positioned one period prior to the first cash flow. Single payment factors are used to position the equivalents as needed.

Applications of series include bonds and loans. Bond problems require mastery of the vocabulary associated with bonds in order to construct the future income stream. This includes the terms maturity date, redemption (face or par) value, coupon rate, and coupon. The most an investor should pay for a bond is the discounted value of its income stream, with discounting done at the investor's desired rate of return.

Loan problems include balloon notes, an extra payment at the end of a loan equal to its balance due. Computing the amount of interest paid on a loan during a year is useful, since it can be a tax deduction. Early loan payments have a large interest component, and early principal payments can be good investments if the interest rate on the loan is greater than the rate of return the investor can earn elsewhere.

Multiple interest rates preclude developing simple factors for uniform series. If the rates continually fluctuate, then the best that can be done is to use the multiple rate compound and discounted amount formulas, factor the series amount, and manually compute the sum of the remaining terms. If rates are stable over regions, then the factors can be used within regions to compute single equivalents. These equivalents can be moved from one region to the next by single payment factors with each region.

Questions

Section 1: Compound Amounts

1.1 Midas earns ¾% per month on his investments. He plans to save $300 per month to buy a used truck. How much truck can he buy with deposits at times:

a) 1, 2, …, 30 and the purchase at time 30? (10,050.87)

b) 0, 1, …, 29 and the purchase at time 30? (10,126.25)

c) 1, …, 30 and the purchase at time 48? (11,497.80)

1.2 Midas has decided that he wants to spend $15,000 on his truck. He earns ¾% per month, so how much is his monthly deposit if it is made at times:

a) 1, 2, …, 30 with the purchase at time 30? (447.72)

b) 0, 1, …, 29 with the purchase at time 30? (444.39)

c) 1, …, 30 with the purchase at time 48? (391.38)

Section 2: Discounted Amounts

2.1 Midas borrows at 1% per month to buy his beloved truck. If he wants to pay $300 per month, then how much can he borrow at time 0 with payments at times:

a) 1, 2, …, 30? (7,742.31)

b) 13, 14, …, 42? (6,870.91)

2.2 At time 0 Midas borrows $10,050.87 at 1% per month to buy the wretched truck. What will be the size of his monthly payments if they are made at times:

a) 1, 2, …, 30? (389.45)

b) 13, 14, …, 42? (438.84)

2.3 Two new graduates have the potential to save $550 each month at ½% per month. Expenses, such as buying a car or stereo must come out of those potential savings. This problem examines the effects of different financial practices on their bank balances after 6 years. Flambeaux buys a new car by borrowing $20,000 at 1% per month with payments at times 1, 2, …, 48. He will keep the car for 6 years and then sell it for $4,000. Marie buys a 3 year old used car for $10,000 by borrowing at 1% with payments at times 1, 2, …, 36. She will sell the 6 year old car after 3 years for $4,000. If she has $10,000 in savings at the end of three years, then she will buy another 3 year old car for $10,000 cash and sell it after 3 years for $4,000. If she does not have the $10,000 in savings, then she will borrow whatever she needs at 1% per month with payments at times 37, 38, …, 72.

a) What are Flambeaux's monthly payments on his car? (526.68)

b) How much will he have in savings after his deposit at time 48? (1,261.56)

c) What will his savings be after selling his car at time 72? (19,409.56)

d) What are Marie's monthly payments on her car? (332.14)

e) How much will she have in savings after she sells her car at time 36? (12,569.76)

f) Will she have to borrow for the next car? If so, how much? (No)

g) How much will her payments be at times 37, 38, …, 72, if any? (0)

h) What will her savings be after selling her car at time 72? (28,710.04)

Section 3: Multiple Series

3.1 Dominique plans to graduate when he is 23 and make yearly deposits at times 24, 25, … 60 into his individual retirement account (IRA). He plans on making withdrawals when he is 66, 67, …, 85.

a) What are his withdrawals if deposits of $3,000 earn 6% per year? (44,546.11)

b) What are his withdrawals if deposits of $3,000 earn 9% per year? (130,648.44)

c) What deposits at 6% per year allow withdrawals of $70,000? (4,714.22)

d) What deposits at 9% per year allow withdrawals of $70,000? (1,607.37)

Section 4: Bonds

4.1 A bond has just been issued with a maturity date 20 years from now. Its face value is $5,000 with a coupon rate of 7% per year and coupons paid each year.

a) What is the most that an investor wanting to earn 8% should pay? Is this above or below par? (4,509.09, below)

b) If 5 years have passed and the fifth coupon has just been paid, then what is the most that an investor wanting to earn 6% should pay? Is this above or below par? (5,485.61, above)

Section 5: Loans

5.1 Dominique borrows $10,000 for a car at 1% per month. The loan is for 36 months, and she wants to keep the notes low until she graduates. She negotiates a $200 per month payment at times 1, 2, …, 36, plus an extra balloon payment at time 36. What is the amount of the balloon payment? (5,692.31)

5.2 Pierre borrows $80,000 at ¾% per month for 15 years for a house.

a) What is the monthly note? (811.41)

b) How much interest is contained in payments 7, 8, …, 18? (6,971.40)

c) What is the total amount of interest over the life of the loan? (66,054.80)

5.3 Taccaux borrows $80,000 at ¾% per month for 25 years for a house.

a) What is his monthly note? (671.36)

b) If he pays $10,000 extra at month 12 and keeps paying the regular note, then how much will he owe after the regular plus extra payment at time 12? (69,107.50)

c) When will the last payment be? (210)

d) How much will the last payment be? (588.23)

e) If he had not paid early, Taccaux would have invested the $10,000 at ½% per month. How much would he have at the end of year 25? (42,055.79)

f) If he invests his savings in note payments at ½% per month, how much will he have at the end of year 25? (76,202.66)

Section 6: Multiple Interest Rates

6.1 Suppose that interest rates will be 7% from year 0 to year 1, 8% from year 1 to year 2, 9% from year 2 to year 3, and 10% from year 3 to year 4.

a) What is the compound amount at time 4 of deposits of $1,000 at times 1, 2, 3, and 4? (4,593.92)

b) If $5,000 is wanted as a compound amount at time 4, then how much must be deposited at times 1, 2, 3, and 4? (1,088.40)

c) If withdrawals of $1,000 are wanted at times 1, 2, 3, and 4, then how much must be deposited at time 0? (3,315.56)

d) If $5,000 is deposited at time 0, then how much can be withdrawn at times 1, 2, 3, and 4? (1,508.04)

6.2 Interest rates are 7% until year 10 and 9% from year 10 to year 20.

a) What is the compound amount at time 20 of deposits of $1,000 at times 6, 7, … 16? (24,233.89)

b) If $50,000 is wanted as a compound amount at time 20, then how much must be deposited at times 6, 7, … 16? (2,063.23)

c) If withdrawals of $1,000 are wanted at times 6, 7, … 16, then how much must be deposited at time 0? (5,203.80)

d) If $50,000 is deposited at time 0, then how much can be withdrawn at times 6, 7, … 16? (9,608.37)